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WHICH SINGULAR K3 SURFACES COVER AN ENRIQUES SURFACE

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ABSTRACT. We determine the necessary and sufficient conditions on the entries of the intersection matrix of the transcendental lattice of a singular K3 surface for the surface to doubly cover an Enriques surface.

1. INTRODUCTION

When X is a singular K3 surface over the field $\mathbb{C}$, the transcendental lattice T_X of X is denoted by its intersection matrix

$$\begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix} \tag{1}$$

with respect to some basis $\{u, v\}$, where $a, b > 0$ and $4ab - c^2 > 0$. For the definitions and basic facts about K3 surfaces we refer to [1].

Following the works of Horikawa on the period map of Enriques surfaces and work of Nikulin on the embeddings of even lattices, Keum gave an integral lattice theoretical criterion for the existence of a fixed point free involution on a K3 surface, [5, 6, 11, 7]. This criterion is then applied in [7] to show that every algebraic Kummer surface is the double cover of some Enriques surface, in which case the a, b, c of T_X are even and $17 \leq \rho(X) \leq 20$; see also [10, 8].

If U denotes the hyperbolic lattice of rank 2 and if E_8 denotes the even unimodular negative definite lattice of rank 8, then a sublattice Λ^- of the K3-lattice Λ is defined as

$$\Lambda^- = U \oplus U(2) \oplus E_8(2).$$

A K3 surface with $12 \leq \rho(X) \leq 20$ covers an Enriques surface if and only if there is a primitive embedding $\phi : T_X \to \Lambda^-$ such that the orthogonal complement of the image in Λ^- contains no self-intersection -2 vector, and when $\rho(X) = 10$ or 11, one also needs to have $length\,(T_X) \leq \rho(X) - 2$, [7, Theorem 1].

We implement this criterion to find explicit necessary and sufficient conditions on the entries of T_X so that X covers an Enriques surface when $\rho(X) = 20$. In practice, if X actually covers an Enriques surface it is sometimes, but by no means always, easy to exhibit an embedding $\phi : T_X \to \Lambda^-$ such that i) it is possible to

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demonstrate that ϕ is primitive and that ii) it is possible to show that the existence of a self-intersection -2 vector in $\phi(T_X)^{\perp}$ leads to a contradiction. However, in case X does not cover an Enriques surface, then it is hard work to demonstrate that for every primitive embedding the orthogonal complement of the image has a self-intersection -2 vector. We resolve this difficulty in

Theorem 1. *If X is a singular K3 surface with transcendental lattice given as in* (1)*, then X covers an Enriques surface if and only if one of the following conditions holds:*
I *a, b and c are even (Keum's result, see* [7]*).*
II *c is odd and ab is even.*
III-1 *c is even. a or b is odd. The form $ax^2+cxy+by^2$ does not represent* 1.
III-2 *c is even. a or b is odd. The form $ax^2+cxy+by^2$ represents* 1*, and* $4ab-c^2 \neq 4, 8, 16$.

Equivalently, X fails to doubly cover an Enriques surface if and only if one of the following conditions holds
III-3 *c is even. a or b is odd. The form $ax^2+cxy+by^2$ represents* 1*, and* $4ab-c^2 = 4, 8, 16$.
IV *abc is odd.*

2. Parities in transcendental lattice

Before we proceed with the proof we show that the parity properties given in Theorem 1 are well defined.

Let $\theta = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in SL_2(\mathbb{Z})$. Then every matrix of the form ${}^t\theta T_X \theta$ represents the transcendental lattice of X with respect to some basis. Setting

$$\begin{aligned} {}^t\theta\, T_X\, \theta &= \begin{pmatrix} 2(ax^2+cxz+bz^2) & 2axy+c(xw+yz)+2bwz \\ 2axy+c(xw+yz)+2bwz & 2(ay^2+cyw+bw^2) \end{pmatrix} \\ &= \begin{pmatrix} 2a' & c' \\ c' & 2b' \end{pmatrix}, \end{aligned}$$

we see by inspection that
I If a, b, and c are even, then a', b' and c' are even.
II If c is odd and ab is even, then c' is odd and $a'b'$ is even.
III If c is even with a or b odd, then c' is even with a' or b' odd.
IV If abc is odd, then $a'b'c'$ is odd.

