PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 1, Pages 145–154
S 0002-9939(04)07520-3
Article electronically published on June 23, 2004

KRYLOV-BOGOLYUBOV AVERAGING OF ASYMPTOTICALLY AUTONOMOUS DIFFERENTIAL EQUATIONS

ANATOLIY SAMOILENKO, MANUEL PINTO, AND SERGEI TROFIMCHUK

(Communicated by Carmen C. Chicone)

ABSTRACT. We apply the Krylov and Bogolyubov asymptotic integration procedure to asymptotically autonomous systems. First, we consider linear systems with quasi-periodic coefficient matrix multiplied by a scalar factor vanishing at infinity. Next, we study the asymptotically autonomous Van-der-Pol oscillator.

1. INTRODUCTION

In this paper, we study the behavior of some classes of asymptotically autonomous differential equations of the form $x' = f(t, x, h(t))$, $x \in \mathbb{R}^n$, applying the Krylov and Bogolyubov (KB) asymptotic integration procedure [3, 16] to the respective quasi-periodic system $x' = f(t, x, \varepsilon)$. We suppose here that $\lim_{t\to+\infty} h(t) = 0$ and that ε is a "small" parameter. We indicate several cases when the procedure of asymptotic integration in ε of $x' = f(t, x, \varepsilon)$ can be used successfully to study solutions of $x' = f(t, x, h(t))$ as $t \to \infty$. For the first time this idea was proposed in [2] (see also the Appendix) for the periodic linear systems. Independently, Skriganov [15] used an analogous approach to describe the eigenfunctions of some self-adjoint Schrödinger operator in $L_2(\mathbb{R}_+)$, induced by the differential expression $hy = -y'' + [v(t) \cos \omega t]y$, where $v^{(j)}(t) = O(t^{-\alpha-j})$ at $t = +\infty$, and $\alpha > 0$, $j = 0, 1, 2$. Next, Harris and Sibuya [9] followed a similar approach, again considering linear periodic systems. See also [6, Chapter 4, §§4.6-4.8, and references on pp. 227-228], where very close ideas were proposed by Cassell, Eastham and McLeod. We notice that Cassell pointed out the possibility of an extension of the methods in [6, 5] to linear systems with almost periodic coefficients. In this respect, it is important to mention the method of asymptotic integration of the linear systems $x' = (A_0 + \sum_{j=1}^{k} A_j(t)t^{-j\alpha})x$, where $\alpha k \in (0, 1]$ and the $A_j(t)$ are finite trigonometric polynomials, proposed by Samokhin and Fomin [14] for $k = \alpha = 1$ and by Burd and Karakulin [4] in the general case. In fact, they used some appropriate expansions in powers of the small parameter ε. Our results develop further these lines of investigation, extending them to quasi-periodic and nonlinear systems.

Received by the editors May 7, 2002 and, in revised form, September 9, 2003.
2000 *Mathematics Subject Classification.* Primary 34E05.
Key words and phrases. Asymptotic integration, asymptotically autonomous equation, Levinson theorem, Krylov-Bogolyubov averaging principle, Van-der-Pol oscillator, adiabatic oscillator.
The first author was supported in part by FONDECYT (Chile), project 7960723.
The second and third authors were supported in part by FONDECYT (Chile), project 8990013.

In the latter case, our approach complements convergence results obtained within the frame of a Poincaré-Bendixson trichotomy theory (see [11] and the references therein) providing us with exact asymptotic formulae.

2. Integration of quasi-periodic linear systems

Here we apply the KB method developed for the linear systems $x' = (P+\varepsilon Q(t))x$ with continuous quasi-periodic matrix $Q(t)$ to integrate asymptotically autonomous systems $x' = (P + h(t)Q(t))x$. We suppose that all solutions of the nonperturbed systems $x' = Px$ are bounded on $\mathbb{R}$ so that the Lyapunov transformation $x = \exp(Pt)y$ reduces the above systems to their respective Bogolyubov's standard forms $y' = \varepsilon A(t)y$ and $y' = h(t)A(t)y$, where $A(t) = \exp(-Pt)Q(t)\exp(Pt)$ is also quasi-periodic. Here we call $x = S(t)y$ the Lyapunov transformation if S is a bounded differentiable invertible matrix function such that S^{-1} is also bounded. The quasi-periodicity of A means that it can be represented in the form $A(t) = \sum_{\mathbf{n}\in\mathbb{Z}^k} a_{\mathbf{n}} \exp(i\mathbf{n}\mathbf{w}t)$, where $||A||_2^2 = \sum |a_{\mathbf{n}}|^2 < \infty, \mathbf{n} = (n_1, n_2, ..., n_k)$, and $\mathbf{w} = (\omega_1, \omega_2, ..., \omega_k)$ is a frequency basis for $A(t)$. If, in addition, $\sum |\mathbf{n}|^{2s}|a_{\mathbf{n}}|^2 < \infty$ where $|\mathbf{n}| = \sum |n_j|$, we will write $A \in H_s$.

