PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 1, Pages 1–10
S 0002-9939(04)07716-0
Article electronically published on August 24, 2004

JOINS OF PROJECTIVE VARIETIES AND MULTISECANT SPACES

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(Communicated by Michael Stillman)

Abstract. Let $X_1, \dots, X_s \subset \mathbf{P}^N$, $s \geq 1$, be integral varieties. For any integers $k_i > 0$, $1 \leq i \leq s$, and $t \geq 0$ set $\vec{k} := (k_1, \dots, k_s)$ and $\vec{X} := (X_1, \dots, X_s)$. Let $\mathrm{Sec}(\vec{X}; t, \vec{k})$ be the set of all linear t-spaces contained in a linear $(k_1 + \cdots + k_s - 1)$-space spanned by k_1 points of X_1, k_2 points of X_2, ..., k_s points of X_s. Here we study some cases where $\mathrm{Sec}(\vec{X}; t, \vec{k})$ *has the expected dimension*. The case $s = 1$ was recently considered by Chiantini and Coppens and we follow their ideas. The two main results of the paper consider cases where each X_i is a surface, more particularly:

$$s = 3, k_1 = k_2 = k_3 = 1 \text{ and } t = 1$$

or

$$s = 2, k_1 = 2, k_2 = 1 \text{ and } t = 1.$$

1. Introduction

L. Chiantini and M. Coppens revived a piece of classical projective geometry (see [6] and references therein): the study of the set of all linear spaces contained in the secant varieties of an integral variety $X \subset \mathbf{P}^N$. For further papers on this topic, see [5], [8] and [9]. Let $G(t+1, N+1)$ be the Grassmannian of all t-dimensional linear subspaces of $\mathbf{P}^N$. The order k secant variety of X is the join of k copies of X. In this paper we fix s varieties $X_i \subset \mathbf{P}^N$, $1 \leq i \leq s$, and consider the closure in $G(t+1, N+1)$ of the set of all t-spaces contained in a $(k_1 + \cdots + k_s - 1)$-space spanned by k_1 points of X_1, k_2 points of X_2, ...,k_s points of X_s. The case $s = 1$ is the case considered in [6] and we will often use the ideas contained in [6].

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$. For any two integral varieties X, Y of $\mathbf{P}^N$, let $[X; Y]$ be the join of X and Y; thus if $X = Y = \{P\}$ with P a point, then $[X; Y] = \{P\}$, while in all other cases $[X; Y]$ is the closure in $\mathbf{P}^N$ of the union of all lines $\langle \{A, B\} \rangle$ spanned by some $A \in X$ and some $B \in Y$ with $A \neq B$.

Fix integers $N \geq 3$, $s > 0$ and $k_i \geq 0$, $1 \leq i \leq s$. Set $\vec{k} := (k_1, \dots, k_s)$ and $|\vec{k}| := k_1 + \cdots + k_s - 1$. Let t be an integer such that $0 \leq t \leq |\vec{k}|$. Fix s irreducible varieties $X_i \subset \mathbf{P}^N$, $1 \leq i \leq s$. Usually, we will be interested in the case $X_i \neq X_j$ for $i \neq j$, since the general case may be reduced to this case by decreasing s, but with the same

Received by the editors August 16, 2002.
2000 *Mathematics Subject Classification.* Primary 14N05, 14M15.
Key words and phrases. Joins, multisecant spaces, secant variety, Grassmannian.
The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

value of $|\vec{k}|$. The $(t,\vec{k})$-secant variety $\mathrm{Sec}(\vec{X};t,\vec{k})$ of $\vec{X}$ is the closure in $G(t+1,N+1)$ of all t-spaces contained in a $|\vec{k}|$-dimensional linear subspace of $\mathbf{P}^N$ spanned by k_1 points of X_1, k_2 points of X_2,...,k_s points of X_s. Set $\mathrm{Sec}(\vec{X};\vec{k}) := \mathrm{Sec}(\vec{X};0,\vec{k}) \subseteq \mathbf{P}^N$. The $(0,\vec{k})$-secant variety $\mathrm{Sec}(\vec{X};\vec{k})$ of $\vec{X}$ will be called the $\vec{k}$-secant variety of $\vec{X}$. Set $n_i := \dim(X_i)$. We have $\dim(\mathrm{Sec}(\vec{X};\vec{k})) \le \min\{N, \sum_{i=1}^s k_i(n_i+1)-1\}$ and $\dim(\mathrm{Sec}(\vec{X};t,\vec{k})) \le \min\{(t+1)(N-t), \sum_{i=1}^s k_in_i + (|\vec{k}|-t)(t+1)\}$. We will say that $\vec{X}$ is $\vec{k}$-defective (resp. $(t,\vec{k})$-defective) if

$$\dim(\mathrm{Sec}(\vec{X};\vec{k})) < \min\{N, \sum_{i=1}^s k_i(n_i+1)-1\}$$

(resp. $\dim(\mathrm{Sec}(\vec{X};t,\vec{k})) < \min\{(t+1)(N-t), \sum_{i=1}^s k_in_i + (|\vec{k}|-t)(t+1)\}$). If $\vec{X}$ is $\vec{k}$-defective (resp. $(t,\vec{k})$-defective) the integer

$$\delta(\vec{X};\vec{k}) := \sum_{i=1}^s k_i(n_i+1)-1-\dim(\mathrm{Sec}(\vec{X};\vec{k}))$$

(resp. $\delta(\vec{X};t,\vec{k}) := \sum_{i=1}^s k_in_i + (|\vec{k}|-t)(t+1) - \dim(\mathrm{Sec}(\vec{X};t,\vec{k}))$) will be called the total order of $\vec{k}$-defectivity (resp. $(t,\vec{k})$-defectivity) of $\vec{X}$.

It seems very natural to start the study of $(t,\vec{k})$-defectivity from the case $\dim(X_i) = 1$ for every i. For the case in which each X_i is a non-degenerate curve, see Corollary 1 and Theorem 4 in section 3. For a more general defectivity result for non-degenerate curves, see [3]; in the quoted paper we considered the set of all flags of linear spaces contained in a $|\vec{k}|$-dimensional linear space instead of the set of all t-dimensional linear spaces. Obviously, the interested reader may do other related cases (e.g. some degenerate curves or a surface and $s-1$ non-degenerate curves). We stress that degenerate varieties may not give defective s-ples (e.g. when $k_i = 1$ for all i take as X_i, $1 \le i \le s$, linearly independent linear subspaces). For degenerate varieties, see also Remark 3. We believe that when $s \ge 2$, the mutual position of the varieties is more important than their structure. For examples of defectivity when one of the varieties is a cone, see Remark 1. For a complete analysis of a toy case, see Example 1. We raise the following question.