3. Two lemmas on integral lattices

For the fundamental concepts related to integral lattices we refer to [1, 3, 4, 9]. We leave the proofs to the reader.[1]

Let $M = (\mathbb{Z}^n, A)$ be an integral lattice where $A = {}^tA$ is the intersection matrix, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a primitive element, i.e., $\gcd(\alpha_1, \ldots, \alpha_n) = 1$. We denote by $\langle \alpha, \beta \rangle_M$ the inner product of the vectors α and β in M. Denote the orthogonal complement of α in M by $\alpha^{\perp}$.

Lemma 2. *The index of $\alpha \oplus \alpha^{\perp}$ in M divides $\langle \alpha, \alpha \rangle_M$.* □

[1] The full proofs can be found in arXiv.math.AG/0205282 v1.

Let L_1 and L_2 be two lattices with base elements $e_1, ..., e_n$ and $f_1, ..., f_m$ respectively where $m \geq n$. Assume that we have an embedding of L_1 into L_2 given by

$$\phi(e_i) = a_{i1} f_1 + \cdots + a_{im} f_m, \quad i = 1, ..., n$$

where the a_{ij}'s are integers. For any choice of integers $1 \leq t_1, \ldots, t_n \leq m$, define

$$\begin{aligned} \Delta(t_1, ..., t_n) &= \det \left(a_{it_j}\right)_{1 \leq i,j \leq n}, \text{ and} \\ d &= gcd\{\Delta(t_1, ..., t_n) \mid 1 \leq t_1, \cdots, t_n \leq m\}. \end{aligned}$$

Lemma 3. *The embedding ϕ is primitive if and only if $d = 1$. In other words, a lattice embedding is primitive if and only if the greatest common divisor of the maximal minors of the embedding matrix with respect to any choice of bases is 1.* □

As an immediate application of this lemma we can indicate that all the mappings in [7, pp. 106-108] have embedding matrices whose maximal minors have greatest common divisor equal to 1.

4. The case when c is even with a or b odd

If a is even, then set $\theta = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. If ${}^t\theta \, T_X \, \theta = \begin{pmatrix} 2a' & c' \\ c' & 2b' \end{pmatrix}$, then a' and b' are odd, and c' is even. If b is even, then $\theta = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ changes T_X into an equivalent form where again a' and b' are odd, and c' is even. So we might assume without loss of generality that ab is odd, and c is even.

We will consider a particular embedding of T_X into $\Lambda^- = U \oplus U(2) \oplus E_8(2)$.

Let $\{u, v\}$ be a basis of T_X, $\{u_1, u_2\}$ be a basis of U and $\{v_1, v_2\}$ be a basis of $U(2)$.

Define $\phi : T_X \to \Lambda^-$ by

$$\begin{aligned} \phi(u) &= u_1 + a u_2, \\ \phi(v) &= u_1 + (c - a) u_2 + v_1 + \frac{1}{2}(a + b - c) v_2. \end{aligned}$$

It can be shown by direct computation that this is an embedding and by Lemma 3 that this embedding is primitive.

4.1. The form $ax^2 + cxy + by^2$ does not represent 1.

Let $f = x u_1 + x' u_2 + y v_1 + y' v_2 + e \in \Lambda^-$, where $e \in E_8(2)$ with $e \cdot e = -4k,\ k \geq 0$ (we will use $\cdot$ to denote the inner product on Λ^-).

Impose the condition that f lies in the orthogonal complement of $\phi(T_X)$ in Λ^- and that $f \cdot f = -2$.

Solving the equations $f \cdot \phi(u) = 0$, $f \cdot \phi(v) = 0$ for x', y' and substituting into the equation $f \cdot f = -2$ gives

$$(2) \qquad 1 - (ax^2 + (c - 2a)xy + (a + b - c)y^2) = 2k \geq 0.$$

The binary quadratic form $ax^2 + (c - 2a)xy + (a + b - c)y^2$ is equivalent to the form $ax^2 + cxy + by^2$. Since $a > 0$ and $c^2 - 4ab < 0$, this is a positive definite form. Equation (2) holds if and only if this form represents 1, and then $k = 0$ (see [12]).