In this section, where some fixed $m \in \mathbb{N}$ will denote the order of approximations, we suppose that

- **(H1)** $A \in H_l$ for some $l > m(k+1)+k/2+1$. If the Fourier sum for A is not finite, then the frequency basis $\mathbf{w}$ is strongly incommensurable: $|\mathbf{n}\mathbf{w}| \geq c|\mathbf{n}|^{-k-1}$ for every $\mathbf{n} \in \mathbb{Z}^k \setminus \mathbf{0}$ and some $c > 0$.

Having assumed this, we can exploit the formalism of the BM method, which was developed for $x' = \varepsilon A(t)x$ in [16], this time with the purpose of asymptotic integration of $x' = h(t)A(t)x$:

Theorem 2.1. *Let* **(H1)** *be satisfied and let* $h : [0, +\infty) \to (0, +\infty)$ *be absolutely continuous,* $h(+\infty) = 0$. *Then there are quasi-periodic functions* Y_j *(each is unique up to the mean value) such that the change of variables* $x = (E + \sum_{j=1}^m \varepsilon^j Y_j(t))\xi$ *transforms the system* $x' = \varepsilon A(t)x$ *into*

$$\xi' = (\sum_{j=1}^{m} \varepsilon^j A_j + \varepsilon^{m+1} A_{m+1}(t, \varepsilon))\xi.$$

Next, $x = (E + \sum_{j=1}^m h^j(t)Y_j(t))\xi$ *transforms* $x' = h(t)A(t)x$ *into*

$$\xi' = (\sum_{j=1}^{m} h^j(t)A_j + h^{m+1}(t)B_{m+1}(t) + h'(t)C_{m+1}(t))\xi$$

where $B_{m+1}(t) = O(1)$ *and* $C_{m+1}(t) = O(1)$.

Remark 2.2. Without restricting the generality, we can assume that $\mathrm{Tr}A(t) = \sum a_{ii}(t) = 0$ in Theorem 2.1; in this case also $\mathrm{Tr}A_j = 0$. To prove the latter, take the fundamental matrix $X(t, \varepsilon)$ of $x' = \varepsilon A(t)x$ and observe that

$$1 = \det X(t, \varepsilon) = b(t, \varepsilon) \exp((\sum_{j=1}^{m} \varepsilon^j \mathrm{Tr}A_j)t + o(\varepsilon^m)),$$

where $b(t, \varepsilon) = \det(E + \sum_{j=1}^m \varepsilon^j Y_j(t))$ is bounded.

Proof. We should find quasi-periodic functions $Y_j(t)$ and constant matrices $A_j,\ j \leq m$, such that the Lyapunov transformation

$$x = (E + h(t)Y_1(t) + h^2(t)Y_2(t) + ... + h^m(t)Y_m(t))\xi \tag{2.1}$$

transforms $x' = h(t)A(t)x$ into

$$\xi' = (h(t)A_1 + h^2(t)A_2 + ... + h^m(t)A_m + A_{m+1}(t))\xi, \tag{2.2}$$

where $A_{m+1}(t)$ is to be described. To this end $Y_i(t), A_i$ should satisfy the identity

$$\begin{aligned} h'\{Y_1 + 2hY_2 + mh^{m-1}Y_m\} + \{hY_1' + h^2Y_2' + ... + h^mY_m'\} \\ +(E + hY_1 + h^2Y_2 + ... + h^mY_m)(hA_1 + h^2A_2 + ... + h^mA_m + A_{m+1}) \\ = hA(E + hY_1 + h^2Y_2 + ... + h^mY_m). \end{aligned} \tag{2.3}$$

By comparing the coefficients at like powers of h, we obtain the following recurrent relations to determine Y_j and A_j:

$$Y_1'(t) + A_1 = A(t), \tag{2.4}$$

$$Y_2'(t) + A_1Y_1(t) + A_2 = A(t)Y_1(t), \tag{2.5}$$

$$Y_j'(t) + A_j = A(t)Y_{j-1}(t) - \sum_{i=1}^{j-1} Y_i(t)A_{j-i},\ j = 3, 4, ..., m.$$

Now, take $A_1 = \mathbf{M}(A)$, where $\mathbf{M}$ is the averaging operator: $\mathbf{M}(A) = a_0$. Then Eq. (2.4) can be written as

$$Y_1'(t) = A(t) - \mathbf{M}(A) = \sum_{\mathbf{n}\neq 0} a_{\mathbf{n}} \exp(i\mathbf{nw}t).$$

Now, using the incommensurability condition from (**H2**), we can find that

$$Y_1(t) = \sum_{\mathbf{n}\neq\mathbf{0}} \frac{a_{\mathbf{n}}}{i\mathbf{nw}} \exp(i\mathbf{nw}t) \in H_{l-(k+1)}$$