Question 1. Fix integers $s \ge 2$, $k_i > 0$, $1 \le i \le s$, $t > 0$, and integral varieties $X_i \subset \mathbf{P}^N$, $1 \le i \le s$. Set $\vec{X} := (X_1,\dots,X_s)$, $\vec{k} := (k_1,\dots,k_s)$, $\vec{Y} := (X_1,\dots,X_{s-1})$ and $\vec{m} := (k_1,\dots,k_{s-1})$. Assume that $\mathrm{Sec}(\vec{Y};t,\vec{m})$ has the expected dimension and that X_s is a curve. Are there reasonable conditions on X_s assuring that $\mathrm{Sec}(\vec{X};t,\vec{k})$ has the expected dimension? More generally, for any $\vec{Y}$ has $\mathrm{Sec}(\vec{X};t,\vec{k})$ the maximal possible dimension compatible with the dimension of $\mathrm{Sec}(\vec{Y};t,\vec{m})$?

It is well known and easy to show that in the case $t = 0$ a sufficient condition for an affirmative answer to Question 1 is that X_s is a non-degenerate curve (see Corollary 1 and Remark 3 for more precise results). See Theorem 4 for the case $t = 1$ and [3] when $\dim(X_i) = 1$ and each X_i is non-degenerate.

Our main results are non-existence results for the $(1,\vec{k})$-defectivity of joins of surfaces. In section 2 we will prove the following results.

Theorem 1. *Let X_1, X_2 and X_3 be integral non-degenerate surfaces of $\mathbf{P}^N$, $N \ge 5$, such that $X_i \ne X_j$ for $i \ne j$. Assume $\dim([X_1;X_2]) = 5$ and that X_3 is not*

a cone. Set $\vec{k} = (1,1,1)$. *Then* $\vec{X} := (X_1, X_2, X_3)$ *is not* $(1,\vec{k})$*-defective, i.e.* $\dim(\mathrm{Sec}(\vec{X}; 1, \vec{k})) = 8$.

Theorem 2. *Let* X_1 *and* X_2 *be integral non-degenerate surfaces of* $\mathbf{P}^N$, $N \geq 5$, *such that* $X_1 \neq X_2$. *Assume that neither* X_1 *nor* X_2 *is a cone. Set* $\vec{k} := (2,1)$. *Then* $\vec{X} := (X_1, X_2)$ *is not* $(1,\vec{k})$*-defective.*

The condition $\dim([X_1; X_2]) = 5$ in the statement of Theorem 1 is very mild (see [1]). It implies that neither X_1 nor X_2 is a cone. We do not know if the condition that no X_i is a cone is always necessary (see Remark 1), but certainly X_1, X_2 and X_3 cannot be cones with the same vertex (see Example 1). We do not have any construction (except cones) to obtain defective s-ples.

In section 3 we will give two general results on the $(t,\vec{k})$-defectivity of varieties of arbitrary dimension: an easy extension of [8] to the case of joins of different varieties (Theorem 3) and a non-defectivity result with respect to lines (Theorem 4).

2. Proofs of Theorems 1 and 2

In this section we will prove Theorems 1 and 2 and give the toy example and the remark on cones promised in the introduction.

For any subset S of $\mathbf{P}^N$, let $\langle S\rangle$ be its linear span. We start with a baby example.

Example 1. Fix integers $s \geq 2$, $n_i \geq 2$, $1 \leq i \leq s$, $k_i > 0$, $1 \leq i \leq s$, $N > \min_{1\leq i\leq s}\{n_i\}$ and t such that $0 \leq t \leq k_1 + \cdots + k_s - 1$. Fix $P \in \mathbf{P}^N$ and non-degenerate varieties $X_i \subset \mathbf{P}^N$, $1 \leq i \leq s$, such that $\dim(X_i) = n_i$. Set $\vec{X} := (X_1, \dots, X_s)$ and $\vec{k} := (k_1, \dots, k_s)$. Assume that each X_i is a cone with vertex containing P. For all hyperplanes H, M of $\mathbf{P}^N$ such that $P \notin H \cup M$ the s-ples $(X_1 \cap H, \dots, X_s \cap H)$ and $(X_1 \cap M, \dots, X_s \cap M)$ (respectively seen as s-ples in H and in M) are projectively equivalent (use the linear projection from P). Fix H and set $Y_i := X_i \cap H$ and $\vec{Y} := (Y_1, \dots, Y_s)$. Fix $S_i \subset Y_i$, $1 \leq i \leq s$, such that $\mathrm{card}(S_i) := k_i$ and $\dim(\langle S_1 \cup \cdots \cup S_s\rangle) = k_1 + \cdots + k_s - 1$. Thus $\dim(\langle S_1 \cup \cdots \cup S_s \cup \{P\}\rangle) = k_1 + \cdots + k_s$, and for every t-dimensional linear space $D \subset \langle S_1 \cup \cdots \cup S_s\rangle$ the $(t+1)$-dimensional linear space $[D; \{P\}]$ contains a $(t+1)$-dimensional family of t-dimensional linear spaces not containing P and mapped isomorphically onto D by the linear projection from P. Fix any such t-dimensional linear space D'. There is a hyperplane M of $\mathbf{P}^N$ such that $D' \subset M$ and $P \notin M$. Set

$$S_i^M := \bigcup_{Q\in S_i} \langle\{Q, P\}\rangle \cap M \subset X_i.$$

Thus $\dim(\langle S_1^M \cup \cdots \cup S_s^M\rangle) = k_1 + \cdots + k_s - 1$, $D' \subset \langle S_1^M \cup \cdots \cup S_s^M\rangle$ and hence $D' \in \mathrm{Sec}(\vec{X}; t, \vec{k})$. Thus if $\mathrm{Sec}(\vec{Y}; t, \vec{k}) = G(t+1, N)$, then

$$\mathrm{Sec}(\vec{X}; t, \vec{k}) = G(t+1, N+1),$$

while if $\mathrm{Sec}(\vec{Y}; t, \vec{k}) \neq G(t+1, N)$, then $\delta(\vec{X}; t, \vec{k}) = \delta(\vec{Y}; t, \vec{k}) + k_1 + \cdots + k_s - 1 - t$. Thus in the former case $(\vec{X}, \vec{k})$ is not $(t,\vec{k})$-defective. In the latter case if $t < k_1 + \cdots + k_s - 1$, then $\vec{X}$ is $(t,\vec{k})$-defective. The fact that in the definition of defectivity we have to take the cut-off function min implies that very natural constructions do not always (but only almost always) give degenerate s-ples.

