DECOMPOSABILITY OF GRAPH $C^*$-ALGEBRAS

JEONG HEE HONG

(Communicated by David R. Larson)

Abstract. We give conditions on an arbitrary directed graph $E$ for the associated Cuntz-Krieger algebra $C^*(E)$ to be decomposable as a direct sum. We describe the direct summands as certain graph algebras.

0. Introduction

Recently various generalizations of Cuntz-Krieger algebras [2] have attracted a lot of attention. In this article we are concerned with generalized Cuntz-Krieger algebras based on directed graphs ([2] and references therein). One of the key advantages in the theory of graph algebras is that a directed graph $E$ is used to conveniently represent generators and relations of the associated graph algebra $C^*(E)$. Thanks to the combined efforts of a number of researchers, it is now known how to read from the graph many of the basic properties and invariants of the algebra.

As with many a mathematical theory, classification of the objects in question presents itself as an important objective. A future classification of graph algebras might be very useful in paving the way for other classifications of more general classes of $C^*$-algebras, similarly to the way the classification of Cuntz-Krieger algebras was the starting point for the Kirchberg-Phillips classification of purely infinite simple algebras. In this context, the class of non-simple purely infinite graph algebras (in the sense of Kirchberg-Rørdam) appears to be of particular interest (cf. [6]). Certainly, the first necessary step towards a classification of non-simple algebras is good understanding of their ideal structure. For graph algebras this has been recently achieved (cf. [1, 5]). These results have already been successfully applied in solutions to some concrete problems in the classification of graph algebras as well as in quantum groups (cf. [1, 8, 3]).

In the present article we consider the question when an ideal of a graph algebra is a direct summand or, in other words, when a graph algebra decomposes as a direct sum. This very natural question turns out to be more complicated than it appears. Obviously, $C^*(E)$ splits as a direct sum when the graph $E$ is disconnected. However, such a splitting also exists for many connected directed graphs. Especially in the context of infinite directed graphs this is a subtle problem requiring careful
analysis, and even for finite graphs it is not a trivial one. The main result of this paper is a necessary and sufficient condition on an arbitrary infinite graph $E$ that guarantees that the associated graph algebra $C^*(E)$ decomposes as a direct sum. Furthermore, we show that the summands are themselves isomorphic to certain graph algebras.

1. Preliminaries on graph $C^*$-algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph with countably many vertices $E^0$ and edges $E^1$, and range, source functions $r, s : E^1 \rightarrow E^0$, respectively. The graph $C^*$-algebra, or simply graph algebra $C^*(E)$ is defined as the universal $C^*$-algebra generated by families of projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$, subject to the following relations:

1. $P_v P_w = 0$ for $v, w \in E^0, v \neq w$.
2. $S_e^* S_f = 0$ for $e, f \in E^1, e \neq f$.
3. $S_e^* S_e = P_{r(e)}$ for $e \in E^1$.
4. $S_e S_e^* \leq P_{s(e)}$ for $e \in E^1$.
5. $P_v = \sum_{e \in E^1 : s(e) = v} S_e S_e^*$ for $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$.

In this case, $\{P_v, S_e : v \in E^0, e \in E^1\}$ is called a Cuntz-Krieger $E$-family. Universality in the definition means that if $\{Q_v : v \in E^0\}$ and $\{T_e : e \in E^1\}$ are families of projections and partial isometries, respectively, satisfying conditions (GA1–GA5), then there exists a $C^*$-algebra homomorphism from $C^*(E)$ to the $C^*$-algebra generated by $\{Q_v : v \in E^0\}$ and $\{T_e : e \in E^1\}$ such that $P_v \mapsto Q_v$ and $S_e \mapsto T_e$ for $v \in E^0, e \in E^1$. It is also equivalent to the existence of a gauge action $\gamma : T \rightarrow \text{Aut}(C^*(E))$, which is characterized by $\gamma_t(S_e) = t S_e$ and $\gamma_t(P_v) = P_v$ for $e \in E^1, v \in E^0, t \in T$.

As usual we denote by $E^*$ the set of all finite paths in $E$ (vertices in $E^0$ are identified with paths of length 0), and by $E^\infty$ the set of all infinite paths in $E$. By writing $v \geq w$ when there is a path from $v$ to $w$, we say that a subset $H$ of $E^0$ is hereditary if $v \in H$ and $v \geq w$ imply $w \in H$. A subset $X$ of $E^0$ is said to be saturated if every vertex $v$ that satisfies $0 < |s^{-1}(v)| < \infty$ and $s(e) = v \Rightarrow r(e) \in X$ itself belongs to $X$. The following definitions come from [1]. For a hereditary and saturated subset $X$ of $E^0$, we denote $X^\infty = \{v \in E^0 \setminus X : |s^{-1}(v)| = \infty, 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus X)| < \infty\}$. If $w \in X^\infty$, $P_{w, X} = \sum_{e \in E^1, s(e) = w, r(e) \notin X} S_e S_e^*$ denotes the subprojection of $P_w$.

Our methods in this article are mainly based on the structure of gauge-invariant ideals of graph algebras (see [1] for details). Let $X$ be a hereditary and saturated subset of $E^0$. For $B \subseteq X^\infty$, we denote by $J_{X,B}$ the ideal of $C^*(E)$ generated by $\{P_v : v \in X\}$ and $\{P_w - P_{w, X} : w \in B\}$. When $B = \emptyset$, we write $J_{X, \emptyset} = I_X$, the ideal generated by $\{P_v : v \in X\}$. We have

$$J_{X,B} = \sum_{\alpha, \eta, \mu, \nu \in E^*} S_\alpha P_\eta S^*_\mu (P_w - P_{w, X}) S^*_\nu :$$

$$\alpha, \eta, \mu, \nu \in E^*, r(\alpha) = r(\eta) = v \in X, r(\mu) = r(\nu) = w \in B \}.$$

