NORMS ON EARTHQUAKE MEASURES AND ZYGMUND FUNCTIONS

JUN HU

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Abstract. The infinitesimal earthquake theorem gives a one-to-one correspondence between Thurston bounded earthquake measures and normalized Zygmund bounded functions. In this paper, we provide an intrinsic proof of a theorem given in an earlier paper by the author; that is, we show that the cross-ratio norm of a Zygmund bounded function is equivalent to the Thurston norm of the earthquake measure in the correspondence.

1. Introduction

Consider the open unit disk $\mathbb{D}$ centered at the origin of the complex plane $\mathbb{C}$ as the hyperbolic plane. A geodesic lamination $\mathcal{L}$ in $\mathbb{D}$ is a collection of geodesics that foliate a closed subset $L$ of $\mathbb{D}$. Here $L$ is called the locus of $\mathcal{L}$, the geodesics are called the leaves of $\mathcal{L}$, the connected components of $\mathbb{D} \setminus L$ are called the gaps, and the gaps and the leaves of $\mathcal{L}$ are called the strata of $\mathcal{L}$. Let $\mathbb{S}^1$ denote the boundary circle of $\mathbb{D}$, and let $X$ be the space $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{\text{the diagonal}\}$ factorized by the equivalence relation $(a,b) \sim (b,a)$. A Borel measure $\sigma$ defined on $X$ is called an earthquake measure if there is a lamination $\mathcal{L}$ such that $\sigma$ is supported on the pairs of the endpoints of the leaves in $\mathcal{L}$.

Let $\mathcal{L}$ be a lamination in $\mathbb{D}$ and $\beta$ a closed geodesic segment of hyperbolic length $\leq 1$. If $\beta$ is transversal to a leaf in $\mathcal{L}$, then $\mathcal{L}$ intersects $\beta$ in a parallel fashion in the sense that there are two geodesic lines $l_1$ and $l_2$ among the lines of $\mathcal{L}$ intersecting $\beta$ such that any other line in $\mathcal{L}$ intersecting $\beta$ separates $l_1$ from $l_2$. Suppose the strip $S$ bounded by $l_1$ and $l_2$ in the unit disk is of the form $[a,b] \times [c,d]$, where $a,d$ and $b,c$ are the endpoints of $l_1$ and $l_2$, respectively, and $a,b,c,d$ are arranged on $\mathbb{S}^1$ in counter-clockwise order. We denote by

$$\sigma(\beta) = \sigma([a,b] \times [c,d]).$$

In [5], Thurston defines the norm of $\sigma$ to be

$$||\sigma||_{Th} = \sup_{t(\beta) \leq 1} \sigma(\beta) = \sup_{t(\beta) = 1} \sigma(\beta),$$

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where $\beta$ is a closed geodesic segment transversal to $\mathcal{L}$ and $l(\beta)$ denotes the hyperbolic length of $\beta$. An earthquake measure is called \textit{Thurston bounded} if it has finite Thurston norm. Let $\mathcal{M}$ be the collection of all Thurston bounded earthquake measures defined on $X$. \forall \sigma \in \mathcal{M}$, define

$$V_\sigma(x) = E(\sigma)(x) = \int \int E_{ab}(x)d\sigma(a,b),$$

where $E_{ab}(x) = \frac{(x-a)(x-b)}{a-b}$ if $x \in [a,b]$ and $E_{ab}(x) = 0$ otherwise. Here we also have the agreement that for each geodesic line $ab$ in $\mathcal{L}$, $[a,b]$ denotes the short arc on $S^1$ in counter-clockwise order. $V_\sigma$ maps each point $x$ on $S^1$ to a vector tangent to $S^1$ at $x$, and then it defines a tangent vector field on $S^1$.

Simply consider the tangent vectors of $S^1$ as complex numbers. A continuous tangent vector field $V$ on $S^1$ is said to be \textit{Zygmund bounded} if

$$|V(e^{2\pi i(\theta + t)}) + V(e^{2\pi i(\theta - t)}) - 2V(e^{2\pi i\theta})| \leq M|t|$$

for a constant $M > 0$ and for all $0 \leq \theta < 1$ and $0 < t < \frac{1}{2}$. It is proved in [1] that for any $\sigma \in \mathcal{M}$, $V_\sigma$ is Zygmund bounded; conversely, for any Zygmund bounded tangent vector field $V$ on $S^1$, there exists a Thurston bounded earthquake measure $\sigma$ such that

$$V(x) = \pi \int \int E_{ab}(x)d\sigma(a,b) \text{ modulo a quadratic polynomial;}$$

furthermore, if two $V$’s differ by a quadratic polynomial, then the corresponding $\sigma$’s are the same. This is the so-called \textit{infinitesimal earthquake theorem} (see Theorem 5.1 in [1]), which gives a one-to-one correspondence between $\mathcal{M}$ and the space of Zygmund bounded tangent vector fields on $S^1$.