Remark 1. Fix $P \in \mathbf{P}^N$ and also fix a locally closed and irreducible subset T of $G(t+1, N+1)$ such that $P \notin A$ for every $A \in T$. Let $T * P$ be the closure in $G(t+1, N+1)$ of the set of all $B \in G(t+1, N+1)$ contained in some $(t+1)$-dimensional linear space $\langle A \cup \{P\}\rangle$ for some $A \in T$. Fix integers $s \geq 1$, $k_i > 0$, $1 \leq i \leq s$, and integral varieties $X_i \subset \mathbf{P}^N$, $1 \leq i \leq s$. Assume that X_s is a positive-dimensional cone with vertex containing $P \in \mathbf{P}^N$, say $X_s = [D; \{P\}]$ with $\dim(D) = \dim(X_s) - 1$ and $P \notin D$. Set $\vec{X} := (X_1, \dots X_{s-1}, X_s)$ and $\vec{Y} := (X_1, \dots, X_{s-1}, D)$. Obviously, $\mathrm{Sec}(\vec{X}; \vec{k})$ is the cone with P as vertex and $\mathrm{Sec}(\vec{Y}; \vec{k})$ as a basis. Hence $\vec{X}$ is $\vec{k}$-defective if $k_s \geq 2$ and $\dim(\mathrm{Sec}(\vec{Y}; \vec{k})) \leq N-2$. Now assume $t > 0$. Let $\mathrm{Sec}(\vec{Y}; t, \vec{k})'$ be the open subset of $\mathrm{Sec}(\vec{Y}; t, \vec{k})$ formed by the t-planes not containing P. It is easy to check that $\mathrm{Sec}(\vec{X}; t, \vec{k}) = \mathrm{Sec}(\vec{Y}; t, \vec{k})' * P$. Since $\dim(G(t+1, t+2)) = t+1$, we obtain $\dim(\mathrm{Sec}(\vec{X}; t, \vec{k})) \leq \dim(\mathrm{Sec}(\vec{Y}; t, \vec{k})) + t+1$. Hence $\mathrm{Sec}(\vec{X}; t, \vec{k})$ is $(t, \vec{k})$-defective if $k_s \geq t+2$ and $\dim(\mathrm{Sec}(\vec{Y}; t, \vec{k})) + t+1 < (t+1)(N-t)$.

Remark 2. Let $X_i \subset \mathbf{P}^m$, $1 \leq i \leq 3$, $m \geq 3$, be integral surfaces such that $\langle X_1 \cup X_2 \cup X_3\rangle = \mathbf{P}^m$ and either $X_1 \neq X_2$ or X_1 is not a plane. It is easy to check that for a general $(A_1, A_2, A_3) \in X_1 \times X_2 \times X_3$ and a general $(B_1, B_2, B_3, B_4) \in X_1 \times X_2 \times X_3 \times X_4$ we have $\dim(\langle\{A_1, A_2, A_3\}\rangle) = 2$ and $\dim(\langle\{B_1, B_2, B_3, B_4\}\rangle) = 3$.

Lemma 1. *Let $X_i \subset \mathbf{P}^4$, $1 \leq i \leq 2$, be integral non-degenerate surfaces. Then for a general $(A_1, A_2) \in X_1 \times X_2$ we have $\langle\{A_1, A_2\}\rangle \cap (X_1 \cup X_2) = \{A_1, A_2\}$.*

Proof. If $X_1 = X_2$, then the lemma is [6], Cor. 1.3, for the invariants $r = 4$, $n = 2$ and $k = 1$. Assume $X_1 \neq X_2$. First assume that $\langle\{A_1, A_2\}\rangle \cap (X_1 \cup X_2)$ contains $B \in X_1 \backslash \{A_1\}$. Then for a general $A_2 \in X_2$ the restriction to X_1 of the linear projection $\mathbf{P}^4 \backslash \{A_2\} \to \mathbf{P}^3$ from A_2 is not birational. Since $X_2 \neq X_1$, this implies that X_2 is contained in the so-called Segre locus $\Sigma(X_1)$ of X_1, contradicting the inequality $\dim(\Sigma(X_1)) \leq 1$ proved in [4], Th. 1. Similarly, if $\langle\{A_1, A_2\}\rangle \cap (X_1 \cup X_2)$ contains $B \in X_2 \backslash \{A_2\}$, we see that X_1 is contained in the Segre locus of X_2, contradicting [4], Th. 1. □

Lemma 2. *Let $X_i \subset \mathbf{P}^5$, $1 \leq i \leq 3$, be integral surfaces such that $X_i \neq X_j$ and $\langle X_i \cup X_j\rangle = \mathbf{P}^5$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$. Then for a general $(A_1, A_2, A_3) \in X_1 \times X_2 \times X_3$ we have $\langle\{A_1, A_2, A_3\}\rangle \cap (X_1 \cup X_2 \cup X_3) = \{A_1, A_2, A_3\}$.*