The ideal $J_{X,B}$ is gauge-invariant, i.e. $\gamma_t(J_{X,B}) = J_{X,B}$ for all $t \in T$. It is now fully known that $J_{X,B}$ and the quotient $C^*(E)/J_{X,B}$ are isomorphic to graph algebras, associated with a directed graph $X E_B$ and a quotient graph $E/X$. To form a directed graph $X E_B$, let $\overline{F}_E(X, B)$ be the collection of all finite paths.
\[ \alpha = (a_1, \ldots, a_{|\alpha|}) \] of positive length such that \( s(\alpha) \in E^0 \setminus X \), \( r(\alpha) \in X \cup B \), and \( r(a_j) \notin X \cup B \) for \( j < |\alpha| \). Set \( F_E(X, B) = \overline{F}_E(X, B) \setminus \{ e \in E^1 : s(e) \in B \text{ and } r(e) \in X \} \). We denote by \( \overline{T}_E(X, B) \) another copy of \( F_E(X, B) \), and write \( \overline{T} \in \overline{T}_E(X, B) \) for the copy of \( \alpha \in F_E(X, B) \). Then the graph \( X_E B \) is given as follows:

\[
\begin{align*}
(x E B)^0 &= x E^0_B = X \cup B \cup F_E(X, B), \\
(x E B)^1 &= x E^1_B \\
&= \{ e \in E^1 : s(e) \in X \} \cup \{ e \in E^1 : s(e) \in B \text{ and } r(e) \in X \} \cup \overline{T}_E(X, B),
\end{align*}
\]

with \( s(\overline{\alpha}) = \alpha \) and \( r(\overline{\alpha}) = r(\alpha) \) for \( \alpha \in F_E(X, B) \), and the source and range as in \( E \) for the other edges of \( X E_B \). When \( B = \emptyset \), we simply denote \( x_E B = x E \).

The quotient graph \( E/X \) is given by \( (E/X)^0 = (E^0 \setminus X) \cup \{ \beta(v) : v \in X_{\text{fin}} \} \) and \( (E/X)^1 = r^{-1}(E^0 \setminus X) \cup \{ \beta(e) : e \in E^1, r(e) \in X_{\text{fin}} \} \), where \( r, s \) are extended by \( s(\beta(e)) = s(e) \) and \( r(\beta(e)) = \beta(r(e)) \). Here \( \beta \) is just a symbol helping to distinguish \( v \) and \( e \) from the extra \( \beta(v) \) and \( \beta(e) \) in \( E/X \), respectively. See [3, Example 1.4] and [1] Example 3.3, which illustrate the graphs \( x E_B \) and \( E/X \) respectively.

**Theorem 1.1.** Let \( E \) be a directed graph. Then there is a 1-1 correspondence between the set of gauge-invariant ideals of \( C^*(E) \) and the set of ideals of the form \( J_{X, B} \) where \( X \) is a hereditary and saturated subset of \( E^0 \) and \( B \subseteq X_{\text{fin}} \). Moreover,

(i) \( [3 \text{ Lemma 1.5}] \) the ideal \( J_{X, B} \) is isomorphic to \( C^*(x E_B) \), and

(ii) \( ) [1 \text{ Corollary 3.5}] \) its quotient \( C^*(E)/J_{X, B} \) is isomorphic to \( C^* ((E/X) \setminus \beta(B)) \).

If \( B = X_{\text{fin}} \), then

\[
C^*(E)/J_{X, X_{\text{fin}}} \cong C^*(E \setminus X),
\]

where \( E \setminus X = (E^0 \setminus X, r^{-1}(E^0 \setminus X), r, s) \).

2. Direct sum decompositions of graph \( C^* \)-algebras

**Definition 2.1.** Let \( E \) be a directed graph. If there exist two non-zero \( C^* \)-algebras \( A, B \) such that \( C^*(E) \cong A \oplus B \), then \( C^*(E) \) is said to be decomposable. Otherwise, \( C^*(E) \) is indecomposable.

Our aim is to find conditions on \( E \) so that \( C^*(E) \) is decomposable. We denote by \( \text{Prim}(A) \) the set of all primitive ideals in a \( C^* \)-algebra \( A \), equipped with the hull-kernel topology.

**Lemma 2.2.** If \( C^*(E) = A \oplus B \) with non-zero closed ideals \( A \) and \( B \), then \( A \) and \( B \) are gauge-invariant.

**Proof.** Since every ideal in \( C^*(E) \) can be realized as the intersection of a family of primitive ideals and \( \text{Prim}(C^*(E)) \) is the disjoint union of \( \text{Prim}(A) \) and \( \text{Prim}(B) \), it suffices to show that \( \text{Prim}(A) \) and \( \text{Prim}(B) \) are invariant under the gauge action \( \gamma \).

If \( J \) is a primitive ideal of \( C^*(E) \) that is not gauge-invariant, then there exists a homeomorphic imbedding \( \phi : T \to \text{Prim}(C^*(E)) \) such that \( J \) belongs to \( \phi(T) \), by combining Lemma 2.8 and Theorem 2.10 of [3]. Furthermore, \( \phi(T) \) is invariant under the gauge action. Since \( T \) is connected, \( \phi(T) \) is connected in \( \text{Prim}(C^*(E)) \). However, both \( \text{Prim}(A) \) and \( \text{Prim}(B) \) are closed and open, and therefore \( \phi(T) \) is entirely contained either in \( \text{Prim}(A) \) or in \( \text{Prim}(B) \). Consequently, both \( \text{Prim}(A) \) and \( \text{Prim}(B) \) are invariant under the gauge action. \( \square \)
Note that in Definition 2.1 for $C^*(E)$ to be decomposable we do not a priori require that $A$ and $B$ be graph algebras. However, this turns out to be true by the following theorem.

**Theorem 2.3.** If $C^*(E) = A \oplus B$ with non-zero closed ideals $A$ and $B$, then there exist non-empty, disjoint, hereditary and saturated subsets $X$ and $Y$ of $E^0$ such that $A = J_{X,X}^{\text{fin}}$, $B = J_{Y,Y}^{\text{fin}}$, and $(X \cup X^{\text{fin}}_\infty) \cap (Y \cup Y^{\text{fin}}_\infty) = \emptyset$. Furthermore, $C^*(E)$ is decomposed into the direct sum of two graph algebras as $C^*(E \setminus Y) \oplus C^*(E \setminus X)$.

**Proof.** By Lemma 2.2 $A$ and $B$ are gauge-invariant. Then by Theorem 1.1 there exist two hereditary and saturated subsets $X$ and $Y$ of $E^0$ such that $A = J_{X,C}$ and $B = J_{Y,D}$ where $C \subseteq X^{\text{fin}}_\infty$ and $D \subseteq Y^{\text{fin}}_\infty$. We have

$$X = \{v \in E^0 : P_v \in A\}, \quad C = \{v \in E^0 \setminus X : P_v - P_{v,X} \in A\}, \quad Y = \{v \in E^0 \setminus Y : P_v - P_{v,Y} \in B\}. $$

It follows from the decomposability of $C^*(E)$ that $X$ and $Y$ are non-empty and disjoint. Then, by the definitions of $X^{\text{fin}}_\infty$ and $Y^{\text{fin}}_\infty$, $X^{\text{fin}}_\infty \cap Y^{\text{fin}}_\infty = \emptyset$.