(see also [13, Theorem 2, p. 24] for more details). Moreover, since $A \in H_l$, we have that $AY_1 \in H_{l-(k+1)}$ (by the Moser inequality, see [13, pp. 5-8]). Proceeding analogously, we can take A_2 as

$$A_2 = \mathbf{M}(AY_1 - A_1Y_1) = \mathbf{M}(AY_1),$$

then find $Y_2 \in H_{l-2(k+1)}$ by integrating (2.5), etc. Finally, in this way, we can find all $A_j,\ j = 1, 2, ..., m$ and $Y_j \in H_{l-j(k+1)} \subset H_{k'}$, with $k' > k/2 + 1$. Since, by [13, Theorem 1, p. 5], all $Y_j, j \leq m$ are continuously differentiable, (2.3) takes the form

$$(E + h(t)Y_1(t) + h^2(t)Y_2(t) + ... + h^m(t)Y_m(t))A_{m+1}(t) = h'(t)\alpha(t) + h^{m+1}(t)\beta(t),$$

where α, β are continuous and bounded. Thus $A_{m+1}(t) = h^{m+1}(t)B_{m+1}(t) + h'(t)C_{m+1}(t)$ where $B_{m+1}(t) = O(1)$ and $C_{m+1}(t) = O(1)$. Obviously, if we replace $h(t)$ with ε in the above considerations, we end the proof of Theorem 2.1. Notice that all $A_j, Y_j(t)$ are real since $A(t)$ is a real matrix-valued function. □

To progress in the analysis of system (2.2) and to develop further the analogy between asymptotic integrations of $x' = \varepsilon A(t)x$ and $x' = h(t)A(t)x$, we should eliminate the trivial case $h \in L_1(\mathbb{R}_+)$ (which is the analogue of $\varepsilon = 0$) and to find an analogue to the fact of constancy of $\varepsilon : \varepsilon' = 0$. To this end, we will assume

(**H2**) To be definite, let $j \le m$ be such that $A_j \ne 0$ while $A_i = 0$ if $i < j$. Let h', $h^{m+1} \in L_1(\mathbb{R}_+)$ while $h^r \notin L_1(\mathbb{R}_+)$ for $r \le j$ (all this implies $\lim_{t\to\infty} h(t) = 0$).

Taking into account **(H2)**, we rewrite (2.2) as

$$\xi'(t) = h^j(A_j + hA_{j+1} + ... + h^{m-j}A_m + h^{m-j+1}B_{m+1} + h'h^{-j}C_{m+1})\xi(t).$$

The change of the time variable $s = \sigma(t) = \int_0^t h^j(u)du$ (note that $\sigma(\mathbb{R}_+) = \mathbb{R}_+$ due to **(H2)**) transforms this system into

$$\zeta'(s) = (A_j + ... + \mu^{m-j}(s)A_m + \mu^{m-j+1}(s)B_{m+1}(\sigma^{-1}(s)) + R(s))\zeta(s), \tag{2.6}$$

where $\zeta(s) = \xi(t)$, $\mu(s) = h(\sigma^{-1}(s))$ and $R(s) = h'(\sigma^{-1}(s))\mu^{-j}(s)C_{m+1}(\sigma^{-1}(s))$. It is clear that $\mu(s) \to 0$ as $s \to +\infty$. Moreover, $\mu', R \in L_1(\mathbb{R}_+)$, while $\mu^q \in L_1(\mathbb{R}_+)$ if and only if $h^{q+j} \in L_1(\mathbb{R}_+)$. In this situation, the possibility of asymptotic integration of (2.6) depends essentially on the spectral properties of the leading term A_j. The most important (and simple) situation occurs when all eigenvalues $\lambda_i = \Re\lambda_i \pm \Im\lambda_i$ of the real matrix A_j have different real parts $\Re\lambda_i$, that is,

(**H3**) All eigenvalues λ_i of A_j are distinct, and every vertical line $l(\alpha) = \{\alpha + it : t \in \mathbb{R}\} \subset \mathbb{C}$ contains at most one pair of eigenvalues λ_i.

In this case there exists a Lyapunov change of variables $\zeta = S(s)z$ with $S' \in L_1(\mathbb{R}_+)$ transforming the matrix $A_j + ... + \mu^{m-j}(s)A_m$ to the diagonal form $\Lambda(s) = \mathrm{diag}\{\lambda_1(s), ..., \lambda_n(s)\}$ with $\lambda_i(s) = \lambda_i + O(\mu(s))$. Moreover, due to **(H3)**, we can apply the Levinson theorem (see [10] and [2, 8, 9, 12]) to obtain the following result:

Theorem 2.3 (Bogolyubov averaging principle)**.** *Assume that $A(t)$ is a real matrix function and that* (**H1**), (**H2**), (**H3**) *are satisfied. Then there exists a fundamental matrix $X(t)$ of the system $x' = h(t)A(t)x,\ x \in \mathbb{R}^n$ that has the following asymptotic representation:*

$$X(t) = (E + o(1))\exp(A_j \int_0^t (h^j(u) + O(h^{j+1}(u)))du).$$

Remark 2.4. The above restriction on the eigenvalues of A_j is not necessary: in fact, supposing that all eigenvalues of $A_j + h(t)A_{j+1} + ... + h^{m-j}(t)A_m$ are different and that they satisfy some additional dichotomy conditions (e.g., see [12, Theorem 2.5]), we can generalize somewhat the result of Theorem 2.3.