Proof. Assume that for a general $(A_1, A_2, A_3) \in X_1 \times X_2 \times X_3$ we have $\langle\{A_1, A_2, A_3\}\rangle \cap (X_1 \cup X_2 \cup X_3) \neq \{A_1, A_2, A_3\}$. Just to fix the notation assume that $\langle\{A_1, A_2, A_3\}\rangle \cap (X_1 \cup X_2 \cup X_3)$ contains $B \in X_1 \backslash \{A_3\}$. Let $f : \mathbf{P}^5 \backslash \{A_3\} \to \mathbf{P}^4$ be the linear projection from the point A_3 and $g : \mathbf{P}^4 \backslash \{f(A_2)\} \to \mathbf{P}^3$ the linear projection from the point $f(A_2)$. Using that $X_3 \neq X_2$, $X_3 \neq X_1$, X_3 is not contained in $\langle X_i\rangle$ $(i = 1, 2)$ if X_i is degenerate and A_3 is general in X_3, we obtain $A_3 \notin X_1 \cup X_2$, $\langle f(X_1) \cup f(X_2)\rangle = \mathbf{P}^4$ and $\dim(\langle f(X_i)\rangle = \min\{4, \dim(\langle X_i\rangle)\}$ for $i = 1, 2$. Hence either $\langle f(X_1)\rangle = \mathbf{P}^4$ or $f(X_2) \nsubseteq \langle f(X_1)\rangle$. If $\langle f(X_1)\rangle \neq \mathbf{P}^4$, by the generality of $f(A_2)$ in $f(X_2)$ we obtain that $g|f(X_1)$ is injective (here we just take any $f(A_2) \notin \langle f(X_1)\rangle$), contradicting the existence of A_1 and B (even if A_1 is not assumed to be general in X_1) such that $A_1 \neq B$ and $B \in \langle\{A_1, A_2, A_3\}\rangle \cap (X_1 \cup X_2 \cup X_3)$. Hence we may assume $\langle f(X_1)\rangle = \mathbf{P}^4$. To obtain a contradiction it is sufficient to show that $g|f(X_1)$ is birational. Assume that $g|f(X_1)$ is not birational. Since $f(A_2)$ is a general point of $f(X_2)$ and $f(X_1) \neq f(X_2)$, we obtain that a general point of $f(X_2) \backslash f(X_1)$ is in the Segre locus $\Sigma(f(X_1))$ of $f(X_1)$, contradicting [4], Th. 1. □

Lemma 3. *Let $X_i \subset \mathbf{P}^5$, $1 \le i \le 2$, be integral surfaces such that $X_1 \ne X_2$, $\langle X_1 \cup X_2\rangle = \mathbf{P}^5$ and $\dim(\langle X_1\rangle) \ge 4$. Then for a general $(A_1, A_2, B) \in X_1 \times X_1 \times X_2$ we have $\langle\{A_1, A_2, B\}\rangle \cap (X_1 \cup X_2) = \{A_1, A_2, A_3\}$.*

Proof. Assume that for a general $(A_1, A_2, B) \in X_1 \times X_1 \times X_2$ we have $\langle\{A_1, A_2, B\}\rangle \cap (X_1 \cup X_2) \ne \{A_1, A_2, B\}$. Let $f : \mathbf{P}^5\backslash\{B\} \to \mathbf{P}^4$ be the linear projection from B. First assume that $\langle\{A_1, A_2, B\}\rangle \cap (X_1 \cup X_2)$ contains $D \in X_1\backslash\{A_1, A_2\}$. Since B is general in X_2, $f(X_1)$ spans $\mathbf{P}^4$. Hence a general secant line of $f(X_1)$ is not a trisecant line ([6], Cor. 1.3). Since $f(D) \in \langle f(A_1), f(A_2)\rangle$, we obtain that either $f(D) = f(A_1)$ or $f(D) = f(A_2)$. Just to fix the notation we assume $f(D) = f(A_1)$. Since A_1 is general in X_1, we obtain that $f|X_1$ is not birational. Hence a general $B \in X_2$ is in the Segre locus $\Sigma(X_1)$ of X_1, contradicting [4], Th. 1. Now assume that $\langle\{A_1, A_2, B\}\rangle \cap (X_1 \cup X_2)$ contains $C \in X_2\backslash\{B\}$. We obtain that any secant line to $f(X_1)$ intersects $f(X_2\backslash\{B\})$. Hence $f(X_2) \subseteq \Sigma(f(X_1))$, a contradiction. □

Lemma 4. *Let $C, D \subset \mathbf{P}^m$, $m \ge 3$, be integral non-degenerate curves. Assume $C \ne D$. Then a general secant line to C is not secant to D.*

Proof. Assume that this is not true and fix a general $P \in C$. By assumption for a general $Q \in C$ the line $\langle\{P, Q\}\rangle$ is secant to D. Hence the linear projection from P is not birational. Thus a general $P \in C$ is contained in the Segre locus $\Sigma(D)$ of D, contradicting [4], Th. 1. □

Lemma 5. *Let $X_i \subset \mathbf{P}^5$, $1 \le i \le 2$, be integral surfaces such that $\langle X_1 \cup X_2\rangle = \mathbf{P}^5$ and neither X_1 nor X_2 is a plane. Then for a general $(A_1, A_2, B_1, B_2) \in X_1 \times X_1 \times X_2 \times X_2$ the set $\langle\{A_1, A_2, B_1, B_2\}\rangle \cap (X_1 \cup X_2)$ is finite.*

Proof. Assume that the lemma is false and that for instance $\langle\{A_1, A_2, B_1, B_2\}\rangle \cap (X_1 \cup X_2)$ contains an integral curve $C \subset X_2$. Let $f : \mathbf{P}^5\backslash\langle\{A_1, A_2\}\rangle \to \mathbf{P}^3$ be the linear projection from the line $\langle\{A_1, A_2\}\rangle$. First assume that C is not contained in a plane containing $\langle\{A_1, A_2\}\rangle$. Thus $f(C\backslash C \cap \langle\{A_1, A_2\}\rangle)$ is a curve. Since $f(C\backslash C \cap \langle\{A_1, A_2\}\rangle)$ is contained in the line $\langle\{f(B_1), f(B_2)\}\rangle$, we obtain $\langle\{f(B_1), f(B_2)\}\rangle \subseteq f(X_2\backslash X_2 \cap \langle\{A_1, A_2\}\rangle)$. Since $(f(B_1), f(B_2))$ is general in $f(X_2) \times f(X_2)$, $f(X_2)$ is a plane. Thus $\dim(\langle X_2 \cup \{A_1, A_2\}\rangle) = 4$. By the generality of A_1 and A_2 and the assumption $\langle X_1 \cup X_2\rangle = \mathbf{P}^5$, we obtain that X_2 is a plane, a contradiction. Now assume that C is contained in a plane M containing $\langle\{A_1, A_2\}\rangle$. Varying A_1 and A_2 we obtain that X_2 contains at least a two-dimensional family of plane curves. If $\dim(\langle X_2\rangle) \ge 4$, we obtain that C is a plane conic and X_2 is either the Veronese surface or a projection of the Veronese surface ([6], Segre's lemma at p. 623). We have $M = \langle C\rangle$, $\{A_1, A_2\} \subset M$ and the scheme-theoretic intersection of $\langle\{A_1, A_2\}\rangle$ with C has length two. Hence any secant line to X_1 is secant to X_2. Take a general hyperplane H of $\mathbf{P}^5$ and apply Lemma 4 to a general projection of the curves $X_1 \cap H$ and $X_2 \cap H$ in $\mathbf{P}^3$ to obtain a contradiction. Now assume $\dim(\langle X_2\rangle) \le 3$. Hence $\dim(\langle X_2\rangle) = 3$. Since $\langle X_1 \cup X_2\rangle = \mathbf{P}^5$, for general $(A_1, A_2) \in X_1 \times X_1$ we have $\langle\{A_1, A_2\}\rangle \cap \langle X_2\rangle = \emptyset$, and hence $\langle\{A_1, A_2, B_1, B_2\}\rangle \cap X_2$ is contained in $\langle\{B_1, B_2\}\rangle \cap X_2$ and hence it is finite, a contradiction. □