To show the fact $C = X^{\text{fin}}_\infty$, suppose that there is a vertex $v \in X^{\text{fin}}_\infty \setminus C$. Then the projection $P_v - P_{v,X} \notin J_{X,C}$. Since $C^*(E) = J_{X,C} \oplus J_{Y,D}$, we must have $P_v - P_{v,X} \in J_{Y,D}$, or $v \in D \subseteq Y^{\text{fin}}_\infty$, a contradiction to the fact $X^{\text{fin}}_\infty \cap Y^{\text{fin}}_\infty = \emptyset$. Thus we must have $C = X^{\text{fin}}_\infty$, and a similar argument yields $D = Y^{\text{fin}}_\infty$. The fact $(X \cup X^{\text{fin}}_\infty) \cap (Y \cup Y^{\text{fin}}_\infty) = \emptyset$ then follows easily from the hereditary and saturated properties of $X$ and $Y$.

Moreover, if $C^*(E) = J_{X,X}^{\text{fin}} \oplus J_{Y,Y}^{\text{fin}}$, then

$$J_{X,X}^{\text{fin}} \cong C^*(E)/J_{Y,Y}^{\text{fin}} \cong C^*(E \setminus Y)$$

and

$$J_{Y,Y}^{\text{fin}} \cong C^*(E)/J_{X,X}^{\text{fin}} \cong C^*(E \setminus X),$$

i.e. $C^*(E) \cong C^*(E \setminus Y) \oplus C^*(E \setminus X)$. \(\Box\)

**Remark 2.4.** By Theorem 1.1 we know that $J_{X,X}^{\text{fin}} \cong C^*(x E_{Y}^{\text{fin}})$. If $C^*(E)$ is decomposable as in Theorem 2.3, then $C^*(x E_{Y}^{\text{fin}}) \cong J_{X,X}^{\text{fin}} \cong C^*(E \setminus Y)$. In general, the two graphs $x E_{X}^{\text{fin}}$ and $E \setminus Y$ are different even though their associated graph algebras are isomorphic.

The following observation will be useful later in this article.

**Lemma 2.5.** Let $X$ and $Y$ be non-empty, disjoint, hereditary and saturated subsets of $E^0$ such that $C^*(E) = J_{X,X}^{\text{fin}} \oplus J_{Y,Y}^{\text{fin}}$. If $u \in E^0 \setminus (X \cup Y)$, then there exists a path in $E$ from $u$ into $X \cup Y$.

**Proof.** Suppose that there exists no paths in $E$ from $u$ into $X \cup Y$. We have $u \notin X^{\text{fin}}_\infty \cup Y^{\text{fin}}_\infty$. Since $P_u$ must be in one of the summands, it suffices to show $P_u J_{X,X}^{\text{fin}} = 0 = P_u J_{Y,Y}^{\text{fin}}$ to obtain a contradiction. We use the description of the ideals $J_{X,X}^{\text{fin}}$ and $J_{Y,Y}^{\text{fin}}$ given by the formula (1). If $w \in X$ (or $Y$) and $\alpha, \eta$ are paths in $E$ with $r(\alpha) = r(\eta) = w$, then there must be no paths in $E$ from $u$ to both $\alpha$ and $\eta$ by assumption. Hence $P_u (S_{\alpha} P_w S_{\eta}^*) = 0$. Let $w \in X^{\text{fin}}_\infty$ and $\mu, \nu$ be paths in $E$ with $r(\mu) = r(\nu) = w$. Again there must be no paths in $E$ from $u$ to both $\mu$ and $\nu$. Hence we get $P_u (S_{\mu} (P_w - P_{w,X}) S_{\nu}^*) = 0$. Similarly we obtain $P_u (S_{\mu} (P_w - P_{w,Y}) S_{\nu}^*) = 0$ for the case of $w \in Y^{\text{fin}}_\infty$. \(\Box\)
3. Certain representations

We now focus on constructing two representations (Lemmas 3.2 and 3.4) of $C^*(E)$, which will play a crucial role in proving our main result. To this end, it is useful to consider a certain subgraph $F$ of $E$. Let $X$ and $Y$ be non-empty, disjoint, hereditary and saturated subsets of $E^0$. Then the subgraph $F = (F^0, F^1, r, s)$ of $E$ is given by

(2) $F^0 = E^0$,

(3) $F^1 = E^1 \setminus \{ (e \in E^1 : s(e) \in X_\text{fin}, r(e) \in X \} \cup \{ f \in E^1 : s(f) \in Y_\text{fin}, r(f) \in Y \}$.

Let $\Omega$ be the collection of all finite paths in $F$ beginning outside $X \cup Y$ and ending inside $X \cup Y$, upon the first entry into $X \cup Y$, i.e.,

$$\Omega := \{ \omega = (e_1, \ldots, e_k) \in F^* : s(\omega) \notin X \cup Y, r(\omega) \in X \cup Y, r(e_i) \notin X \cup Y \text{ for } i < k \ (k \in \mathbb{N} \setminus \{0\}) \},$$

and let $\mathcal{H}_\Omega$ be the Hilbert space with an orthonormal basis $\{ \xi_\omega : \omega \in \Omega \}$ indexed by $\Omega$. We define projections $\{ Q_v : v \in E^0 \}$ and partial isometries $\{ T_e : e \in E^1 \}$ on $\mathcal{H}_\Omega$ as follows:

(4) $Q_v(\xi_\omega) = \begin{cases} \xi_\omega & \text{if } v = s(\omega), \\ 0 & \text{otherwise}, \end{cases}$

(5) $T_e(\xi_\omega) = \begin{cases} \xi_{e;\omega} & \text{if } r(e) = s(\omega), \\ 0 & \text{otherwise}. \end{cases}$

Note that for $v \in E^0 \setminus (X \cup Y)$, the projection $Q_v$ has finite rank if and only if there exist finitely many paths in $F$ from $v$ to $X \cup Y$. Indeed, the vertex $v \in E^0 \setminus (X \cup Y)$ corresponds to a projection $Q_v$ that maps onto $\text{span} \{ \xi_\omega : s(\omega) = v, \omega \in \Omega \}$.

Lemma 3.1. Let $X$ and $Y$ be non-empty, disjoint, hereditary and saturated subsets of $E^0$. Let $F$ be the subgraph of $E$ as defined in (2) and (3). If for every $v \in E^0 \setminus (X \cup Y)$ there exist finitely many (and at least one) paths in $F$ with source $v$ to $X \cup Y$, then there exist only finitely many edges in $F$ emitting from $v$.