Example 2.5 (Adiabatic oscillator)**.** Let $a(t) = \sum a_{\mathbf{n}} \exp(i\mathbf{nw})t$ be real (so that $a_{\mathbf{n}} = a_{-\mathbf{n}}$) and analytic ($|a_{\mathbf{n}}| \le c_1 \exp(-c_2|\mathbf{n}|)$ for some $c_1, c_2 > 0$ and all $\mathbf{n}$). Simplifying, suppose that $a_{\mathbf{0}} = 0$. In the case when the above sum is not finite, we will need also the following incommensurability inequality: $|\mathbf{nw}| \ge c_3|\mathbf{n}|^{-k-1}$, where $c_3 > 0,\ \mathbf{n} \in \mathbb{Z}^k \setminus \mathbf{0}$. Now take $\lambda > 0$ and consider

$$y'' + (1 + a(\lambda t)h(t))y = 0,\ y \in \mathbb{R},\ t \ge 1, \tag{2.7}$$

which can be written as $z' = (P + a(\lambda t)h(t)Q)z$, where $z = (y, y')^T$ and

$$P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (\text{so that } \exp(Pt) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}), \quad Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Applying the Lyapunov transformation $z = \exp(Pt)x$, we find that

$$x' = \frac{a(\lambda t)h(t)}{2}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} -\sin 2t & \cos 2t \\ \cos 2t & \sin 2t \end{pmatrix}\right]x. \tag{2.8}$$

Now, let λ satisfy $|\lambda\mathbf{nw} \pm 2| \geq c_4|\mathbf{n}|^{-l}$ with some integer l and $c_4 > 0$. Since $a(t)$ is analytic, we can use Theorem 2.1 for every $m \in \mathbb{N}$. We find easily that all $A_j = 0$ in (2.6) since $\lambda \notin \mathcal{R} = \{\pm 2/(\mathbf{nw}), \mathbf{n} \in \mathbb{Z}^k \setminus \mathbf{0}\}$. Therefore, in the case when $|\lambda\mathbf{nw}\pm 2| \geq c_4|\mathbf{n}|^{-l}$ for all $\mathbf{n} \in \mathbb{Z}^k \setminus \mathbf{0}\}$, and $h^s \in L_1(\mathbb{R}_+)$ for some integer s, the fundamental matrix solution for (2.7) can be represented as $X(t) = \exp(Pt)(E + o(1))$.

Next, assume that $\lambda \in \mathcal{R}$: in this case, there exists a unique index $\mathbf{d} = \mathbf{d}(\lambda)$ such that $\lambda\mathbf{dw} + 2 = 0$. We will use Theorems 2.1, 2.3 with $m = 2$ reducing Eq. (2.8) to $\xi' = (h(t)A_1 + h^2(t)A_2 + h^3(t)B_3(t) + h'(t)C_3(t))\xi$, where

$$A_1(\lambda) = \frac{1}{2}\begin{pmatrix} \alpha_1(\lambda) & -\alpha_2(\lambda) \\ -\alpha_2(\lambda) & -\alpha_1(\lambda) \end{pmatrix},$$
$$A_2(\lambda) = \frac{1}{4}\begin{pmatrix} \phi_1(\lambda) & -\theta(\lambda) + \phi_2(\lambda) \\ \theta(\lambda) + \phi_2(\lambda) & -\phi_1(\lambda) \end{pmatrix},$$

and $\alpha_1 = \mathbf{M}(a(\lambda t)\sin 2t) = \Im a_{\mathbf{d}},\ \alpha_2 = \mathbf{M}(a(\lambda t)\cos 2t) = \Re a_{\mathbf{d}}$,

$$\phi_1 = \sum_{\mathbf{n}+\mathbf{j}+\mathbf{d}=\mathbf{0}} \frac{-2\Im(a_{\mathbf{n}}a_{\mathbf{j}})}{(\lambda\mathbf{jw} - 2)\lambda\mathbf{jw}},\ \phi_2 = \sum_{\mathbf{n}+\mathbf{j}+\mathbf{d}=\mathbf{0}} \frac{-2\Re(a_{\mathbf{n}}a_{\mathbf{j}})}{(\lambda\mathbf{jw} - 2)\lambda\mathbf{jw}},\ \theta = \sum_{\mathbf{j}\neq\mathbf{d}} \frac{|a_{\mathbf{j}}|^2}{\lambda\mathbf{jw} + 2}.$$

Assume now that $\det A_1(\lambda) = -\alpha_1^2(\lambda) - \alpha_2^2(\lambda) = -|a_{\mathbf{d}}|^2 \neq 0$. Then, by Theorem 2.3, the final asymptotic formula is

$$Z(t) = \exp(Pt)X(t) = \exp(Pt)\exp(0.5A_1(\lambda)\int_1^t (h(s) + O(h^2(s)))ds)[E + o(1)].$$

Since $\mathrm{Tr}A_1(\lambda) = 0$ and $\det A_1(\lambda) < 0$, there are two linearly independent solutions of (2.7), one of which is unbounded while the other converges to zero (we have the so-called resonant state of (2.7)).