Look at the set-up of Lemma 5. If X_2 is a plane, then $\langle\{A_1, A_2, B_1, B_2\}\rangle \cap X_2$ is a line.

Proof of Theorem 1. We divide the proof into 9 steps. Steps 1 to 7 are just the translation in our set-up of the corresponding steps in the proof of the Theorem in

section 2 of [6]. The degree 3 curve arising in Step 10 of [6] does not appear in our proof of Theorem 1 because the integer 3 is now distributed between X_1, X_2 and X_3. Instead, in our proof of Theorem 1 we obtain a one-dimensional family Φ of lines contained in X_3. Furthermore, in Step 9 we will use again that $X_2 \neq X_3$. Therefore the proof of Theorem 1 is shorter and easier than the proof of the Theorem in [6], §2.

Step 1. Taking a general linear projection into $\mathbf{P}^5$ we reduce to the case $N = 5$. By assumption we have $\dim([X_1; X_2]) = 5$, i.e. $[X_1; X_2] = \mathbf{P}^5$. Let $J := \{(\Pi, Q) : Q \in \Pi\} \subset \mathrm{Sec}(\vec{X}; \vec{k}) \times \mathbf{P}^5$ be the incidence variety and $q : J \to \mathrm{Sec}(\vec{X}; \vec{k})$, $p : J \to \mathbf{P}^5$ the projections. We have $\dim(J) = 8$ (see e.g. the proof of [6], Prop. 1.1). Since $[X_1; X_2] = \mathbf{P}^5$, p is surjective. Thus for a general $P \in \mathbf{P}^5$ every irreducible component of $p^{-1}(P)$ has dimension 3. Fix a general $P \in \mathbf{P}^5$ and choose one irreducible component L_P of $p^{-1}(P)$.

Step 2. Let $W_P := p(q^{-1}(q(L_P)))$ be the union of all planes belonging to L_P. In this step we will check the existence of a choice of the component L_P of $p^{-1}(P)$ such that W_P is an irreducible variety containing X_3. W_P is irreducible because L_P is irreducible and q is equidimensional and with irreducible fibers. Since $[X_1; X_2] = \mathbf{P}^5$ and P is general, there are $A \in X_1$ and $B \in X_2$ such that $P \in \langle\{A, B\}\rangle$. Hence for a general $Q \in X_3$ the plane $\langle\{A, B, Q\}\rangle$ belongs to $\mathrm{Sec}(\vec{X}; \vec{k})$ and contains P, i.e. $\langle\{A, B, Q\}\rangle \in p^{-1}(P)$. Thus X_3 is contained in $p(q^{-1}(q(p^{-1}(P))))$. Since X_3 is irreducible, there is at least one irreducible component L_P of $p^{-1}(P)$ such that $X_3 \subseteq p(q^{-1}(q(L_P)))$.

Step 3. In order to obtain a contradiction, from now on we assume that $\vec{X}$ is $(1, \vec{k})$-defective. Here we will check that $\dim(W_P) = 4$. Assume $\dim(W_P) = 5$. Then for a general $Q \in \mathbf{P}^5$ there is $\Pi \in L_P$ such that $Q \in \Pi$. Thus the line $\langle\{P, Q\}\rangle$ is contained in W_P. By the generality of P and Q we obtain $G(2, 6) = \mathrm{Sec}(\vec{X}; 1, \vec{k})$, a contradiction. Now assume $\dim(W_P) \leq 3$. Since W_P is irreducible and contains X_3 (Step 2) and $P \notin X_3$, $\dim(W_P) = 3$. For a general $Q \in X_3$ there is $\Pi \in W_P$ such that $Q \in \Pi$. Thus $\langle\{P, Q\}\rangle \subset \Pi$. Hence W_P is the cone $[X_3; \{P\}]$. Since W_P contains a 3-dimensional family of planes, the projection of X_3 from P is a surface Y containing a 3-dimensional family of lines. No such surface Y exists because any two general points of it would be contained in a line contained in Y; hence Y would be a plane, while a plane does not contain a 3-dimensional family of lines.

Step 4. Choose $A \in X_1$ and $B \in X_2$ such that $P \in \langle\{A, B\}\rangle$. Since P is general, the pair (A, B) is general in $X_1 \times X_2$. From now on we fix a general $(A, B) \in X_1 \times X_2$ and a general $P \in \langle\{A, B\}\rangle$. Let Ψ be the rational map from X_3 into $G(3, 6)$ that sends a general $C \in X_3$ into the plane $\langle\{A, B, C\}\rangle \in G(3, 6)$. Call L'_P the closure of $\mathrm{Im}(\Psi)$. Clearly, L'_P is irreducible and by construction it lies in $q(p^{-1}(P))$. We choose as L_P a component of $q(p^{-1}(P))$ containing L'_P.

First Claim: We have $\dim(L'_P) = 2$, $p(q^{-1}(L'_P)) = W_P = [\{A\}; [\{B\}; X_3]]$. With this choice of L_P we have $X_3 \subseteq W_P$, i.e., the statement of Step 2 holds for this component of $q(p^{-1}(P))$.