Proof. Let $v \in E^0 \setminus (X \cup Y)$. Suppose that there are infinitely many edges $e$ in $F$ with $s(e) = v$. By passing this into the graph $E$, we see that $v \notin X_\text{fin} \cup Y_\text{fin}$ in $E$. Hence only two cases are possible.

(i) If all edges $e \in s^{-1}(v)$ satisfy $r(e) \in E^0 \setminus (X \cup Y)$ except finitely many edges, then there are finitely many paths in $F$ from $r(e)$ to $X \cup Y$ by assumption. These would produce infinitely many paths in $F$ from $v$ to $X \cup Y$, a contradiction.

(ii) If there are infinitely many $e \in E^1$ with $s(e) = v$ and $r(e) \in X$ and infinitely many $f \in E^1$ with $s(f) = v$ and $r(f) \in Y$, then all these edges still remain in the graph $F$ to yield infinitely many paths in $F$ from $v$ to $X \cup Y$, a contradiction.

We now examine whether the family $\{ Q_v, T_e : v \in E^0, e \in E^1 \}$ satisfies the Cuntz-Krieger relations for $E$. Conditions (GA1) and (GA2) are obvious. Condition (GA3) follows from the fact that $T_e^* T_e(\xi_\omega) = Q_v(\xi_\omega)$ if and only if $s(\omega) = v = r(e)$. Similarly, condition (GA4) is fulfilled. Unfortunately this family may not satisfy (GA5). Indeed, suppose $v \in E^0$ has the property $0 < |s^{-1}(v)| < \infty$ in $E$. For any $e \in s^{-1}(v)$, $T_e T_e^*$ is a projection onto $\text{span} \{ \xi_\omega : \omega = (e, \omega') \}$ for some $\omega' \in
\( \Omega \) and \( r(e) = s(\omega') \), while the range of the projection \( Q_v \) includes the vector \( \xi_{(f, \omega')} \) for some \( \omega' \in \Omega \) and \( r(f) = s(\omega') \), whenever there is an edge \( f \in E^1 \) such that \( s(f) = s(e) = v \) and \( e \neq f \). Bearing this in mind, we define \( R_v \) as the projection onto \( \text{span}\{\xi_e \mid e \in E^1, s(e) = v, r(e) \in X \cup Y\} \). Then \( R_v \) is a projection of finite rank in \( \mathcal{B}(\mathcal{H}_\Omega) \) by the construction, and

\[
Q_v = \sum_{e \in E^1, s(e) = v, r(e) \notin X \cup Y} T_e T_e^* + R_v.
\]

By hint of this, we pass the generating family into \( \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega) \) so that the image forms a Cuntz-Krieger \( E \)-family. To this end let \( \pi : \mathcal{B}(\mathcal{H}_\Omega) \to \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega) \) be the canonical quotient map. Then \( \{\pi(Q_v), \pi(T_e) : v \in E^0, e \in E^1\} \) forms a Cuntz-Krieger \( E \)-family in \( \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega) \). Indeed, it is enough to check (GA5). So let \( v \in E^0 \) be \( 0 < |s^{-1}(v)| < \infty \). Since the rank of \( R_v \) onto \( \text{span}\{\xi_e \mid e \in E^1, s(e) = v, r(e) \in X \cup Y\} \) is finite in \( \mathcal{B}(\mathcal{H}_\Omega) \), the formula (6) yields \( \pi(Q_v) = \sum_{e \in E^1, s(e) = v} \pi(T_e)\pi(T_e)^* \) in \( \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega) \), and hence (GA5) holds.

**Lemma 3.2.** Let \( X \) and \( Y \) be non-empty, disjoint, hereditary and saturated subsets of \( E^0 \). Then there is a \( * \)-homomorphism

\[
\rho : C^*(E) \to \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega),
\]

satisfying \( \rho(S_e) = \pi(T_e) \), \( \rho(P_v) = \pi(Q_v) \), \( e \in E^1, v \in E^0 \), and \( \rho(J_{X,X_{\infty}}) = \rho(J_{Y,Y_{\infty}}) = \{0\} \).

**Proof.** Since \( \{\pi(Q_v), \pi(T_e) : v \in E^0, e \in E^1\} \) forms a Cuntz-Krieger \( E \)-family in \( \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega) \) by the above argument, universality of \( C^*(E) \) implies that there exists a \( * \)-homomorphism \( \rho : C^*(E) \to \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega) \) such that \( \rho(S_e) = \pi(T_e) \), \( \rho(P_v) = \pi(Q_v) \), \( e \in E^1, v \in E^0 \).

For the remaining part \( \rho(J_{X,X_{\infty}}) = \{0\} \), we check on the generators \( P_v \) and \( P_w - P_{w,X} \) of \( J_{X,X_{\infty}} \), when \( v \in X \) and \( w \in X_{\infty} \). If \( v \in X \), then \( \rho(P_v) = \pi(Q_v) = 0 \) because there is no \( \alpha \in \Omega \) satisfying \( s(\alpha) = v \). Let \( w \in X_{\infty} \). If \( \alpha = (e_1, \ldots, e_k) \in \Omega \) with \( s(\alpha) = w \), then \( r(e_1) \notin X \). Hence we have \( \pi(Q_v)\xi_\alpha = \pi(Q_{w,X})\xi_\alpha \) for all \( \alpha \in \Omega \), and thus \( \rho(P_w - P_{w,X}) = \pi(Q_w - Q_{w,X}) = 0 \). The similar argument gives \( \rho(J_{Y,Y_{\infty}}) = \{0\} \).