Next, if $\det A_1(\lambda) = 0$, then $A_1(\lambda) = \mathrm{O}$, so that A_2 should be considered. First, by the Levinson result [10], $Z(t) = \exp(Pt)(E + o(1))$ if $h^2 \in L_1(\mathbb{R}_+)$. Assume now that $h^2 \notin L_1(\mathbb{R}_+)$. Since the eigenvalues $\lambda_{1,2} = \pm\sqrt{-\det A_2(\lambda)}$ of $A_2(\lambda)$ are different when $\det A_2(\lambda) \neq 0$, we can apply again Theorem 2.3 in this case to obtain

$$Z(t) = \exp(Pt)\exp(A_2(\lambda)\int_1^t (h^2(s) + O(h^3(s)))ds)[E + o(1)].$$

Finally, suppose that $a(t)$ is either a periodic function or a quasi-periodic polynomial. In both cases, it can be easily proved that $\lim_{\lambda\to 0, \lambda\in\mathcal{R}} \det A_2(\lambda) = ||a||_2^4/64$, so that we obtain a resonant state for sufficiently small λ if and only if $\lambda\mathbf{wd} = 2$ and $a_{\mathbf{d}} \neq 0$. (This interesting observation is due to Eastham [7], who proved that $\det A_2(\lambda) > 0$ for $\lambda \ll 1$ and sufficiently smooth periodic $a(t)$.) In particular, for every quasi-periodic polynomial $a(t)$ the number of resonant values of λ is finite.

3. Asymptotic integration of the van der Pol oscillator

In this section, we consider the asymptotically autonomous equation

$$x'' + x = h(t)(1 - x^2)x', \tag{3.1}$$

where $h(t)$ is a nonnegative scalar function vanishing at infinity. We have

$$x'' + x = 0 \tag{3.2}$$

as the limit equation to (3.1). Using the Marcus theorem [11], we conclude that the ω-limit set of every bounded trajectory to Eq. (3.1) is a connected compact annulus in $\mathbb{R}^2$ consisting of solutions to (3.2). Moreover, Eq. (3.1) is an integrable asymptotically autonomous one [11] when $h \in L_1(\mathbb{R}_+)$. Since the limit equation (3.2) has smooth Lyapunov function $V(x, x') = x^2 + (x')^2$, we can conclude, by [11, Theorem 2.7], that the ω-limit set of every bounded solution of (3.1) is a single periodic orbit of (3.2). It can be proved that for integrable h, the inverse statement also holds: every periodic orbit of (3.2) attracts some trajectory of the nonautonomous system (3.1). Now, if h is not integrable, the asymptotic behavior of the solutions to Eq. (3.1) can change drastically. For example, we believe that all nontrivial solutions of the nonautonomous van der Pol equation with $h \notin L_1(\mathbb{R}_+)$ and $h' \in L_1(\mathbb{R}_+)$ converge to the single periodic orbit $x = 2\sin t, x' = 2\cos t$. Here we prove a local version of this statement, investigating solutions with initial values near the indicated circle. Again our studies are based on the KB method, this time on its nonlinear version.

Theorem 3.1. *Let $h : [0, +\infty) \to [0, +\infty)$ be such that $h' \in L_1(\mathbb{R}_+)$, $h(+\infty) = 0$ while $h \notin L_1(\mathbb{R}_+)$. Then there are $\delta > 0$ and $T > 0$ such that every solution to (3.1) satisfying $4 - \delta \le x^2(T) + (x')^2(T) \le 4 + \delta$ has the following asymptotic form:*

$$x(t) = (2 + o(1)) \sin(t + O(\int_0^t h(s)ds)), \quad x'(t) = (2 + o(1)) \cos(t + O(\int_0^t h(s)ds)).$$

To prove this theorem, we will need the following.

Lemma 3.2. *Let $a \in C(\mathbb{R}_+, \mathbb{R}_+) \setminus L_1(\mathbb{R}_+)$ and $b \in L_1(\mathbb{R}_+, \mathbb{R}_+)$. Then*

$$\lim_{t\to+\infty} \int_0^t \exp(-\int_s^t a(s)ds)b(s)ds = 0.$$

Moreover, for every $\varepsilon > 0$ there exists μ_0 such that, for all $t \ge \mu \ge \mu_0$, we have

$$0 \le \int_\mu^t \exp(-\int_s^t a(s)ds)b(s)ds < \varepsilon.$$

Proof. Take $\mu \ge 0$. By the mean value therem, there exists a continuous increasing function $\sigma : [\mu, \infty) \to [\mu, \infty)$ such that $\int_{\sigma(t)}^t a(s)ds = \int_\mu^{\sigma(t)} a(s)ds$ and $\lim\limits_{t\to+\infty} \sigma(t) = +\infty$. Hence, since $a \notin L_1(\mathbb{R}_+)$, we conclude that $\int_{\sigma(t)}^t a(s)ds \to +\infty$ as $t \to +\infty$. Now, Lemma 3.2 follows from the following simple estimations:

$$\int_\mu^t \exp(-\int_s^t a(s)ds)b(s)ds = \int_\mu^{\sigma(t)} \exp(-\int_s^t a(s)ds)b(s)ds$$

$$+ \int_{\sigma(t)}^t \exp(-\int_s^t a(s)ds)b(s)ds \le \exp(-\int_{\sigma(t)}^t a(s)ds) \int_\mu^\infty b(s)ds + \int_{\sigma(t)}^\infty b(s)ds.$$