If we assume that the form $ax^2 + cxy + by^2$ does not represent 1, then equation (2) cannot be solved. So there is no self-intersection -2 vector in the orthogonal complement of $\phi(T_X)$. This proves **III-1**.

4.2. **The form $ax^2 + cxy + by^2$ does represent 1.**
In this case the binary quadratic form $ax^2 + cxy + by^2$ is equivalent to the form $x^2 + (ab - c^2/4)y^2$; see [12, p. 174]. Then a basis $\{u, v\}$ of the transcendental lattice exists such that with respect to that basis the matrix

$$T_X = \begin{pmatrix} 2(1) & 0 \\ 0 & 2(\frac{\Delta}{4}) \end{pmatrix}$$

where $\Delta = 4ab - c^2$.

Let ϕ be a primitive embedding of T_X into Λ^-, and set $\phi(u) = \alpha$ with

$$\alpha = a_1u_1 + a_2u_2 + a_3v_1 + a_4v_2 + \omega_1$$

where $\omega_1 \in E_8(2)$ with $\omega \cdot \omega = -4k \leq 0$.

$\alpha \cdot \alpha = 2$ forces a_1 and a_2 to be odd.

If $\beta = b_1u_1 + b_2u_2 + b_3v_1 + b_4v_2 + \omega_2$ is in the orthogonal complement $\alpha^\perp$ of α in Λ^-, then $\beta \cdot \alpha = 0$ forces b_1 and b_2 to be of the same parity. This in turn implies the following.

Lemma 4. *If $\beta, \gamma \in \alpha^\perp$, then $\beta \cdot \gamma \equiv 0 \mod 2$.* □

Let $\beta_1, \ldots, \beta_{11}$ be basis elements for $\alpha^\perp$, and $B' = (2b_{ij})$, $2b_{ij} = \beta_i \cdot \beta_j$ the intersection matrix for this basis. Set $B = (b_{ij})$.

Let C be the 12×12 matrix whose rows are the coordinates of $\alpha, \beta_1, \ldots, \beta_{11}$ with respect to the standard basis of Λ^-. Finally, let A denote the intersection matrix of Λ^- with respect to its standard basis. We have

$$CA\,{}^t C = \begin{pmatrix} 2 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{pmatrix}. \tag{3}$$

Since $\alpha, \beta_1, \ldots, \beta_{11}$ is not a basis of Λ^-, $|\det C| > 1$. By Lemma 2, $|\det C|$ divides 2, hence is equal to 2. By interchanging β_1 by β_2 if necessary, we can assume without loss of generality that $\det C = 2$.

It then follows from equation (3) that $\det B = 1$.

Define a new lattice $L = \big(\mathbb{Z}^{11}, B(-1)\big)$. L has signature $(\tau^+, \tau^-) = (10, 1)$. Since $\tau^+ - \tau^- \not\equiv 0 \mod 8$, L is odd. Then L is an indeterminate, odd, unimodular lattice, and as such is isomorphic to $\langle -1 \rangle^1 \oplus \langle 1 \rangle^{10}$.

There is an isomorphism $F : \alpha^\perp \to L$ that sends β_i to $e_i = (0, \ldots, 1, \ldots, 0)$, where 1 is in the i-th place. This isomorphism respects inner products in the sense that

$$-2[F(\lambda_1) \cdot F(\lambda_2)] = \lambda_1 \cdot \lambda_2, \text{ for all } \lambda_1, \lambda_2 \in \alpha^\perp.$$

Let $e'_1, \ldots, e'_{11}$ be a basis of L diagonalizing its intersection matrix. Then the intersection matrix of $\alpha \oplus \alpha^\perp$ with respect to the basis $\alpha, F^{-1}(e'_1), \ldots, F^{-1}(e'_{11})$ is

$$\begin{pmatrix} 2 & 0 & 0 & \ldots & 0 \\ 0 & 2 & 0 & \ldots & 0 \\ 0 & 0 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -2 \end{pmatrix}.$$