Motivated by the work of [3], we introduce a norm to measure the Zygmund boundedness of $V$ and show it is equivalent to the Thurston norm of $G$. Given a quadruple $Q = \{a, b, c, d\}$ consisting of four points $a, b, c, d$ on the unit circle $S^1$ arranged in counter-clockwise order, we define

$$\text{cr}(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}$$

and

$$V(Q) = \frac{V(b) - V(a)}{b-a} - \frac{V(c) - V(b)}{c-b} + \frac{V(d) - V(c)}{d-c} - \frac{V(a) - V(d)}{a-d}.$$

Then the \textit{cross-ratio norm} $||V||_{cr}$ of $V$ is defined to be

$$||V||_{cr} = \sup_{cr(Q)=1} |V(Q)|.$$

In fact, $||V||_{cr}$ is finite if and only if $V$ is Zygmund bounded.

It is proved in [2] that $V_\sigma$ is the initial derivative to the parameter $t$ of the earthquake curve determined by $t\sigma$, $t \geq 0$. By means of an implicit method, it is deduced in [4] that the cross-ratio norm of $V_\sigma$ is equivalent to the Thurston norm of $\sigma$. On the other hand, it is a general principle that strategies in the study of earthquake theory (see [3] and [2]) transfer to parallel strategies in the infinitesimal theory. In this paper, we construct an intrinsic and direct proof for the following theorem.
Main Theorem. There exists a universal constant $C > 0$ such that $\forall \sigma \in \mathcal{M}$,
\[
\frac{1}{C} ||\sigma||_{Th} \leq ||V_\sigma||_{cr} \leq C ||\sigma||_{Th}.
\]

2. Proof

We divide the proof into two parts. We first prove that there exists $C > 0$ such that $||\sigma||_{Th} \leq C ||V_\sigma||_{cr}$; we then show that there exists $C > 0$ such that $||V_\sigma||_{cr} \leq C ||\sigma||_{Th}$. Before we start the proofs, let us summarize some techniques into lemmas.

Let $\sigma$ denote a Thurston bounded earthquake measure, and let $V = V_\sigma$. Let $B$ denote an orientation-preserving M"obius transformation from the upper half-plane $\mathbb{H}$ or the unit open disk $\mathbb{D}$ onto $\mathbb{D}$, and let $\tilde{\sigma} = B^{-1}(\sigma)$ be the pullback of $\sigma$ under $B$ (i.e., the pushforward of $\sigma$ under $B^{-1}$). Also, define
\[
\tilde{V}(x) = V_\tilde{\sigma}(x) = E(\tilde{\sigma})(x) = \int \int E_{ab}(x)d\tilde{\sigma}(a,b).
\]

**Lemma 1.**
\[
\tilde{V}(x) = \frac{V(B(x))}{B'(x)}.
\]

**Proof.** Define $x' = B(x)$, $a' = B(a)$ and $b' = B(b)$. By using the identity
\[
[B(a) - B(b)]^2 = (a - b)^2B'(a)B'(b)
\]
and the assumption that $B$ is orientation-preserving, we have
\[
\frac{[B(x) - B(a)][B(x) - B(b)]}{[B(a) - B(b)]} = \frac{B'(x)(x-a)(x-b)}{(a-b)}.
\]

Then
\[
\begin{align*}
V(B(x)) &= V(x') = \int \int E_{a'b'}(x')d\sigma(a',b') = \int \int \frac{(x' - a')(x' - b')}{a' - b'} d\sigma(a',b') \\
&= \int \int \frac{[B(x) - B(a)][B(x) - B(b)]}{B(a) - B(b)} d\sigma(B(a),B(b)) \\
&= \int \int \frac{B'(x)(x-a)(x-b)}{(a-b)} d\tilde{\sigma}(a,b) = B'(x)\tilde{V}(x).
\end{align*}
\]

Therefore,
\[
\tilde{V}(x) = \frac{V(B(x))}{B'(x)}.
\]

□

**Lemma 2.** For any quadruple $Q$ of four points $a, b, c, d$ on the real line or the unit circle in counter-clockwise order,
\[
\tilde{V}[Q] = V[B(Q)] \quad \text{or} \quad V[Q] = \tilde{V}[B^{-1}(Q)].
\]

**Proof.** Let $h_t = Id + t\tilde{V}$, where $t$ is a real parameter near zero. Clearly, for each $x$, $h_t(x)$ is differentiable on $t$ with $V = \frac{d}{dt}h_t|_{t=0}$. Define
\[
\text{cr}(h_t(Q)) = \frac{[h_t(b) - h_t(a)][h_t(d) - h_t(c)]}{[h_t(c) - h_t(b)][h_t(d) - h_t(a)]}.
\]

Observe first that
\[
V[Q] = \frac{d}{dt} \ln \text{cr}(h_t(Q))|_{t=0}.
\]
Corollary 1. $||\tilde{V}||_{cr} = ||V||_{cr}$.

Lemma 3. Assume $\lambda > 0$, $-\infty < a < b < c < d$, and $c \leq s \leq d \leq t$. Let $V(x) = \lambda E_{s,t}(x)$ and $Q = \{a,b,c,d\}$. Consider $V[Q]$ as a function of $s$ and $t$. Then $V[Q] \geq 0$, and $V[Q]$ is an increasing function on $t$ for each fixed $s$ and a decreasing function on $s$ for each fixed $t$.

Proof. It is easy to check that $\frac{\partial}{\partial s}E_{s,t}(x) > 0$ and $\frac{\partial}{\partial s}E_{s,t}(x) < 0$ for each $s < x < t$. Clearly,

$$V[Q] = \frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} + \frac{V(d) - V(c)}{d - c} - \frac{V(a) - V(d)}{a - d}$$

$$= \frac{V(d)}{d - c} - \frac{V(d)}{d - a} = \frac{1}{d - c} - \frac{1}{d - a} \lambda E_{s,t}(d).$$

Since $\frac{1}{d - c} - \frac{1}{d - a} > 0$,

$$\frac{\partial}{\partial t}V[Q] > 0 \text{ and } \frac{\partial}{\partial s}V[Q] < 0.$$

Therefore $V[Q]$ is an increasing function on $t$ for each fixed $s$ and is a decreasing function on $s$ for each fixed $t$. 

Lemma 4. Assume $\lambda > 0$, $-\infty < a < b < c < d \leq \infty$, and $b \leq s \leq c$ and $t \geq d$. Let $V(x) = \lambda E_{s,t}(x)$ and $Q = \{a,b,c,d\}$. Consider $V[Q]$ as a function of $s$ and $t$. Then $V[Q] \geq 0$, and $V[Q]$ is increasing on $s$ for each fixed $t$ and also increasing on $t$ for each fixed $s$.

Proof. It is straightforward to check

$$V[Q] = \lambda \left[ -\frac{d - b}{(c - b)(d - c)} E_{s,t}(c) + \frac{c - a}{(d - c)(d - a)} E_{s,t}(d) \right]$$

$$= \lambda \left[ -\frac{d - b}{(c - b)(d - c)} \frac{(c - s)(c - t)}{s - t} + \frac{c - a}{(d - c)(d - a)} E_{s,t}(d) \frac{(d - s)(d - t)}{s - t} \right].$$
Lemma 2 implies the lines in the lamination $L$ denote the geodesic perpendicular to both $a < b$ and also an increasing function on $t$ for each fixed $t$.

Then,

$$\frac{\partial}{\partial s} V[Q] = \lambda \left[ \frac{(d - b)(c - s)^2}{(c - b)(d - c)(s - t)^2} + \frac{(c - a)(d - s)^2}{(d - a)(s - t)^2} \right]$$

$$= \frac{\lambda(t - d)^2 (d - b)(t - c) (s - t)^2}{(s - t)(d - c)} - \frac{c}{(c - d)(s - t)^2} - \frac{c - a}{d - a}.$$

Clearly, $\frac{d - b}{c - a} > 1$ and $\frac{c - a}{d - a} < 1$. Hence $\frac{\partial}{\partial s} V[Q] > 0$ when $t > d$ and $\frac{\partial}{\partial s} V[Q] \geq 0$ when $t = d$. Therefore $V[Q]$ is an increasing function on $s$ for each fixed $t$.

Similarly,

$$\frac{\partial}{\partial t} V[Q] = \lambda \left[ \frac{(s - t)(d - c)(c - s)}{(s - t)(d - c)(s - t)} \right]$$

$$= \frac{\lambda(c - s)^2 (d - b)(t - c)(s - t)^2}{(s - t)(d - c)} - \frac{c}{(c - d)(s - t)^2} - \frac{c - a}{d - a}.$$

Since $a < b \leq s \leq c < d$,

$$\frac{d - s}{c - s} = \frac{d - b}{c - b} > 1 \quad \text{and} \quad \frac{d - s}{c - s} = \frac{d - a}{c - a} > 1.$$

Then

$$-1 + \left( \frac{d - s}{c - s} \frac{d - b}{c - b} \frac{d - s}{c - s} \frac{d - a}{c - a} \right) > 0.$$ 

Hence $\frac{\partial}{\partial t} V[Q] > 0$ when $b \leq s < c$ and $\frac{\partial}{\partial t} V[Q] = 0$ when $s = c$. Therefore $V[Q]$ is also an increasing function on $t$ for each fixed $s$.

**Theorem 1.** There exists a universal constant $C > 0$ such that

$$\|\sigma\|_{T^h} \leq C\|\sigma\|_{cr}.$$ 

In fact, we can take $C = \frac{(1+e)^2}{3e+6e+1}$. 

**Proof.** Let $D$ denote a closed disk in $\mathbb{D}$ of hyperbolic diameter 1, $l_1$ and $l_2$ denote the lines in the lamination $L$ of $\sigma$ that bound all the lines in $L$ intersecting $D$. Let $\beta$ denote the geodesic perpendicular to both $l_1$ and $l_2$ (in the case that $l_1$ and $l_2$ share at least one endpoint, we let $\beta$ be a geodesic perpendicular to $l_1$ such that the hyperbolic length of the segment on $\beta$ between $l_1$ and $l_2$ is less than or equal to $\frac{1}{2}$). Label the endpoints of $\beta$ by $x$ and $y$ so that the arc $[x, y]$ from $x$ to $y$ going in the counter-clockwise direction is not longer than the arc $[y, x]$ from $y$ to $x$. Let $B : \mathbb{D} \to \mathbb{H}$ be the Möbius transformation that maps $x$ to 0, $y$ to $\infty$, the arc $[x, y]$ to the positive half real line, and the geodesics $l_1$ and $l_2$ to the geodesics connecting $-1$ to 1 and $s$ to $\infty$ with $s > 1$ (in the case that $l_1$ and $l_2$ share at least one endpoint, then $B(l_1)$ or $B(l_2)$ connects $s$ to 1 or $-1$ to $s$ with $s \geq 1$). Since the hyperbolic distance between $l_1$ and $l_2$ is less than or equal to 1, $s \leq e$ (in the case that $l_1$ and $l_2$ share one endpoint, the requirement on $\beta$ also implies that $s \leq e$). Let $\tilde{\sigma}$ denote the pushforward of $\sigma$ under $B$ and $\tilde{V} = E(\tilde{\sigma})$. It is clear that $\|\tilde{\sigma}\|_{T^h} = \|\sigma\|_{T^h}$, and Lemma 2 implies $\|\tilde{V}\|_{cr} = \|\tilde{V}\|_{T^h}$. 

Assume $a = 1$, $b = \infty$, $c = -s$ and $d = \frac{1}{s}$. Denote by $Q = \{a, b, c, d\}$ and $Q' = B^{-1}(Q) = \{a', b', c', d'\}$. Clearly, $cr(Q) = cr(Q') = 1$, and Lemma 2 implies $\tilde{V}[Q] = V[Q']$. Denote by $u'$ and $v'$ the other endpoints of $l_1$ and $l_2$ such that $a', u', b', c', v', d'$ are arranged on $S^1$ in counter-clockwise order. We divide the lines
in the lamination $\mathcal{L}$ that affect the value of $V[Q']$ into three groups. Let $\mathcal{L}_m$ denote the collection of the lines in $\mathcal{L}$ intersecting $D$, $\mathcal{L}_b$ denote the collection of the lines in $\mathcal{L} \setminus \mathcal{L}_m$ connecting points in $(u', b')$ to points in $(b', c']$, and $\mathcal{L}_d$ the collection of the lines in $\mathcal{L} \setminus \mathcal{L}_m$ connecting points in $(v', d')$ to points in $(d', a')$. Denote by $\sigma_i = \sigma|_{\mathcal{L}_i}$ and $V_i = E(\sigma_i)$ for $i = m, b, d$. By the linearity of the operator $E$, we have

$$V[Q'] = V_m[Q'] + V_b[Q'] + V_d[Q'].$$  

By Lemma 3

$$V_d[Q'] = V_d[\{a', b', c', d'\}] \geq 0 \quad \text{and} \quad V_b[\{c', d', a', b'\}] \geq 0.$$  

Then

$$V_b[Q'] = V_b[\{a', b', c', d'\}] = -V_b[\{b', c', d', a'\}] = V_b[\{c', d', a', b'\}] \geq 0.$$  

Therefore,

$$V[Q'] \geq V_m[Q'].$$  

To complete the proof, we need to work out an explicit lower bound for $V_m[Q']$. Denote by $\bar{\sigma}_m$ the pushforward of $\sigma_m$ under $B$ and $\bar{V}_m = E(\bar{\sigma}_m)$. By Lemma 2

$$V_m[Q'] = \bar{V}_m[B(Q')] = \bar{V}_m[Q].$$  

By Lemma 4, if we move the weights of the geodesic lines in the lamination $\tilde{\mathcal{L}}_m$ of $\bar{\sigma}_m$ to the geodesic line connecting $-1$ to $s$, then the value of $\bar{V}_m[\{b, c, d, a\}]$ is possibly increased, and hence the value of $\bar{V}_m[Q] = -\bar{V}_m[\{b, c, d, a\}]$ is possibly decreased. Therefore

$$\tilde{V}_m[Q] \geq (\lambda E_{-1,s})[Q],$$  

where $\lambda = \sigma(\mathcal{L}_m)$. It is easy to check that

$$(\lambda E_{-1,s})[Q] = \lambda \frac{2E_{-1,s}(d) - E_{-1,s}(a)}{a - d}$$

and

$$\frac{2E_{-1,s}(d) - E_{-1,s}(a)}{a - d} = \frac{-3s^2 + 6s + 1}{(1 + s)^2} \geq \frac{-3\epsilon^2 + 6\epsilon + 1}{(1 + \epsilon)^2} > 0$$

for $1 \leq s \leq \epsilon$. Letting $C = \frac{(1 + \epsilon)^2}{-3\epsilon^2 + 6\epsilon + 1}$, we have

$$||V||_{cr} \geq V[Q'] \geq V_m[Q'] = \tilde{V}_m[Q] \geq (\lambda E_{-1,s})[Q] \geq \frac{\lambda}{C}.$$  

Hence

$$\lambda \leq C||V||_{cr},$$  

which implies

$$||\sigma||_{\mathcal{T}_h} \leq C||V||_{cr}.$$  

$\square$

**Theorem 2.** There exists a universal constant $C > 0$, independent of $\sigma$, such that

$$||V|_{\cr} \leq C||\sigma||_{\mathcal{T}_h}. \quad (10)$$  

In fact, we can take $C = C_0 + 2C_1C_2$, where $C_0$ is the smallest positive integer greater than or equal to $\ln(3 + 2\sqrt{2})$, $C_1 = \frac{\epsilon}{\epsilon - 1}$, and $C_2$ is the smallest positive integer greater than or equal to $\ln(e + \sqrt{\epsilon^2 - 1})$.  

Remark. The inequality (10) was obtained in [1] for a different cross-ratio norm on $V$ through a complex method which considered the holomorphic differential arising from the third derivative of $V + iH(V)$, where $H(V)$ denotes the Hilbert transformation of $V$. The proof of the inequality (10) in this paper is purely real. In addition, our Theorem 1 and that inequality in [1] imply that the two cross-ratio norms on $V$ are actually equivalent (see [2]).

Lemma 5. Let $Q$ be a quadruple consisting of four points $a, b, c, d$ on $\mathbb{R}^1$ or $S^1$ arranged in counter-clockwise order. Take $\frac{|dz|}{y}$ as the hyperbolic metric on the upper half-plane $\mathbb{H}$. Then $cr(Q) = 1$ if and only if the geodesic $\overline{ac}$ from $a$ to $c$ is perpendicular to the geodesic $\overline{bd}$ from $b$ to $d$, and if and only if the hyperbolic distance from $ab$ to $cd$ (or $bc$ to $da$) is equal to $\ln(3 + 2\sqrt{2})$.

Proof. It is straightforward (see [3] for details).

Lemma 6. Consider the upper half-plane $\mathbb{H}$. Let $l_n$ denote the geodesic connecting $-e^{-n}$ to $e^{-n}$ for each $n \in \{0\} \cup \mathbb{N}$ and $\mathcal{L}$ the lamination consisting of $l_n$’s. Suppose that $\sigma$ is an earthquake measure supported on $\mathcal{L}$, and let $\lambda_n = \sigma(l_n)$. Let $Q = \{1, \infty, -1, 0\}$, and

$$V(x) = \int \int E_{a,b}(x) d\sigma(a, b).$$

There exists a constant $C_1 > 0$ such that

$$0 \leq V[Q] \leq C_1 \max_{n \geq 0} \lambda_n.$$

Proof. Define $\lambda = \max_{n \geq 0} \lambda_n$, $a_n = -e^{-n}$ and $b_n = e^{-n}$. Clearly, $V(1) = V(\infty) = V(-1) = 0$, and then $V[Q] = 2V(0)$. We need to work out $V(0)$, that is,

$$0 \leq V(0) = \int \int E_{a_n, b_n}(0) d\sigma(a_n, b_n) = \sum_{n \geq 0} \lambda_n \frac{a_n b_n}{b_n - a_n} = \sum_{n \geq 0} \lambda_n \frac{e^{-n}}{2} \leq \frac{1}{2} \frac{e}{e-1} \lambda.$$

Let $C_1 = \frac{e}{e-1}$. Then $0 \leq V[Q] = 2V(0) \leq C_1 \lambda$.

We reduce the proof of Theorem 2 to Propositions 1 and 2. Let $\mathcal{L}$ be the lamination that supports $\sigma$. Given a quadruple $Q$ of four points $a, b, c, d$ on $S^1$ in counter-clockwise order with $cr(Q) = 1$, we first assume that three points $a, b$ and $c$ belong to the same stratum $A$ and estimate $V[Q]$ in this case. By a Möbius change of coordinates and Lemma 2 we may assume that $a = 1, b = \infty, c = -1$, and $d = 0$. We will show that there is a constant $C$ such that

$$0 \leq V[Q] \leq C||\sigma||_{\mathcal{L}h}.$$

Let $x_n$ denote the point $-e^{-n}$ and $y_n$ the point $e^{-n}$ on the real axis for each $n \in \{0\} \cup \mathbb{N}$. Let $\mathcal{L}'$ denote the collection of the lines in $\mathcal{L}$ that connect points of the interval $[-1, 0)$ to points of $(0, 1)$, $\mathcal{L}^-_0$ the collection of the lines in $\mathcal{L}'$ that connect points of $[x_0, x_1)$ to points of $(0, y_0)$, and $\mathcal{L}^+_0$ the collection of the lines in $\mathcal{L}'$ that connect points of $[x_0, 0)$ to points of $(y_1, y_0)$. Finally, let $\mathcal{L}_0 = \mathcal{L}^-_0 \cup \mathcal{L}^+_0$. Then any line in $\mathcal{L}' \setminus \mathcal{L}_0$ must connect a point in $[x_1, 0)$ to a point in $(0, y_1)$. Inductively, for each $n \in \mathbb{N}$, let $\mathcal{L}^-_n$ denote the collection of the lines in $\mathcal{L}' \setminus (\mathcal{L}_0 \cup \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_{n-1})$ that connect points of $[x_n, x_{n+1})$ to points of $(0, y_n)$, and $\mathcal{L}^+_n$ the collection of the lines in $\mathcal{L}' \setminus (\mathcal{L}_0 \cup \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_{n-1})$ that connect points of $[x_n, 0)$ to points of $(y_{n+1}, y_n)$, and $\mathcal{L}_n = \mathcal{L}^-_n \cup \mathcal{L}^+_n$. We have the following three lemmas.
Lemma 7. For each $n \in \{0\} \cup \mathbb{N}$, any line in $\mathcal{L}_n$ must connect a point in $[x_n, 0)$ to a point in $(0, y_n]$. 

Proof. It can be easily proved by an induction on $n$. 