Now let \( v \in E^0 \setminus (X \cup Y) \) and suppose there is an infinite path, say \( \alpha = (a_1, a_2, a_3, \ldots) \) in \( E \) such that \( s(\alpha) = v \). We define a corresponding representation of \( C^*(E) \). Let \( \Lambda \) be the collection of all infinite paths in \( E \) that are shift-tail equivalent to \( \alpha \) (recall that an infinite path \( (e_1, e_2, e_3, \ldots) \) is shift-tail equivalent to \( \alpha \) if there exist \( k, m \in \mathbb{N} \) such that \( e_{k+i} = a_{m+i} \) for all \( i \in \mathbb{N} \)), i.e.,

\[
\Lambda := \{\beta = (b_1, b_2, b_3, \ldots) \in E^\infty : \exists k, m \in \mathbb{N} \text{ s.t. } b_{k+i} = a_{m+i} \text{ for all } i \in \mathbb{N}\},
\]

and let \( \mathcal{H}_\Lambda \) be the Hilbert space with an orthonormal basis \( \{\xi_\beta : \beta \in \Lambda\} \) indexed by \( \Lambda \). For \( w \in E^0 \) and \( e \in E^1 \) we define a projection \( U_w \) and a partial isometry \( D_e \) on \( \mathcal{H}_\Lambda \) as follows:

\[
U_w(\zeta_\beta) = \begin{cases} 
\zeta_\beta & \text{if } w = s(\beta), \\
0 & \text{otherwise,}
\end{cases}
\]

\[
D_e(\zeta_\beta) = \begin{cases} 
\zeta_{(e,b_1,b_2,\ldots)} & \text{if } r(e) = s(\beta), \text{ where } \beta = (b_1, b_2, \ldots) \in \Lambda, \\
0 & \text{otherwise.}
\end{cases}
\]
Lemma 3.3. Let $X$ and $Y$ be non-empty, disjoint, hereditary and saturated subsets of $E^0$. If a vertex $v \in E^0 \setminus (X \cup Y)$ has an infinite path $\alpha = (a_1, a_2, a_3, \cdots)$ in $E$ with $s(\alpha) = v$, then there is a $*$-homomorphism $\varrho : C^*(E) \to B(\mathcal{H}_{\Lambda})$ such that $\varrho(P_w) = U_w$ and $\varrho(S_e) = D_e$ for all $w \in E^0$, $e \in E^1$. Furthermore, if $\alpha$ never enters $X \cup Y$, then $\varrho(J_{X,X^\infty}) = \varrho(J_{Y,Y^\infty}) = \{0\}$.

Proof. It is easy to show that the family $\{U_w, D_e \mid w \in E^0, e \in E^1\}$ as defined in $(7)$ and $(8)$ is a Cuntz-Krieger $E$-family, and thus universality of $C^*(E)$ implies that there is a $*$-homomorphism $\varrho : C^*(E) \to B(\mathcal{H}_{\Lambda})$ such that $\varrho(P_w) = U_w$ and $\varrho(S_e) = D_e$ for all $w \in E^0$, $e \in E^1$.

Now suppose that $\alpha$ never enters $X \cup Y$, that is, $r(a_j) \not\in X \cup Y$ for all $j$. It suffices to check that $\varrho$ kills all the generating projections of the ideals $J_{X,X^\infty}$ and $J_{Y,Y^\infty}$. First we show that $\varrho$ kills all generating projections $P_w$ and $P_u - P_{u,X}$ of $J_{X,X^\infty}$ for $w \in X$ and $u \in X^\infty$. If $w \in X$ satisfies $\varrho(P_w) \neq 0$, then there exists an infinite path $\beta = (b_1, b_2, \cdots)$ such that $s(\beta) = w$ and $b_{k+i} = a_{m+i}$ for some $k, m \in \mathbb{N}$. By the hereditary property of $X$, $r(b_k) = r(a_m) \in X$, a contradiction. Now let $u \in X^\infty$ satisfy $\varrho(P_{u,X}) \neq 0$. Since $P_{u,X} = P_u - \sum_{e \in E^1, s(e) = u, r(e) \not\in X} S_e$, there exists an infinite path $\beta = (b_1, b_2, \cdots)$ such that $s(\beta) = u$, $b_{k+i} = a_{m+i}$ for some $k, m \in \mathbb{N}$, and $r(b_1) \in X$. This is a contradiction, since the hereditary property of $X$ would then imply that $\alpha$ enters $X$.

By replacing $X$ with $Y$ in the above argument we can easily show that $\varrho$ kills the generating projections $P_w$ and $P_u - P_{u,Y}$ of $J_{Y,Y^\infty}$ for $w \in Y$ and $u \in Y^\infty$. \hfill $\square$

Lemma 3.4. Let $X$ and $Y$ be non-empty, disjoint, hereditary and saturated subsets of $E^0$ such that $C^*(E) = J_{X,X^\infty} \oplus J_{Y,Y^\infty}$, and let $F$ be the subgraph of $E$ as defined in $(2)$ and $(3)$. If $v \in E^0 \setminus (X \cup Y)$ and there are no paths in $F$ from $v$ to $X \cup Y$, then there exists a representation $\varrho$ of $C^*(E)$ such that $\varrho(J_{X,X^\infty}) = \varrho(J_{Y,Y^\infty}) = \{0\}$ but $\varrho(P_v) \neq 0$.

Proof. Let $v \in E^0 \setminus (X \cup Y)$. Assume that there are no paths in $F$ from $v$ into $X \cup Y$. We first show that there exists an infinite path in $E$ which begins at $v$ and never enters $X \cup Y$. By Lemma 2.5 there exists a path in $E$ from $v$ to $X \cup Y$. Suppose that it enters, say $X$. Let $w$ be the last vertex on this path which is not in $X$. Because of our assumption on $v$, the vertex $w$ must satisfy the following:

(i) $w$ belongs to $X^\infty$,
(ii) there is no edge in $E$ which begins at $w$ and ends inside $Y$.

Condition (i) is obvious. Indeed, if $w \not\in X^\infty$, then it would yield a path in $E$ from $v$ to $X$, a contradiction. Condition (ii) also holds obviously by the same reason as the case of (i). (Note that if the path enters $Y$, then the conditions become (i) $w \in Y^\infty$ and (ii) there is no edge in $E$ which begins at $w$ and ends inside $X$.) This means that there are finitely many edges $e \in E^1$ such that $s(e) = w$ and $r(e) \not\in X \cup Y$. Then, by applying Lemma 2.5 again to the vertex $r(e)$, we see that there exists a path in $E$ from $r(e)$ to $X \cup Y$. Now the same argument with $r(e)$ as before yields a vertex $u$ satisfying the conditions (i) and (ii). Continuously we can produce an infinite path in $E$ from $v$ that never enters $X \cup Y$. Call it $\alpha = (a_1, a_2, a_3, \cdots)$.

Now consider the corresponding representation $\varrho$ on the Hilbert space $\mathcal{H}_{\Lambda}$ as defined in $(7)$ and $(8)$. Since the path $\alpha$ begins at $v$, it follows that $\varrho(P_v) \neq 0$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Moreover, since $\alpha$ never enters $X \cup Y$, $\varrho$ is zero on both $J_{X,X^\infty}$ and $J_{Y,Y^\infty}$ by Lemma 3.3.

4. THE MAIN RESULTS

We now have all the ingredients to prove our main theorem.