We are looking for the existence of a primitive embedding

$$\phi : T_X \longrightarrow \alpha \oplus \alpha^{\perp} \subset \Lambda^{-}$$

such that with respect to this new basis of $\alpha \oplus \alpha^{\perp}$,

$$\begin{aligned}\phi(u) &= (1, 0, \ldots, 0),\\ \phi(v) &= (0, x_0, \ldots, x_{10})\end{aligned}$$

such that

$$\phi(v) \cdot \phi(v) = 2x_0^2 - 2x_1^2 - \cdots - 2x_{10}^2 = 2\left(\frac{\Delta}{4}\right).$$

Using Lemma 3, the problem reduces to a problem in the lattice L, that of investigating the existence of integers $x_0, \ldots, x_{10}$ such that if $x = (x_0, \ldots, x_{10}) \in L$, then the following conditions are satisfied:

$$\begin{aligned} gcd(x_0, \ldots, x_{10}) &= 1,\\ x \cdot x &= -x_0^2 + x_1^2 + \cdots + x_{10}^2\\ &= -\left(\frac{\Delta}{4}\right), \text{ and}\\ y \cdot x = 0 &\implies y \cdot y \neq 1, \text{ for every } y \in L. \end{aligned} \tag{4}$$

The existence of such integers is equivalent to X covering an Enriques surface.

The set of negative self-intersection elements of L spans an open convex cone in $\mathbb{Z}^{11} \otimes \mathbb{R}$, and we refer to [2] for details. We will utilize the techniques of Vinberg from [15] to investigate the existence of integers as above.

All automorphisms of L are generated by reflections, and a fundamental region for negative self-intersecting vectors in L is bounded by reflecting hyperplanes. Each reflecting hyperplane consists of vectors orthogonal to some vector $e \in \mathbb{Z}^{11}$ with $e \cdot e = 1$. A negative self-intersecting vector $v \in L$ has no vector of self-intersection 1 in its orthogonal complement if and only if it is not on one of these reflecting hyperplanes. Such a vector can then be mapped by an automorphism to the interior of the fundamental region. So it suffices to consider only vectors on the interior of the fundamental region. This finally amounts to saying that the conditions in the set of equations (4) hold if and only if (see [15])

$$\begin{aligned} gcd(x_0, \ldots, x_{10}) = 1,\\ -x_0^2 + x_1^2 + \cdots + x_{10}^2 = -\left(\frac{\Delta}{4}\right),\\ x_1 \geq \cdots \geq x_{10} > 0,\\ x_0 \geq x_1 + x_2 + x_3, \text{ and}\\ 3x_0 > x_1 + \cdots + x_{10}. \end{aligned}$$

Let P denote the set of all $x \in L$ satisfying the above conditions.

The rest of this case is elementary, and we summarize the results in two technical lemmas.

Lemma 5. *There is no $x \in P$ with $x \cdot x = -1, -2, -4$.*

Proof. Let $P(m) = \{x \in P \mid x = (m, x_1, \ldots, x_{10})\ \}$. Then it is easy to show that $\max_{x \in P(3m)} (x \cdot x) = 5 - 4m$, $m > 2$,

$\max_{x\in P(6)}(x\cdot x) = -5,$
$\max_{x\in P(3m+1)}(x\cdot x) = 1-4m,\ m\geq 1,$
$\max_{x\in P(3m+2)}(x\cdot x) = 9-8m,\ m\geq 3,$
$\max_{x\in P(8)}(x\cdot x) = -12,$
$\max_{x\in P(5)}(x\cdot x) = -7.$

These maximum values are achieved by the vectors
$[3m, m, \dots, m, m-2, 1]$,
$[6, 2, \dots, 2, 1, 1, 1]$,
$[3m+1, m+1, m, \dots, m, 1]$,
$[3m+2, m+2, m, \dots, m, 3]$,
$[8, 4, 2, \dots, 2]$,
$[5, 3, 1, \dots, 1]$, respectively.