Lemma 8. There exists a constant $C_2 > 0$, independent of $h$ and $\sigma$, such that $\sigma(\mathcal{L}_n) \leq C_2\|\sigma\|_{T_h}$ for any $n \in \{0\} \cup \mathbb{N}$.

Proof. For each $n \in \{0\} \cup \mathbb{N}$, let $l^-_n$ denote the geodesic line connecting the point $x_n$ to the point $y_n$. Also, for any $n \in \mathbb{N}$, let $l^+_n$ denote the geodesic connecting the point $x_n$ to 0, and $l^+_n$ the geodesic connecting 0 to the point $y_n$. The hyperbolic distance from $l^-_n$ to $l^-_{n+1}$ (or $l_+^{n+1}$), $n \in \{0\} \cup \mathbb{N}$, is equal to a constant, that is equal to $\ln(e + \sqrt{e^2 - 1})$. Let $C_2$ denote the smallest positive integer that is greater than or equal to $\ln(e + \sqrt{e^2 - 1})$. Then 

$$\sigma(\mathcal{L}_n) \leq C_2\|\sigma\|_{T_h}$$

for each $n \in \{0\} \cup \mathbb{N}$. 

Let $\tilde{V}$ be the same map as defined in Lemma 9 with $\lambda_n = \sigma(\mathcal{L}_n)$.

Lemma 9. We have the following inequality:

$$V[Q] \leq \tilde{V}[Q].$$

Proof. Define $\sigma_n = \sigma|_{\mathcal{L}_n}$ and $V_n = E(\sigma_n)$. Let $\tilde{\sigma}_n$ denote the atomic earthquake measure with weight $\lambda_n$ supported on the geodesic $l^-_n$ and let $\tilde{V}_n = E(\tilde{\sigma}_n)$. By the linearity of the operator $E$,

$$V[Q] = \sum_{n=0}^{\infty} V_n[Q] \quad \text{and} \quad \tilde{V}[Q] = \sum_{n=0}^{\infty} \tilde{V}_n[Q].$$

By Lemma 3 and Lemma 7, if we move the weights of the geodesic lines in $\mathcal{L}_n$ to the geodesic line $l^-_n$, we only increase $V_n[Q]$, that is, $V_n[Q] \leq \tilde{V}_n[Q]$. Therefore 

$$V[Q] \leq \tilde{V}[Q].$$

Proposition 1. If $cr(Q) = 1$ and $a, b, c$ belong to the same stratum of an earthquake measure $(E, \mathcal{L})$, then 

$$0 \leq V_\sigma[Q] \leq C_1 C_2\|\sigma\|_{T_h}.$$ 

Proposition 2. Suppose $cr(Q) = 1$, and assume that there exists at least one geodesic line in the lamination $\mathcal{L}$ of $\sigma$ that separates the vertices $a$ and $b$ from the vertices $c$ and $d$. Then

$$|V_\sigma[Q]| \leq (C_0 + 2C_1 C_2)\|\sigma\|_{T_h}.$$ 

Proof. Given two points $x$ and $y$ on the unit circle, we use $[x, y]$ to denote the arc on $S^1$ from $x$ to $y$ in the counter-clockwise direction. We divide the geodesic lines in $\mathcal{L}$ that affect $V[Q]$ into five groups. Let $\mathcal{L}_m$ denote the collection of the geodesic lines in $\mathcal{L}$ that connect points of the arc $[d, a]$ to points of the arc $[b, c]$. Let $\mathcal{L}_a$ denote the collection of the lines in $\mathcal{L}$ that connect points of the arc $(d, a)$ to points of the arc $(a, b)$, $\mathcal{L}_b$ the collection of the lines in $\mathcal{L}$ that connect points of the arc $(a, b)$ to points of the arc $(b, c)$, $\mathcal{L}_c$ the collection of the lines in $\mathcal{L}$ that connect
points of the arc \((b, c)\) to points of the arc \((c, d)\), and finally \(\mathcal{L}_d\) the collection of the lines in \(\mathcal{L}\) that connect points of the arc \((c, d)\) to points of the arc \((d, a)\) (see Figure 1). Let \(\sigma_i\) denote the restriction of \(\sigma\) on \(\mathcal{L}_i\) and \(V_i = E(\sigma_i)\), where \(i = m, a, b, c, d\). Clearly \(\|\sigma_i\|_{TH} \leq \|\sigma\|_{TH}\)

\[
\]