Proof of the First Claim It is easy to check (see Lemma 2 or Lemma 5 for stronger statements) that Ψ has finite fibers. Hence $\dim(L'_P) = 2$. Since $L'_P \subseteq L_P$, we have $p(q^{-1}(L'_P)) \subseteq W_P$. By the very definition of the rational map Ψ we have $p(q^{-1}(L'_P)) = [\{A\}; [\{B\}; X_3]]$. Hence to prove the First Claim it is sufficient to prove that the cone $[\{A\}; [\{B\}; X_3]]$ has dimension 4, i.e. that X_3 is not a cone with vertex containing B and that the vertex of the cone $[\{B\}; X_3]$ does not contain

A. Since X_1 and X_2 are non-degenerate and the pair (A, B) is general in $X_1 \times X_2$, both assertions are obvious.

Step 5. For any $\Pi \in L_P \backslash L'_P$ write Λ_Π for the linear span of Π and the line $\langle\{A, B\}\rangle$.

Second Claim: Λ_Π is a 3-dimensional linear space contained in W_P, and W_P is the closure of the union of the spaces Λ_Π as Π varies in $L_P \backslash L'_P$. For a general $\Pi \in L_P$ the scheme $\Lambda_\Pi \cap X_3$ contains a curve.

Proof of the Second Claim Since $P \in \langle\{A, B\}\rangle \cap \Pi$ and $\Pi \notin L'_P$, we have $\dim(\Lambda_\Pi) = 3$. By the First Claim for a general $\Pi \in L_P$ and a general $Q \in \Pi$ there is $C \in X_3$ such that $Q \in \langle\{A, B, C\}\rangle$. Thus $\langle\{A, B, Q\}\rangle \subseteq W_P$ and hence $\Lambda_\Pi \subseteq W_P$. Since $\Lambda_\Pi \neq \Lambda_{\Pi'}$ for a general pair $(\Pi, \Pi') \in L_p \times L_P$ and $\dim(W_P) = 4$, W_P is the closure of the union of the spaces Λ_Π as Π varies in $L_P \backslash L'_P$. Since $X_3 \subset W_P$, for a general Π the set $\Lambda_\Pi \cap X_3$ contains a curve, proving the Second Claim.

Step 6. Here we will check that $\Lambda_\Pi \cap \Lambda_{\Pi'} = \langle\{A, B\}\rangle$. Assume on the contrary that $\Lambda_\Pi \cap \Lambda_{\Pi'}$ is a plane, V. By the Linear Lemma in [6], §1, this implies that either all 3-spaces Λ_Π are contained in a 4-dimensional linear space M or for every $R \in L_P \backslash L'_P$ the 3-space Λ_R contains V. The first possibility cannot occur because X_3 is non-degenerate and contained in W_P (Step 2) and W_P is the closure of the union all Λ_R (First Claim). Assume that for every $R \in L_P \backslash L'_P$ the 3-space Λ_R contains V. The linear projection $\alpha : X_3 \backslash X_3 \cap V \to \mathbf{P}^2$ is dominant because the last assertion of the Second Claim implies that α does not contract infinitely many lines. Hence the linear projection of X_3 from the line $\langle\{A, B\}\rangle$ into $\mathbf{P}^3$ is a cone. By the Lemma proved in [6], Step 6 at p. 625, X_3 is a cone, a contradiction.

Step 7. Here we will check that $\langle\{A, B\}\rangle$ is the only line containing P and intersecting $X_1 \backslash X_1 \cap X_2$ and $X_2 \backslash X_1 \cap X_2$. Since the tangent developable of X_3 has dimension 4 and P is general, P is not contained in any line tangent to X_3 at one of its smooth points. Since $[X_1; X_2] = \mathbf{P}^5$, the set $\mathcal{D}$ of all lines containing P and intersecting both $X_1 \backslash X_1 \cap X_2$ and $X_2 \backslash X_1 \cap X_2$ is finite. Now we will check that $\mathcal{D} = \{\langle\{A, B\}\rangle\}$. Take any $D \in \mathcal{D}$. By the finiteness of $\mathcal{D}$, D must be fixed as Π varies. Hence $D \subseteq \Lambda_\Pi \cap \Lambda_{\Pi'} = \langle\{A, B\}\rangle$ (Step 6).

Step 8. Call Γ_Π the union of the one-dimensional components of $\Lambda_\Pi \cap X_3$. Here we will check that for general Π the curve Γ_Π is a line. Recall that W_P is the closure of the union of all spaces Λ_Π with $\Pi \in L_P$. Let $Y \subseteq \mathbf{P}^r$ be an irreducible m-dimensional variety, $m \geq 2$, containing a two-dimensional family of $(m-1)$-dimensional linear spaces. By [6], Lemma in Step 9 of §2, Y is a linear space. Thus W_P contains only a one-dimensional family of distinct 3-spaces Λ_Π. Since $\dim(L_P) = 3$ and each plane of L_P belongs to some 3-space Λ_Π contained in W_P, it follows that the general plane U of Λ_Π containing P intersects X_1, X_2 and X_3 and that, for general P, Π and U, it intersects each X_i exactly at one point (see [6], Cor. 1.3). Hence Γ_Π is a line. Hence the variety X_3 contains an irreducible family Φ of lines Γ_Π, Π general in L_P, with $\Gamma_\Pi \subset \Lambda_\Pi$. Since $\Lambda_\Pi \cap \Lambda_{\Pi'} = \langle\{A, B\}\rangle$ for a general pair (Π, Π') (Step 6), we have $\dim(\Phi) > 0$. Since X_3 is not a plane, we have $\dim(\Phi) = 1$. If all lines Γ_Π pass through a common point Q, then X_3 is a cone with vertex Q, contradicting our assumptions. Since not all lines Γ_Π pass through a common point and X_3 is not a plane, we have $\Gamma_\Pi \cap \Gamma_{\Pi'} = \emptyset$ for a general pair (Π, Π') (Linear Lemma in [6], §1). We now give a side remark. Since X_3 is neither a plane nor a smooth quadric surface, Φ is the only positive-dimensional

irreducible family of lines contained in X_3. Hence Φ does not depend on the choice of P, A and B. For a general $B_3 \in X_3$ there is a unique line $D(B_3)$ such that $B_3 \in D(B_3) \subset X_3$. Since Φ covers X_3, we have $D(B_3) \in \Phi$. Notice that $D(B_3)$ depends only on B_3 and X_3, not on X_1, X_2 and the choices of A, B and P that we made to construct Φ.