It is now clear that -1 and -2 are never achieved. Also, if $x\cdot x = -4$, then $x \in P(4)$. But none of the vectors in $P(4)$ achieves -4. □

Lemma 6. *For every positive integer N, other than $1, 2$ and 4, there is an $x \in P$ such that $x \cdot x = -N$.*

Proof. In the following table each of the given vectors is in P, and moreover $X_m(k)\cdot X_m(k) = -(m+24k)$, $Y_m \cdot Y_m = -m$, $Z(n)\cdot Z(n) = -(4n-1)$, and $W(n)\cdot W(n) = -(4n-3)$. This then proves the lemma.
$X_0(k) = [9k+4, 3k+2, 3k+1, \dots, 3k+1, 3k-1, 2]$, $k \geq 1$.
$X_2(k) = [9k+4, 3k+2, 3k+1, \dots, 3k+1, 3k, 3k, 2]$, $k \geq 1$.
$X_4(k) = [12k+4, 4k+2, 4k+1, \dots, 4k+1, 4k, 1]$, $k \geq 1$.
$X_6(k) = [9k+5, 3k+2, 3k+2, 3k+1, \dots, 3k+1, 2]$, $k \geq 1$.
$X_8(k) = [9k+7, 3k+3, 3k+2, \dots, 3k+2, 3k, 2]$, $k \geq 1$.
$X_{10}(k) = [12k+7, 4k+3, 4k+2, \dots, 4k+2, 4k+1, 1]$, $k \geq 0$.
$X_{12}(k) = [12k+9, 4k+3, \dots, 4k+3, 4k+2, 4k+1, 1]$, $k \geq 0$.
$X_{14}(k) = [9k+8, 3k+3, 3k+3, 3k+2, \dots, 3k+2, 2]$, $k \geq 0$.
$X_{16}(k) = [12k+10, 4k+4, 4k+3, \dots, 4k+3, 4k+2, 1]$, $k \geq 0$.
$X_{18}(k) = [12k+12, 4k+4, \dots, 4k+4, 4k+3, 4k+2, 1]$, $k \geq 0$.
$X_{20}(k) = [6k+12, 2k+6, 2k+3, \dots, 2k+3, 4]$, $k \geq 1$.
$X_{22}(k) = [9k+11, 3k+4, 3k+4, 3k+3, \dots, 3k+3, 2]$, $k \geq 0$.
$Y_6 = [4, 1, \dots, 1]$.
$Y_8 = [6, 2, \dots, 2, 1, 1, 1, 1]$.
$Y_{20} = [6, 2, 2, 1, \dots, 1]$.
$Z(n) = [3n+1, n+1, n, \dots, n, 1]$, $n \geq 1$.
$W(n) = [3n, n, \dots, n, n-1, n-1, 1]$, $n \geq 2$. □

Recalling that $\Delta = 4ab - c^2$, these two lemmas complete the proofs of **III-2** and **III-3**.

5. The other cases

Let $\{u, v\}$ be a basis of the transcendental lattice giving the matrix representation as in (1), and as before let $\{u_1, u_2\}$ be the basis of U, and $\{v_1, v_2\}$ the basis of $U(2)$.

5.1. abc is odd.

Consider the mapping $\phi : T_X \to \Lambda^-$ defined generically as

$$\begin{aligned}\phi(u) &= a_1u_1 + a_2u_2 + a_3v_1 + a_4v_2 + \omega_1,\\ \phi(v) &= b_1u_1 + b_2u_2 + b_3v_1 + b_4v_2 + \omega_2\end{aligned}$$

where the a_i's and b_i's are integers, $\omega_i \in E_8(2)$. If $\phi(u)\cdot\phi(u) = 2a$ and $\phi(v)\cdot\phi(v) = 2b$, then a_1, a_2, b_1 and b_2 are odd. But this forces $\phi(u)\cdot\phi(v)$ to be even. Hence T_X has no embedding into Λ^-.

This proves **IV**.

5.2. c is odd and ab is even.

Consider the mapping $\phi : T_X \to \Lambda^-$ defined as

$$\begin{aligned}\phi(u) &= au_1 + u_2 + \frac{1}{2}(c - ab - 1)v_1,\\ \phi(v) &= u_1 + bu_2 + v_2.\end{aligned}$$

This is an embedding, and by Lemma 3 it is primitive. Let

$$f = c_1u_1 + c_2u_2 + c_3v_1 + c_4v_2 + \omega \in \Lambda^-$$

where $\omega \in E_8(2)$.

$f\cdot\phi(u) = 0$ and $f\cdot\phi(v) = 0$ forces c_1c_2 to be even. Then $f\cdot f \equiv 0 \mod 4$ and hence cannot be -2.

This proves **II** and completes the proof of Theorem 1.

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