For the lamination \(\mathcal{L}_d\), three points \(a, b, c\) are contained in the same stratum. By Proposition 1, we have

\[
0 \leq V_d[Q] \leq C_1 C_2 \|\sigma_d\|_{TH} \leq C_1 C_2 \|\sigma\|_{TH}.
\]

For the lamination \(\mathcal{L}_h\), three points \(c, d, a\) are contained in the same stratum. Again by Proposition 1 we have

\[
0 \leq V_h[\{c, d, a, b\}] \leq C_1 C_2 \|\sigma_b\|_{TH} \leq C_1 C_2 \|\sigma\|_{TH}.
\]

Therefore,

\[
0 \leq V_h[Q] = V_h[\{c, d, a, b\}] \leq C_1 C_2 \|\sigma\|_{TH}.
\]

Similarly, we obtain

\[
-C_1 C_2 \|\sigma\|_{TH} \leq V_a[Q] \leq 0 \quad \text{and} \quad -C_1 C_2 \|\sigma\|_{TH} \leq V_c[Q] \leq 0.
\]

It remains to consider \(V_m[Q]\). Because of Lemma 2, by a Möbius change of coordinates, we may assume \(a = -\infty, b = -1, c = 0, d = 1\). Then the geodesic lines in \(\mathcal{L}_m\) connect the points of \([-1, 0]\) to the points of \([1, +\infty]\). By Lemma 4 if we move all the lines in \(\mathcal{L}_m\) to the geodesic line from 0 to \(\infty\) without changing the weights of the lines in \(\mathcal{L}_m\) to obtain a new measure \(\sigma'_m\), then \(V'_m[Q] = E(\sigma'_m)[Q]\) is possibly bigger, that is, \(V'_m[Q] \leq V_m[Q]\). Clearly, \(V'_m(x) = 0\) for \(x \in [-\infty, 0]\) and \(V'_m(x) = \sigma(\mathcal{L}_m)x\) for \(x \in (0, +\infty)\). Then \(V'_m[Q] = V'_m(1) = \sigma(\mathcal{L}_m)\). If we let \(C_0\)
denote the smallest integer $\geq \ln(3 + 2\sqrt{2})$, then by Lemma 4, $\sigma(L_m) \leq C_0 \|\sigma\|_{Th}$.

Therefore,

$$V_m[Q] \leq V_m^\prime[Q] = \sigma(L_m) \leq C_0 \|\sigma\|_{Th}.$$ 

Again by Lemma 4, if we move all the lines in $L_m$ to the geodesic line from $-1$ to $1$ without changing the weights of the lines in $L_m$ to obtain a new measure $\sigma_m^\prime$, then $V_m^\prime[Q] = E(\sigma_m^\prime)[Q]$ is possibly smaller, that is, $V_m[Q] \geq V_m^\prime[Q]$. Clearly, $V_m^\prime(x) = 0$ for $x \notin (-1, 1)$ and $V_m^\prime(x) = \sigma(L_m)\{x + 1\}(1 - x)$ for $x \in (-1, 1)$. Then $V_m^\prime[Q] = -2V_m^\prime(0) = -\sigma(L_m)$. Therefore,

$$V_m[Q] \geq V_m^\prime[Q] = -\sigma(L_m) \geq -C_0 \|\sigma\|_{Th}.$$ 

Collecting together these estimates, we obtain

$$-(C_0 + 2C_1C_2)\|\sigma\|_{Th} \leq V_0[Q] + V_c[Q] + V_m[Q] + V_b[Q] + V_d[Q] \leq (C_0 + 2C_1C_2)\|\sigma\|_{Th};$$

that is,

$$|V[Q]| \leq (C_0 + 2C_1C_2)\|\sigma\|_{Th}.$$ 

Propositions 1 and 2 imply Theorem 2.

Proof. Consider the pattern of the geodesic lines in $L$ with respect to $Q$, there either exists a line in $L$ separating two adjacent vertices in $Q$ from the other two or no such line exists. The former case is treated by Proposition 2 and the fact that $V[\{b, c, d, a\}] = -V[Q]$; the latter one is treated by Proposition 1 and the summability of $V[Q] = E(\sigma)[Q]$ over four subcollections of $L$ that contain three vertices of $Q$ in the same stratum.

Finally, Theorems 1 and 2 imply our Main Theorem.

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REFERENCES


Department of Mathematics, Brooklyn College, CUNY, Brooklyn, New York 11210

E-mail address: jun@sci.brooklyn.cuny.edu