Theorem 4.1. Let $E$ be a directed graph. Then $C^*(E)$ is decomposable if and only if there exist two non-empty, disjoint, hereditary and saturated subsets $X$ and $Y$ of $E^0$ such that for every $v \in E^0 \setminus (X \cup Y)$ there exist finitely many (and at least one) paths in $F$ from $v$ to $X \cup Y$, where $F$ is the subgraph of $E$ as defined in (2) and (3).

If this is the case, $C^*(E)$ is decomposed into two graph algebras as $C^*(E \setminus Y) \oplus C^*(E \setminus X)$.

Proof. ($\implies$) Let $C^*(E)$ be decomposable. Then by Theorem 2.3 there exist two non-empty, disjoint, hereditary and saturated subsets $X$ and $Y$ of $E^0$ such that $C^*(E) = J_{X,X^\infty} \oplus J_{Y,Y^\infty}$ and $(X \cup X^\infty) \cap (Y \cup Y^\infty) = \emptyset$. We now form the subgraph $F$ of $E$ as defined in (2) and (3), and let $v \in E^0 \setminus (X \cup Y)$. By Lemma 3.2 there is a representation $\varrho$ of $C^*(E)$ on the Hilbert space $\mathcal{H}_\Omega$ such that $\varrho(J_{X,X^\infty}) = \varrho(J_{Y,Y^\infty}) = \{0\}$.

Suppose that there are infinitely many paths in $F$ from $v$ into $X \cup Y$. Then the projection $Q_v$, as defined in (4), has infinite rank in $\mathcal{B}(\mathcal{H}_\Omega)$, and hence $\pi(Q_v) \neq 0$ in $\mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)$. Since $\varrho(Q_v) = \pi(Q_v)$, it follows from Lemma 3.2 that $P_v \neq 0$ in $C^*(E)/(J_{X,X^\infty} \oplus J_{Y,Y^\infty})$, or $P_v \notin J_{X,X^\infty} \oplus J_{Y,Y^\infty} = C^*(E)$, a contradiction. Thus there exist only finitely many paths in $F$ from $v$ to $X \cup Y$.

Now suppose that there are no paths in $F$ from $v$ to $X \cup Y$. Then by Lemma 3.3 there exists a representation $\varrho$ of $C^*(E)$ on the Hilbert space $\mathcal{H}_\Lambda$ such that $\varrho(J_{X,X^\infty}) = \varrho(J_{Y,Y^\infty}) = \{0\}$, but $\varrho(P_v) \neq 0$. This means $P_v \notin J_{X,X^\infty} \oplus J_{Y,Y^\infty} = C^*(E)$, a contradiction. Hence there must exist at least one path in $F$ from $v$ to $X \cup Y$.

($\impliedby$) Let $X$ and $Y$ be non-empty subsets of $E^0$ satisfying the above conditions. We show that $J_{X,X^\infty} \cap J_{Y,Y^\infty} = \{0\}$ and $J_{X,X^\infty} \cup J_{Y,Y^\infty}$ generates $C^*(E)$.

Step 1. For $J_{X,X^\infty} \cap J_{Y,Y^\infty} = \{0\}$, it is enough to check the following cases by virtue of (1).

(i) We show $(S_\alpha P_v S_{\beta})(S_\mu P_v S_{\mu}) = 0$ for paths $\alpha, \beta, \mu, \nu$ such that $r(\alpha) = r(\beta) = w \in Y$, $r(\mu) = r(\nu) = v \in X$. Suppose that either $\beta$ is an initial subpath of $\mu$ or $\mu$ is an initial subpath of $\beta$. This implies that either there is a path from $Y$ to $X$ or vice versa, a contradiction since $X$ and $Y$ are hereditary and disjoint. Thus $S_\beta S_\mu = 0$.

(ii) We show $(S_\alpha P_w S_{\beta})(S_\mu (P_v - P_{v,X}) S_v) = 0$ for paths $\alpha, \beta, \mu, \nu$ such that $r(\alpha) = r(\beta) = w \in Y$ and $r(\mu) = r(\nu) = v \in X^\infty$. Note that $\beta$ cannot be an initial subpath of $\mu$. Otherwise this creates a path from $w \in Y$ to $v$, and the hereditary property of $Y$ implies $v \in Y$; but now $v \in X^\infty \cap Y$ and $v \geq X$ implies that $X \cap Y \neq \emptyset$ since $X$ and $Y$ are hereditary. This contradicts the fact $X^\infty \cap Y = \emptyset$. Next, assume that $\mu$ is an initial subpath of $\beta$. We write $\beta = (\mu, \beta_1)$, and $\beta_1 = \langle e_1, e_2, \cdots, e_k \rangle$. Note that $r(e) \notin X$ because $X$ is hereditary and $r(\beta) = w \in Y$. Thus $S_\mu P_v X = S_e \sum_{f \in E^1, r(f) = w \notin X} S_f S_f^* = S_e^*$. This implies $S_\beta (P_v - P_{v,X}) = S_e^* P_v - S_e^* = 0$, and hence

\[ S_\beta S_\mu (P_v - P_{v,X}) = S_\beta S_\mu (P_v - P_{v,X}) = S_e^* S_e^* (P_v - P_{v,X}) = 0. \]
(iii) The same argument as (ii) gives $(S_\alpha (P_w - P_{w,Y}) S^*_\beta) (S_\mu P_{v,X} S^*_\nu) = 0$ if $\alpha, \beta, \mu, \nu$ are paths such that $r(\alpha) = r(\beta) = w \in Y^\text{fin}_\infty$ and $r(\mu) = r(\nu) = v \in X$.

(iv) We show $(S_\alpha (P_w - P_{v,X}) S^*_\beta) (S_\mu P_{v,X} S^*_\nu) = 0$ for paths $\alpha, \beta, \mu, \nu$ such that $r(\alpha) = r(\beta) = w \in Y^\text{fin}_\infty$ and $r(\mu) = r(\nu) = v \in X$. It is enough to show this when there would be a path from $v$ to $w$ or vice versa. If $v \leq w$, then write $\beta = (\mu, \beta_1)$ with a subpath $\beta_1$ (if $w \geq v$, then write $\mu = (\beta, \mu_1)$ with a subpath $\mu_1$). The same argument as (ii) yields the result.

**Step 2.** To show that the ideal $J$ generated by $J_{X,Y} \cap J_{Y,Y}$ equals $C^*(E)$, it suffices to observe that $J$ contains all projections $P_v, v \in E^0$. Clearly, we only need to examine vertices $v \in E^0 \setminus (X \cup Y)$. To this end, we use the induction on the number of finite paths in $F$ from $v \in E^0 \setminus (X \cup Y)$ to $X \cup Y$.