□

Corollary 3.3. *Let the conditions of Lemma 3.2 be satisfied and $a' \in L_1(\mathbb{R}_+)$. Set $I(t, \mu) = \int_\mu^t \exp(-\int_s^t a(s)ds)a^2(s)ds$. Then $\lim_{t\to+\infty} I(t, \mu) = 0$ for every $\mu \ge 0$. Moreover, $\lim_{\mu\to+\infty} I(t, \mu) = 0$ uniformly for $t \ge \mu$.*

Proof. Integrating $I(t,\mu)$ by parts, we obtain that

$$I(t,\mu) = a(t) - a(\mu)\exp(-\int_\mu^t a(s)ds) - \int_\mu^t a'(s)\exp(-\int_s^t a(s)ds)ds.$$

Now, Corollary 3.3 follows from Lemma 3.2. $\square$

Proof of Theorem 3.1. The change of variables $x = \rho\cos\phi,\ y = -\rho\sin\phi$, where $\rho > 0,\ \phi \in S^1$, transforms the initial system $x' = y,\ y' = -x + h(t)(1-x^2)y$ into

$$\begin{aligned} \rho' &= h(t)\rho\sin^2\phi(1-\rho^2\cos^2\phi), \\ \phi' &= 1 + h(t)\sin\phi\cos\phi(1-\rho^2\cos^2\phi). \end{aligned}$$

After elementary trigonometric transformations, this system takes the form

$$\begin{aligned} \rho' &= 0.5h(t)\rho\{1-\rho^2/4-\cos 2\phi+(\rho^2\cos 4\phi)/4\}, \\ \phi' &= 1 + h(t)\{(\sin 2\phi)/2-(\rho^2\sin 2\phi)/4-(\rho^2\sin 4\phi)/8\}. \end{aligned}$$

Now, following the KB procedure [3], we realize the change of variables

$$\rho = r(1-0.25h(t)[\sin 2\phi-(r^2\sin 4\phi)/8])$$

to obtain

$$\begin{aligned} r' &= 0.5h(t)r(1-r^2/4) + h^2(t)R_1(r,\phi,h) + h'(t)R_2(r,\phi,h), \\ \phi' &= 1 + h(t)R_3(r,\phi,h), \end{aligned}$$

where all R_i are analytical, 2π-periodic in ϕ, and bounded in some domain $\phi \in S^1,\ |r| < d_1,\ |h| < d_2$ where, given $d_1 > 2$, we choose $d_2 = d_2(d_1)$ small enough. Finally, introducing $z = r - 2$ (where $|z| < d_3$ for some small $d_3 < d_2 - 2$), we transform Eq. (3.1) into the system

$$\begin{aligned} z' &= -zh + h^2R_1(z+2,\phi,h) + h'R_2(z+2,\phi,h) + z^2hR_6(z), \\ \phi' &= 1 + hR_3(z+2,\phi,h), \end{aligned}$$

where, for sufficiently small $d_3 > 0$ and sufficiently large $T > 0$, all $|R_i|$ are uniformly bounded (say, by some $\Delta > 0$) in the domain $\{|z| \le d_3\}\times S^1\times\{h : |h| \le \sup_{t\ge T}|h(t)|\}$.

In the next stage of the proof, we show that for every $\varepsilon \in (0,(4\Delta)^{-1})$, there exists $\delta \in (0,\varepsilon)$ and $T > 0$ such that $|z(T)| < \delta$ implies $|z(t)| < \varepsilon$ for all $t > T$. Indeed, using the integral representation

$$z(t) = z(T)\exp(-\int_T^t hds) + \int_T^t \exp(-\int_s^t hdu)[h^2R_1 + h'R_2 + hz^2R_6]ds, \tag{3.3}$$

we find out that, while $|z(t)| < \varepsilon$ (which is true at least for small values of $t-T > 0$),

$$|z(t)| \le \delta\exp(-\int_T^t h(s)ds) + \Delta\int_T^t \exp(-\int_s^t h(u)du)[h^2(s) + |h'(s)| + \varepsilon^2h(s)]ds.$$

Now, by Lemma 3.2 and Corollary 3.3, we can choose $\mu = T > 0$ sufficiently large and $\delta > 0$ small enough to have $|z(t)| < \varepsilon$ for all $t > T$. In particular, all solutions such that $|z(T)| < \delta$, are defined over all $\mathbb{R}_+$. Finally, considering again (3.3), we find easily that $z(t) = o(1)$ at infinity. $\square$

4. Appendix: Nonresonant periodic case

The similarity existing between asymptotic integration of the linear systems $x' = (P + \varepsilon Q(t))x$ and $x' = (P + h(t)Q(t))x$ becomes more evident if we restrict our considerations to the case when $Q(t)$ is periodic: $Q(t + 2\pi) = Q(t)$. Indeed, in this case, the small denominators problem [13] does not arise; moreover, it is possible to make use of the Floquet theory (the analogue of which is not valid for systems with quasi-periodic $Q(t)$). For the first time, this idea was developed in [2, 9]. Here, we show how some results of "nonresonant" theory (e.g., see [6]) can be obtained as a consequence of the following result from [2, Theorem 1].