Step 9. Take a general triple $(A_1, A_2, A_3) \in X_1 \times X_2 \times X_3$. Hence $\langle\{A_1, A_2, A_3\}\rangle$ is a plane and a general element of $\mathrm{Sec}(\vec{X}; 2, \vec{k})$, and hence $\langle\{A_1, A_2, A_3\}\rangle \cap X_i = \{A_i\}$ (Lemma 2). A general $P \in \langle\{A_1, A_2, A_3\}\rangle$ may be considered as a general element of $\mathbf{P}^5$ because $[X_1; [X_2; X_3]] = \mathbf{P}^5$. Since $[X_1; X_2] = \mathbf{P}^5$, there is $(B_1, B_2) \in X_1 \times X_2$ such that $B_1 \neq B_2$ and $P \in \langle\{B_1, B_2\}\rangle$; furthermore, there are only finitely many such pairs (B_1, B_2). Conversely, given a general quadruple $(A'_1, A'_2, B'_1, B'_2) \in X_1 \times X_1 \times X_2 \times X_2$, the 3-dimensional linear space $\langle\{A_1, A_2, B_1, B_2\}\rangle$ intersects X_3 in a non-empty set and one point of this set, for fixed A'_1, A'_2 but for a general pair (B'_1, B'_2), may be considered as a general point A'_3 of X_3; furthermore, for general A'_1 and A'_2 we may find A'_3 not collinear with A'_1 and A'_2. Hence $\langle\{A'_1, A'_2, A'_3\}\rangle$ is a plane of $\langle\{A_1, A_2, B_1, B_2\}\rangle$, and thus it intersects the line $\langle\{B'_1, B'_2\}\rangle$. Since $\dim(\langle\{A_1, A_2, B_1, B_2\}\rangle) = 3$, for general B'_1 and B'_2 we have $\langle\{A'_1, A'_2\}\rangle \cap \langle\{B'_1, B'_2\}\rangle = \emptyset$. Thus we may do the construction of Step 4 starting from B'_1, B'_2 and P instead of A, B and P. By construction $\langle\{A'_1, A'_2, A'_3\}\rangle \in \Lambda_P \backslash \Lambda_{P'}$. Thus by Step 8 the 3-dimensional linear space G spanned by B'_1, B'_2 and $\langle\{A'_1, A'_2, A'_3\}\rangle$ intersects X_3 in a line $D(A'_1, B'_1, A'_2, B'_2) \in \Phi$. However, $G = \langle\{A'_1, A'_2, B'_1, B'_2\}\rangle$. Thus for a general quadruple $(A'_1, A'_2, B'_1, B'_2) \in X_1 \times X_1 \times X_2 \times X_2$ the set $X_3 \cap \langle\{A_1, A_2, B_1, B_2\}\rangle$ contains a line. Furthermore, $\langle\{A_1, A_2, B_1, B_2\}\rangle$ is spanned by $D(A'_1, B'_1, A'_2, B'_2)$, A'_1 and A'_2. Since $\dim(\Phi) = 1$, we may find a one-dimensional irreducible family, Δ, of pairs $(B''_1, B''_2) \in X_1 \times X_2$ such that $X_3 \cap \langle\{A_1, A_2, B_1, B_2\}\rangle = D(A'_1, B'_1, A'_2, B'_2)$ for every $(B''_1, B''_2) \in \Delta$. Hence $\langle\{A'_1, A'_2, B'_1, B'_2\}\rangle$ contains all such pairs (B''_1, B''_2), contradicting Lemma 5 and hence concluding the proof.

Proof of Theorem 2. If $X \subset \mathbf{P}^m$, $m \geq 5$, is a non-degenerate surface such that $\dim([X; X]) = 4$, then X is either a cone or a Veronese surface ([7]). Notice that the role of the surface X_3 in the proof of Theorem 1 was quite different from the roles of X_1 and X_2, while the roles of X_1 and X_2 were exactly the same. The same proof works in the case $X_1 = X_2$ (and proves Theorem 2), except that in Steps 4 and 9 we need to quote Lemma 3 instead of Lemma 2 and that in the proof of Theorem 1 we assumed and heavily used that $\dim([X_1; X_2]) = 5$. Hence to complete the proof of Theorem 2 it is sufficient to check it when $\dim([X_1; X_1]) \leq 4$ and X_1 is not a cone, i.e. when X_1 is the Veronese surface. Assume $N = 5$ and let $S \subset \mathbf{P}^5$ be the Veronese surface. Let $Y \subset \mathbf{P}^5$ be an integral non-degenerate surface with $Y \neq S$ and Y not a cone. The pair (S, Y) is not a $(1, (2, 1))$-defective pair if and only if a general line $D \subset \mathbf{P}^5$ is contained in a plane spanned by two points of S and one point of Y. Let $Z \subset \mathbf{P}^5$ be the secant variety of S. Thus S is the hypersurface of $\mathbf{P}^5$ union all planes spanned by the conics contained in S. Let $D \subset \mathbf{P}^5$ be a general line. Fix $P \in D \cap Z$ and call $E \subset Z$ the plane such that $P \in E$ and $E \cap S = C$, where C is a smooth conic. Set $M := \langle E \cup D\rangle$. Thus $\dim(M) = 3$. Take $Q \in M \cap Y$ and set $F := E \cap \langle\{Q\} \cup D\rangle$. Hence F is the intersection of two planes contained in M. The linear space M moves if we move D among the lines through P. For a general line D containing P the set F is a line not tangent to C.

Hence the general line D containing P is contained in the plane spanned by Q and the two points of $F \cap C \subset S$, concluding the proof. $\square$

3. Further results on $(t, \vec{k})$-defectivity

Let $\sigma : \mathbf{P}^t \times \mathbf{P}^N \to \mathbf{P}^{tN+t+N}$ be the Segre embedding. The proof of [8], Th. 2.1 (i.e. a computation of a certain Jacobian matrix), gives the following result.

Theorem 3. *$\vec{X}$ is $(t, \vec{k})$-defective if and only if $\sigma(\mathbf{P}^t \times \vec{X}) := (\sigma(\mathbf{P}^t \times X_1), \dots, \sigma(\mathbf{P}^t \times X_s))$ is $\vec{k}$-defective and the total order of $(t, \vec{k})$-defectivity of $\vec{X}$ and the $\vec{k}$-defectivity of $\sigma(\mathbf{P}^t \times \vec{X})$ are the same.*

To obtain results for the $(t, \vec{k})$-defectivity it is essential to have results for the $\vec{k}$-defectivity of other joins; Theorem 3 is just one reason. Since the role of the varieties $X_1, \dots, X_s$ is not the same if $k_i \neq k_j$ for some i, j or the geometric properties (and even the dimensions) of the varieties $X_1, \dots, X_s$ may be quite different (as, for instance, in Question 1), we introduce the following definition.

Definition 1. Fix $N, s, k_i, n_i, 1 \le i \le s$, and t as above and assume that $\sum_{i=1}^{s} k_i(n_i + 1) \le N - 1$. Fix an integer i with $1 \le i \le s$ such that $k_i > 0$. Set $\vec{k}(i) := (k'_1, \dots, k'_s)$ with $k'_j = k_j$ if $j \neq i$ and $k'_i = k_i - 1$. Set $\delta(\vec{X}; \vec{k}; i) := \delta(\vec{X}; \vec{k}) - \delta(\vec{X}; \vec{k}(i))$. The integer $\delta(\vec{X}; \vec{k}; i)$ will be called the $\vec{k}$-defect of $\vec{X}$ for the factor i.

The proof of [2], Th. 1.1, gives the following result.

Lemma 6. *Fix N, $\vec{k}$, i and $\vec{X}$ as above and assume that each variety X_j is non-degenerate. Assume $\delta(\vec{X}; \vec{k}; i) > 0$. Fix a general $P_{u,v} \in X_u$, $1 \le u \le s$ and $1 \le v \le k_u$, and let A be the linear span of the tangent spaces $(TX_u)_{P_{u,v}}$, $1 \le u \le s$ and $1 \le v \le k_u$. Then the general hyperplane H of $\mathbf{P}^N$ containing A is tangent to X_i at least along an irreducible variety of dimension $\delta(\vec{X}; \vec{k}; i)$ containing one of the points $P_{i,j}$ with $1 \le j \le k_i$.*

Corollary 1. *Fix N, $\vec{k}$, i and $\vec{X}$ as above and assume that each variety X_j is non-degenerate. Assume $\delta(\vec{X}; \vec{k}; i) > 0$. Then $\dim(X_i) > \delta(\vec{X}; \vec{k}; i)$ and in particular X_i is not a curve.*

Remark 3. In Lemma 6 and Corollary 1 we may substitute the condition that each variety X_j is non-degenerate with the condition that for each proper subspace M of $\mathbf{P}^N$ containing some of the varieties X_j, say X_j for $j \in S \subset \{1, \dots, s\}$, we have $\sum_{j \in S} k_j(\dim(X_j) + 1) \le \dim(M)$.

Theorem 4. *Fix an integer $s \ge 2$, positive integers $k_1, \dots, k_s$ such that $k_s = 1$ and integral subvarieties $X_i \subset \mathbf{P}^N$ such that X_j is non-degenerate for every $j < s$ and $\dim(X_s) = 1$. Set $\vec{k} := (k_1, \dots, k_s)$, $\vec{k}(s) := (k_1, \dots, k_{s-1})$, $\vec{X} := (X_1, \dots, X_s)$ and $\vec{X}(s) := (X_1, \dots, X_s)$. Assume that $\vec{X}(s)$ is neither $\vec{k}(s)$-defective nor $(1, \vec{k}(s))$-defective and that $\mathrm{Sec}(\vec{X}(s); \vec{k}(s))$ is not a cone with vertex containing X_s. Then $\vec{X}$ is not $(1, \vec{k})$-defective.*

Proof. Assume that $\vec{X}$ is $(1, \vec{k})$-defective. A general line $L \in \mathrm{Sec}(\vec{X}; 1, \vec{k})$ is obtained by taking general $P_{i,j} \in X_i$, $1 \le i \le s$, $1 \le j \le k_i$, and then taking a general line L contained in the $|\vec{k}|$-dimensional linear space spanned by the points $P_{i,j}$. Let

$A(s)$ be the join of all points $P_{i,j}$ with $1 \le i \le s-1$ and $1 \le j \le k_i$. Set $Q(L) := A(s) \cap L$. Hence $Q(L) \in \mathrm{Sec}(\vec{X}(s); \vec{k}(s))$. By the generality of the points $P_{i,j}$ and of the line L the point $Q(L)$ may be considered as a general point of $\mathrm{Sec}(\vec{X}(s); \vec{k}(s))$. By assumption there is a one-dimensional family of $|\vec{k}|$-planes, say $\{\Pi_t\}_{t \in T}$ with T irreducible curve such that each Π_t intersects each X_i at k_i distinct points, say $P_{i,j}(t)$, $L \subset \Pi_t$ for every t, and there is $o \in T$ such that $P_{i,j}(o) = P_{i,j}$ for all i, j. Let $A(s,t)$ be the join of all points $P_{i,j}(t)$ with $1 \le i \le s-1$ and $1 \le j \le k_i$. Set $Q(L,t) := A(s,t) \cap L$. Hence $Q(L,t) \in A(s,t)$. First assume $Q(L,t) = Q(L)$ for every t. Since $Q(L,t) := A(s,t) \cap L$, this implies that $Q(L)$ is contained in infinitely many $(|\vec{k}|-1)$-dimensional linear spaces belonging to $\mathrm{Sec}(\vec{X}(s); |\vec{k}(s)|-1, \vec{k}(s))$. Since $Q(L)$ is general in $\mathrm{Sec}(\vec{X}(s); \vec{k}(s))$, this implies $\dim(\mathrm{Sec}(\vec{X}(s); \vec{k}(s)))) \le\le \sum_{i=1}^{s-1} k_i(\dim(X_i)+1) - 2$, a contradiction. If $Q(L,t) \neq Q(L)$ for general $t \in T$, then L is contained in $\mathrm{Sec}(\vec{X}(s); \vec{k}(s))$. By the generality of L we obtain $\mathrm{Sec}(\vec{X}; \vec{k}) = \mathrm{Sec}(\vec{X}(s); \vec{k}(s))$, i.e. $\mathrm{Sec}(\vec{X}(s); \vec{k}(s))$ is a cone with vertex containing X_s, a contradiction. □

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