(i) Suppose that there is only one path $(e_1, \ldots, e_k)$ in $F$ such that $v = s(e_1)$, $r(e_i) \notin X \cup Y$ for $i < k$, and $r(e_k) \in X \cup Y$. We show by reverse induction on $i = 1, \ldots, k$ that $P_{s(e_i)}$ belongs to the ideal $J$ generated by $J_{X,Y} \cap J_{Y,Y}$. (To simplify notation, we agree that $e_{k+1} = 1$.) Indeed, suppose that $P_{s(e_{i+1})} \in J$. If $s(e_i)$ does not belong to $X^\text{fin}_\infty \cup Y^\text{fin}_\infty$, then $e_i$ is the only edge in $E$ emitted by $s(e_i)$, and hence $r(e_i) = s(e_{i+1})$. Now $S_{s(e_i)} = S_{s(e_i)} P_{e_i} = S_{s(e_i)} P_{s(e_{i+1})} \in J$, and it follows that $P_{s(e_i)} = S_{s(e_i)} S^*_{s(e_i)} \in J$, and we are done. If $s(e_i) \in X^\text{fin}_\infty$, then $e_i$ is the only edge emitted by $s(e_i)$ whose range lies outside $X$. For otherwise there existed more than one path in $F$ from $v$ to $X \cup Y$. Thus $S_{s(e_i)} S^*_{s(e_i)} \in J$ by the previous observation. Hence we have $P_{s(e_i)} = (P_{s(e_i)} - P_{s(e_i),X}) + S_{s(e_i)} S^*_{s(e_i)} \in J$, and the claim follows.

(ii) Now suppose that $P_w \in J$ for all vertices $w \in E^0 \setminus (X \cup Y)$ for which there are at most $n$ paths in $F$ from $w$ to $X \cup Y$. Let $v$ be a vertex in $E^0 \setminus (X \cup Y)$ with $n + 1$ paths in $F$ from $v$ to $X \cup Y$. Let $(e_1, \ldots, e_k)$ be one of these paths. Since there are at least two paths in $F$ from $v$ to $X \cup Y$, it follows that there exists an index $i$ such that $s(e_i)$ emits at least two edges in $F$. Let $m$ be the smallest such an index. Then for every edge $f$ in $F^1$ with $s(f) = s(e_m)$ we have $P_{r(f)} \in J$, by the inductive hypothesis. Indeed, for each such $f$ the number of paths in $F$ from $r(f)$ to $X \cup Y$ is not greater than $n$.

If $s(e_m)$ does not belong to $X^\text{fin}_\infty \cup Y^\text{fin}_\infty$, then every edge emitted by $s(e_m)$ is in $F$, and there are only finitely many such edges by Lemma 3.1. Thus $P_{s(e_m)} = \sum_{s(f) = s(e_m)} S_f S^*_f$ belongs to $J$. If $s(e_m) \in X^\text{fin}_\infty$, then every edge emitted by $s(e_m)$ with range outside $X$ is in $F$. Thus in this case we again see that $P_{s(e_m)} = (P_{s(e_m)} - P_{s(e_m),X}) + \sum_{s(f) = s(e_m)} S_f S^*_f$ belongs to $J$. In either case, by the choice of $m$, there exists a unique path $(e_1, \ldots, e_{m-1})$ in $F$ from $v = s(e_1)$ to $s(e_m)$. Thus, a reasoning as in part (i) above shows that $P_{s(e_j)} \in J$ for all $j = 1, \ldots, m$. Consequently $P_v \in J$.
Corollary 4.2. Let $E$ be a directed graph with finitely many vertices. Then there exist finitely many subgraphs $F_1, \ldots, F_n$ of $E$ such that $C^*(E) \cong C^*(F_1) \oplus \cdots \oplus C^*(F_n)$ and each $C^*(F_k)$ is indecomposable.

Proof. The decomposition of $C^*(E)$ depends on the existence of two non-empty, disjoint, hereditary and saturated subsets $X$ and $Y$ of $E^0$ satisfying conditions in Theorem 4.1. If no such subsets $X$ and $Y$ exist, then the algebra $C^*(E)$ is itself indecomposable. Otherwise, $C^*(E) \cong C^*(F_1) \oplus C^*(F_2)$ for some graphs $F_1$ and $F_2$. If both $C^*(F_1)$ and $C^*(F_2)$ are indecomposable, we are done. Otherwise, find a direct sum decomposition of the algebras $C^*(F_k)$ whenever they are decomposable. Continuing this process yields the result after a finite number of steps. \qed

5. $C^*$-algebras of finite graphs

This section is devoted to giving a more feasible criterion for $C^*$-algebras of finite directed graphs, even though it can be covered by Theorem 4.1.

Theorem 5.1. Let $E$ be a finite directed graph. Then $C^*(E)$ is decomposable if and only if there exist two non-empty, disjoint, hereditary and saturated subsets $X$ and $Y$ of $E^0$ such that for every vertex $v \in E^0 \setminus (X \cup Y)$ (i) there exist paths from $v$ to both $X$ and $Y$, and (ii) there is no loop passing through $v$. If this is the case, $C^*(E) = I_X \oplus I_Y \cong C^*(E \setminus Y) \oplus C^*(E \setminus X)$.

Proof. (\(\implies\)) By taking $B = \emptyset$ in Theorem 2.3, we see that $C^*(E) = I_X \oplus I_Y$ with two non-empty, disjoint, hereditary and saturated subsets $X$ and $Y$ of $E^0$.

(i) Consider a vertex $v \in E^0 \setminus (X \cup Y)$ and suppose for a moment that there is no path in $E$ from $v$ to a vertex in $Y$. Since $I_Y = \sum_{\alpha, \beta \in E^*, w \in Y} \{P \alpha \beta w: r(\alpha) = r(\beta) = w\}$, it is clear that $P \alpha I_Y = \{0\}$. Since $C^*(E) = I_X \oplus I_Y$, we must have $P \alpha \in I_X$ and consequently $v \in X$, a contradiction. Hence there is a path from $v$ to $X$. Similarly, there must be a path from $v$ to $X$.

(ii) Suppose $v \in E^0 \setminus (X \cup Y)$ and there is a loop $\mu = (\mu_1, \ldots, \mu_k)$ in $E$ through $v$. Since the ideal $I_{X\cup Y}$, generated by $I_X$ and $I_Y$, equals $C^*(E)$, we have $I_{X\cup Y} = I_{E^0}$. But $X \cup Y$ is a hereditary set and $I_{X\cup Y}$ equals $I_{\Sigma(X \cup Y)}$, where $\Sigma(X \cup Y)$ denotes the saturation of $X \cup Y$. Since a gauge-invariant ideal determines the corresponding hereditary and saturated subset of $E^0$, we must have $\Sigma(X \cup Y) = E^0$ and, in particular, $v$ belongs to the saturation of $X \cup Y$. Now $\Sigma(X \cup Y) = E^0$ is the union of the sequence $\Sigma_n(X \cup Y)$, defined inductively by $\Sigma_0(X \cup Y) = X \cup Y$ and $\Sigma_{n+1} = \Sigma_n(X \cup Y) \cup \{w \in E^0: 0 < |s^{-1}(w)| \text{ and } s(e) = w \text{ imply } r(e) \in \Sigma_n(X \cup Y)\}$ (cf. [4, Remark 3.1]). Due to $v \notin \Sigma_0(X \cup Y)$ we can choose the smallest integer $n > 0$ such that $v \in \Sigma_n(X \cup Y)$. Since $\Sigma_n(X \cup Y)$ is hereditary, $v = r(\mu_k) \in \Sigma_{n-1}(X \cup Y)$. But it is easy to see that $\Sigma_{n-1}(X \cup Y)$ is hereditary. Therefore $v = r(\mu_k) \in \Sigma_{n-1}(X \cup Y)$, a contradiction to our choice of $n$.

(\(\impliedby\)) Let $X$ and $Y$ be subsets of $E^0$ satisfying the above conditions. The fact that $I_X \cap I_Y = \{0\}$ follows by an argument similar to the proof of Theorem 4.1. To show that the ideal generated by $I_X \cup I_Y$ equals $C^*(E)$, i.e. this ideal contains all projections $P_v$, $v \in E^0$, we show that the saturation $\Sigma(X \cup Y)$ equals $E^0$. Suppose, by way of contradiction, that $\Sigma(X \cup Y) \neq E^0$. We define a partial order $\preceq$ for vertices in $E^0 \setminus \Sigma(X \cup Y)$ so that $v \preceq w$ if and only if there is a path from $v$ to $w$. Note that since there are no loops through vertices in $E^0 \setminus \Sigma(X \cup Y)$ it follows that $v \preceq w$ implies $w \not\preceq v$. Since the set $E^0 \setminus \Sigma(X \cup Y)$ is finite, there exists a maximal element $v_0$ with respect to $\preceq$. This vertex $v_0$ is not a sink by assumption. Also,
$v_0$ does not emit any edges into $E_0 \setminus \Sigma(X \cup Y)$, and hence it emits all its finitely many edges into $\Sigma(X \cup Y)$. Consequently $v_0 \in \Sigma(X \cup Y)$, a contradiction. It is then clear that \( C^*(E) = I_X \oplus I_Y \cong C^*(E \setminus Y) \oplus C^*(E \setminus X) \). \qed

6. Examples

Example 6.1. Let $E_1, E_2$ be the following finite directed graphs. The algebra $C^*(E_1)$ is indecomposable by Theorem 5.1 because $X = \{v, w\}$ and $Y = \{u, w\}$ are the only hereditary and saturated subsets of $E_1^0$, and they are not disjoint.

In the graph $E_2$, $X = \{v\}$ and $Y = \{u\}$ are disjoint hereditary and saturated subsets of $E_2^0$. Note that $C^*(E_2)$ is decomposable by Theorem 5.1 because $w \in E_2^0 \setminus (X \cup Y)$ has paths to both $X$ and $Y$, and there is no loop based at $w$. The decomposition depends on the graph $E_2 \setminus Y = E_2 \setminus X$, and hence $C^*(E_2) \cong (M_2(\mathbb{C}) \otimes C(\mathbb{T})) \oplus (M_2(\mathbb{C}) \otimes C(\mathbb{T}))$.

\[ \begin{array}{c}
E_1 \\
\bullet v \quad w \quad u \\
\end{array} \quad \begin{array}{c}
E_2 \\
\bullet v \quad u \\
\end{array} \quad \begin{array}{c}
E_2 \setminus Y \\
\quad \circ \\
\end{array} \]

Example 6.2. Here $\infty$ denotes that there are infinitely many edges from $w$ to $u$. $X = \{v\}$ and $Y = \{u\}$ are disjoint, hereditary and saturated subsets of $E^0$. From the corresponding subgraph $F$ of $E$ as defined in (2) and (3), we see that $C^*(E)$ is decomposable by Theorem 4.1.

\[ \begin{array}{c}
E \\
\bullet v \quad w \quad \infty \quad u \\
\end{array} \quad \begin{array}{c}
F \\
\bullet v \quad w \quad u \\
\end{array} \]

Hence $C^*(E \setminus X) \cong \mathcal{K} \otimes C(\mathbb{T})$ and $C^*(E \setminus Y) \cong M_2(\mathbb{C}) \otimes C(\mathbb{T})$, and both are indecomposable.

\[ \begin{array}{c}
E \setminus X \\
\bullet w \quad \infty \quad u \\
\end{array} \quad \begin{array}{c}
E \setminus Y \\
\bullet v \quad \infty \quad w \\
\end{array} \]

That is, $C^*(E) \cong (\mathcal{K} \otimes C(\mathbb{T})) \oplus (M_2(\mathbb{C}) \otimes C(\mathbb{T}))$.

Example 6.3. The sets $X = \{v\}$ and $Y = \{u\}$ are disjoint, hereditary and saturated subsets of $E^0$. Here the corresponding subgraph $F$ of $E$ is the same as $E$. Since there are infinitely many path in $F$ from $w \in E^0 \setminus (X \cup Y)$ entering $X \cup Y$, $C^*(E)$ is indecomposable by Theorem 4.1.

\[ \begin{array}{c}
E = F \\
\bullet v \quad \infty \quad w \quad \infty \quad u \\
\end{array} \]
Example 6.4. The sets $X = \{x\}$ and $Y = \{y\}$ are disjoint, hereditary and saturated subsets of $E_0$. Since there exist no paths in $F$ from $v \in E_0 \setminus (X \cup Y)$ (and from $w \in E_0 \setminus (X \cup Y)$) entering $X \cup Y$, $C^*(E)$ is indecomposable by Theorem 4.1.

!![Diagram]

**Acknowledgements**

I would like to thank the referee for valuable suggestions and comments which contributed to improvement of the exposition.

**References**


Applied Mathematics, Korea Maritime University, Busan 606–791, South Korea
E-mail address: hongjh@hanara.kmaritime.ac.kr