Proposition 4.1. *Assume that $h(t)$ is absolutely continuous, and of bounded variation with $h(+\infty) = 0$, and suppose that all Floquet multipliers $\mu_j(\varepsilon), j = 1, ..., n$, of the linear periodic system $x' = (P + \varepsilon Q(\lambda t))x,\ \lambda \neq 0$ are simple and belong to the unit circle for small ε. Then all solutions of the system $x' = (P + h(t)Q(\lambda t))x$, $Q(t + 2\pi) = Q(t)$, are bounded.*

As an immediate consequence of Proposition 4.1 and the standard facts of the theory of parametric resonance (see [1] for definitions), we obtain

Corollary 4.2. *Assume that all solutions of the real linear Hamiltonian system $x' = JH_0x,\ x \in \mathbb{R}^{2n}, H_0^* = H_0$ are oscillatory (with frequencies $\omega_j \geq 0,\ j = 1, ..., n$) and that the symplectic monodromy matrix $M = X(2\pi\lambda^{-1})$ is strongly stable. Then all solutions of $x' = J(H_0 + h(t)B(\lambda t))x,\ B^*(t) = B(t) = B(t + 2\pi)$ are bounded.*

Here $X(t)$, with $X(0) = I$, denotes the fundamental matrix solution of $x' = JH_0x$ and H^* is the adjoint to H. It should be noted also that the following simple nonresonance conditions $\lambda k \neq \omega_j \pm \omega_i,\ k \in \mathbb{Z}$ are sufficient (however, not necessary) for the strong stability of M. In particular, the adiabatic oscillator

$$x'' + (1 + h(t)p(\lambda t))x = 0,\ \ p(t + 2\pi) = p(t),\ t > t_0, x \in \mathbb{R}, \tag{4.1}$$

has all solutions bounded when $\lambda \neq 2/k,\ k \in \mathbb{Z} \setminus \{0\}$. The characteristic property of the linear real Hamiltonian system which allows us to prove Corollary 4.2 is that $X(t)JX^*(t) = J$. We can generalize it by requiring the relation

$$X^*(t)DX(t) = D \text{ or } X(t)DX^*(t) = D \tag{4.2}$$

with some invertible matrix D.

Lemma 4.3. *Assume that all multipliers $\mu_i(0)$ of the T-periodic system $x' = A(t, 0)x$ are distinct and belong to the unit circle S^1. In addition, if the fundamental matrix solution $X(t, \varepsilon)$ of $x' = A(t, \varepsilon)x$ satisfies* (4.2)*, then $\mu_i(\varepsilon) \in S^1$ for small ε.*

Proof. (4.2) implies that the spectrum $\sigma(\varepsilon) \in \mathbb{C}$ of the monodromy matrix $M(\varepsilon)$ satisfies $\bar{\sigma} = \sigma^{-1}$ and, in view of the distinctness of the eigenvalues from $\sigma(0)$, implies immediately $\sigma(\varepsilon) \in S^1$. □

Lemma 4.4. (4.2) *holds for the following systems (complex or real):*

$$x' = U(t)Dx \text{ or } x' = DU(t)x,\ \text{when } D^* = D, U^*(t) = -U(t);$$
$$x' = DS(t)x \text{ or } x' = S(t)Dx,\ \text{when } S^*(t) = S(t), D^* = -D.$$

Proof. Since $Z(t) = X(t)D - D(X^*(t))^{-1}, Z(0) = 0$ satisfies $X' = DU(t)X$ and $X' = DS(t)X$, we have $Z(t) \equiv 0$. If $\det D \neq 0$, then the same idea works for $x' = U(t)Dx$ and $X' = S(t)DX$ with $Z_1(t) = Y(t) - D^{-1}(Y^*(t))^{-1}D$. If D is noninvertible, we take a sequence of invertible $D_j \to D$ to obtain (4.2) with D_j instead of D. Finally, a continuity argument completes the proof of the lemma in this case. □

Corollary 4.5. *If $B(t) = B(t + 2\pi)$ and all eigenvalues of $\exp(2\pi\lambda^{-1}AC)$ are distinct and belong to S^1, then all solutions of the systems*

$$x' = (A + h(t)B(\lambda t))Cx \text{ and } x' = C(A + h(t)B(\lambda t))x$$

are bounded if either one of the following two conditions holds: either (i) $A^* = -A,\ B^*(t) = -B(t),\ C^* = C$ or (ii) $A^* = A,\ B^*(t) = B(t),\ C^* = -C$.

Example 4.6 (by Eastham and McLeod [6])**.** Take the real constant diagonal matrices Q and $\Lambda_0 = \mathrm{diag}(a_1, ..., a_n)$, and consider the system

$$x' = (i\Lambda_0 + \varepsilon QP(\lambda t))x,\ P(t + 2\pi) = P(t),\ P^*(t) = -P(t).$$

If Q is invertible, then this system has one of the forms given in Corollary 4.5 with $\exp(2\pi\lambda^{-1}AC) = \exp(i2\pi\lambda^{-1}\Lambda_0) = \mathrm{diag}(\exp(2\pi i\lambda^{-1}a_1), ..., \exp(2\pi i\lambda^{-1}a_n)) = \mathrm{diag}(\mu_1, ..., \mu_n)$. Therefore, if $\lambda k \neq (a_j - a_i)$ for $k \in \mathbb{Z},\ i, j = 1, n$, then all multipliers $\mu_j(\varepsilon) \in S^1$ are different. To get the same result for the degenerate Q, it suffices to see that there exists a sequence of the invertible diagonal matrices Q_n converging to Q and then to use the theorem about continuous dependence on parameters. Thus all solutions of $x' = (i\Lambda_0 + h(t)QP(\lambda t))x$ are bounded for every $h(t)$ that is absolutely continuous, and of bounded variation with $\lim_{t\to+\infty} h(t) = 0$. In this way, we get a somewhat improved version of Theorem 4.6.2 from [6].

Acknowledgment

The authors are indebted to the referee for pointing out the references [4, 14].

References

[1] V.I. Arnold, *Mathematical methods of classical mechanics*, Springer, 1978. MR57:14033b

[2] F.V. Atkinson, 'On asymptotically periodic linear systems', *J. Math. Anal. Appl.* 24 (1968), 646-653. MR39:3095

[3] N.N. Bogolyubov, Yu.A. Mitropolskii, *Asymptotic methods in the theory of nonlinear oscillations, (in Russian)*, Fourth edition, Nauka, Moscow, 1974. MR51:10750

[4] V. Sh. Burd and V.A. Karakulin, 'Asymptotic integration of systems of linear differential equations with oscillatorily decreasing coefficients', Math. Notes 64 (1999), 571-578. MR2000a:34097

[5] J.S. Cassell, 'The asymptotic integration of some oscillatory differential equations', *Quart. J. Math. Oxford Ser. (2)* 33 (1982), 281-296. MR84c:34079

[6] M.S.P. Eastham, *The asymptotic solution of linear differential systems*, London Mathematical Society Monographs, Clarendon Press, Oxford, 1989. MR91d:34001

[7] M.S.P. Eastham, 'The number of resonant states in perturbed harmonic oscillation', *Quart. J. Math. Oxford(2)* 42 (1991), 49-55. MR92d:34075

[8] W.A. Harris and D.A. Lutz, 'A unified theory of asymptotic integration,' *J. Math. Anal. Appl.* 57 (1977), no. 3, 571-586. MR55:3441

[9] W.A. Harris and Y. Sibuya, 'Asymptotic behaviors of solutions of a system of linear ordinary differential equations as $t \to \infty$', *Lecture Notes in Math.*, 1475, Springer, 1991, pp. 210-217. MR93k:34119

[10] N. Levinson, 'The asymptotic nature of solutions of linear differential equations', Duke Math. J., 15 (1958), 111-126. MR9:509h

[11] K. Mischaikow, H. Smith and H. Thieme, 'Asymptotically autonomous semiflows: chain recurrence and Lyapunov functions', Trans. Amer. Math. Soc., 347 (1995), 1669-1685. MR95h:34063

[12] S. Trofimchuk and M. Pinto, 'L_p-perturbations of invariant subbundles for linear systems', Journal of Dynamics and Differential Equations, 14 (2002), 743-761. MR2003m:34116

[13] A.M. Samoilenko, *Elements of the mathematical theory of multi-frequency oscillations*, Kluwer, Dordrecht, 1991, 313 pp. MR93d:34050

[14] Yu. A. Samokhin and V.N. Fomin, 'A method for studying the stability of the oscillations of linear systems that are subject to the action of parametric loads with continuous spectrum', Siberian Math. J. 17 (1976), 926-931. MR54:10767

[15] M.M. Skriganov, 'The eigenvalues of the Schrödinger operator situated on the continuous spectrum', J. Soviet Math. 8 (1977), 464-467. MR49:887

[16] I.Z. Shtokalo, *Linear differential equations with variable coefficients,* Gordon and Breach Science Publishers, Inc., New York, 1961, 100 pp. MR27:5979

Institute of Mathematics, National Academy of Sciences, Tereshchenkyvs'ka str., 3, Kiev, 252601, Ukraine

E-mail address: sam@imath.kiev.ua

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile

E-mail address: pintoj@uchile.cl

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile

E-mail address: trofimch@uchile.cl

Current address: Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile