JOINS OF PROJECTIVE VARIETIES
AND MULTISEcant SPACES

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Abstract. Let $X_1, \ldots, X_s \subset \mathbb{P}^N$, $s \geq 1$, be integral varieties. For any integers $k_i > 0$, $1 \leq i \leq s$, and $t \geq 0$ set $\vec{k} := (k_1, \ldots, k_s)$ and $\vec{X} := (X_1, \ldots, X_s)$. Let $\text{Sec}(\vec{X}; t, \vec{k})$ be the set of all linear $t$-spaces contained in a linear $(k_1 + \cdots + k_s - 1)$-space spanned by $k_1$ points of $X_1$, $k_2$ points of $X_2$, ..., $k_s$ points of $X_s$. Here we study some cases where $\text{Sec}(\vec{X}; t, \vec{k})$ has the expected dimension. The case $s = 1$ was recently considered by Chiantini and Coppens and we follow their ideas. The two main results of the paper consider cases where each $X_i$ is a surface, more particularly:

1. Introduction

L. Chiantini and M. Coppens revived a piece of classical projective geometry (see [6] and references therein): the study of the set of all linear spaces contained in the secant varieties of an integral variety $X \subset \mathbb{P}^N$. For further papers on this topic, see [5], [8] and [9]. Let $G(t + 1, N + 1)$ be the Grassmannian of all $t$-dimensional linear subspaces of $\mathbb{P}^N$. The order $k$ secant variety of $X$ is the join of $k$ copies of $X$. In this paper we fix $s$ varieties $X_i \subset \mathbb{P}^N$, $1 \leq i \leq s$, and consider the closure in $G(t + 1, N + 1)$ of the set of all $t$-spaces contained in a $(k_1 + \cdots + k_s - 1)$-space spanned by $k_1$ points of $X_1$, $k_2$ points of $X_2$, ..., $k_s$ points of $X_s$. The case $s = 1$ is the case considered in [6] and we will often use the ideas contained in [6].

Fix integers $N \geq 3$, $s > 0$ and $k_i \geq 0$, $1 \leq i \leq s$. Set $\vec{k} := (k_1, \ldots, k_s)$ and $|\vec{k}| := k_1 + \cdots + k_s - 1$. Let $t$ be an integer such that $0 \leq t \leq |\vec{k}|$. Fix $s$ irreducible varieties $X_i \subset \mathbb{P}^N$, $1 \leq i \leq s$. Usually, we will be interested in the case $X_i \neq X_j$ for $i \neq j$, since the general case may be reduced to this case by decreasing $s$, but with the same

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value of $|\tilde{k}|$. The $(t, \tilde{k})$-secant variety $\text{Sec}(\tilde{X}; t, \tilde{k})$ of $\tilde{X}$ is the closure in $G(t+1, N+1)$ of all $t$-spaces contained in a $|\tilde{k}|$-dimensional linear subspace of $\mathbb{P}^N$ spanned by $k_i$ points of $X_1$, $k_2$ points of $X_2$, ..., $k_s$ points of $X_s$. Set $\text{Sec}(\tilde{X}; 0, \tilde{k}) := \text{Sec}(\tilde{X}; 0, \tilde{k}) \subseteq \mathbb{P}^N$. The $(0, k)$-secant variety $\text{Sec}(\tilde{X}; k)$ of $\tilde{X}$ will be called the $k$-secant variety of $\tilde{X}$. Set $n_i := \dim(X_i)$. We have $\dim(\text{Sec}(\tilde{X}; k)) \leq \min\{N, \sum_{i=1}^s k_i(n_i + 1) - 1\}$ and $\dim(\text{Sec}(\tilde{X}; t, \tilde{k})) \leq \min\{(t + 1)(N - t), \sum_{i=1}^s k_i n_i + (|\tilde{k}| - t)(t + 1)\}$. We will say that $\tilde{X}$ is $\tilde{k}$-defective (resp. $(t, \tilde{k})$-defective) if
\[
\dim(\text{Sec}(\tilde{X}; \tilde{k})) \leq \min\{N, \sum_{i=1}^s k_i(n_i + 1) - 1\}
\]
(resp. $\dim(\text{Sec}(\tilde{X}; t, \tilde{k})) \leq \min\{(t + 1)(N - t), \sum_{i=1}^s k_i n_i + (|\tilde{k}| - t)(t + 1)\}$). If $\tilde{X}$ is $\tilde{k}$-defective (resp. $(t, \tilde{k})$-defective) the integer
\[
\delta(\tilde{X}; \tilde{k}) := \sum_{i=1}^s k_i(n_i + 1) - 1 - \dim(\text{Sec}(\tilde{X}; \tilde{k}))
\]
(resp. $\delta(\tilde{X}; t, \tilde{k}) := \sum_{i=1}^s k_i n_i + (|\tilde{k}| - t)(t + 1) - \dim(\text{Sec}(\tilde{X}; t, \tilde{k}))$) will be called the total order of $\tilde{k}$-deficiency (resp. $(t, \tilde{k})$-deficiency) of $\tilde{X}$.

It seems very natural to start the study of $(t, \tilde{k})$-deficiency from the case $\dim(X_i) = 1$ for every $i$. For the case in which each $X_i$ is a non-degenerate curve, see Corollary 1 and Theorem 4 in section 3. For a more general deficiency result for non-degenerate curves, see [3]: in the quoted paper we considered the set of all flags of linear spaces contained in a $|\tilde{k}|$-dimensional linear space instead of the set of all $t$-dimensional linear spaces. Obviously, the interested reader may do other related cases (e.g. some degenerate curves or a surface and $s - 1$ non-degenerate curves). We stress that degenerate varieties may not give defective $s$-ples (e.g. when $k_i = 1$ for all $i$ take as $X_i$, $1 \leq i \leq s$, linearly independent linear subspaces). For degenerate varieties, see also Remark 3. We believe that when $s \geq 2$, the mutual position of the varieties is more important than their structure. For examples of deficiency when one of the varieties is a cone, see Remark 1. For a complete analysis of a toy case, see Example 4.

We raise the following question.

**Question 1.** Fix integers $s \geq 2$, $k_i > 0$, $1 \leq i \leq s$, $t > 0$, and integral varieties $X_i \subset \mathbb{P}^N$, $1 \leq i \leq s$. Set $\tilde{X} := (X_1, \ldots, X_s)$, $\tilde{k} := (k_1, \ldots, k_s)$, $\tilde{Y} := (X_1, \ldots, X_{s-1})$, and $\tilde{m} := (k_1, \ldots, k_{s-1})$. Assume that $\text{Sec}(\tilde{Y}; t, \tilde{m})$ has the expected dimension and that $X_s$ is a curve. Are there reasonable conditions on $X_s$ assuring that $\text{Sec}(\tilde{X}; t, \tilde{k})$ has the expected dimension? More generally, for any $\tilde{Y}$ has $\text{Sec}(\tilde{X}; t, \tilde{k})$ the maximal possible dimension compatible with the dimension of $\text{Sec}(\tilde{Y}; t, \tilde{m})$?

It is well known and easy to show that in the case $t = 0$ a sufficient condition for an affirmative answer to Question 1 is that $X_s$ is a non-degenerate curve (see Corollary 1 and Remark 3 for more precise results). See Theorem 4 for the case $t = 1$ and 3 when $\dim(X_1) = 1$ and each $X_i$ is non-degenerate.

Our main results are non-existence results for the $(1, \tilde{k})$-deficiency of joins of surfaces. In section 2 we will prove the following results.

**Theorem 1.** Let $X_1$, $X_2$ and $X_3$ be integral non-degenerate surfaces of $\mathbb{P}^N$, $N \geq 5$, such that $X_i \neq X_j$ for $i \neq j$. Assume $\dim([X_1; X_2]) = 5$ and that $X_3$ is not
a cone. Set $\vec{k} = (1, 1, 1)$. Then $\vec{X} := (X_1, X_2, X_3)$ is not $(1, \vec{k})$-defective, i.e. $\dim(\Sec(\vec{X}; 1, \vec{k})) = 8$.

**Theorem 2.** Let $X_1$ and $X_2$ be integral non-degenerate surfaces of $\mathbb{P}^N$, $N \geq 5$, such that $X_1 \neq X_2$. Assume that neither $X_1$ nor $X_2$ is a cone. Set $\vec{k} := (2, 1)$. Then $\vec{X} := (X_1, X_2)$ is not $(1, \vec{k})$-defective.

The condition $\dim([X_1; X_2]) = 5$ in the statement of Theorem 2 is very mild (see Remark 1). It implies that neither $X_1$ nor $X_2$ is a cone. We do not know if the condition that no $X_i$ is a cone is always necessary (see Remark 1), but certainly $X_1, X_2$ and $X_3$ cannot be cones with the same vertex (see Example 1). We do not have any construction (except cones) to obtain defective $s$-ples.

In section 3 we will give two general results on the $(t, \vec{k})$-defectivity of varieties of arbitrary dimension: an easy extension of 8 to the case of joins of different varieties (Theorem 3) and a non-defectivity result with respect to lines (Theorem 4).

## 2. Proofs of Theorems 1 and 2

In this section we will prove Theorems 1 and 2 and give the toy example and the remark on cones promised in the introduction.

For any subset $S$ of $\mathbb{P}^N$, let $(S)$ be its linear span. We start with a baby example.

**Example 1.** Fix integers $s \geq 2$, $n_i \geq 2$, $1 \leq i \leq s$, $k_i > 0$, $1 \leq i \leq s$, $N > \min_{1 \leq i \leq s}\{n_i\}$ and $t$ such that $0 \leq t < k_1 + \cdots + k_s - 1$. Fix $P \in \mathbb{P}^N$ and non-degenerate varieties $X_i \subset \mathbb{P}^N$, $1 \leq i \leq s$, such that $\dim(X_i) = n_i$. Set $\vec{X} := (X_1, \ldots, X_s)$ and $\vec{k} := (k_1, \ldots, k_s)$. Assume that each $X_i$ is a cone with vertex containing $P$. For all hyperplanes $H, M$ of $\mathbb{P}^N$ such that $P \notin H \cup M$ the $s$-ples $(X_1 \cap H, \ldots, X_s \cap H)$ and $(X_1 \cap M, \ldots, X_s \cap M)$ (respectively seen as $s$-ples in $H$ and in $M$) are projectively equivalent (use the linear projection from $P$). Fix $H$ and set $Y_i := X_i \cap H$ and $\vec{Y} := (Y_1, \ldots, Y_s)$. Fix $S_i \subset Y_i$, $1 \leq i \leq s$, such that $\text{card}(S_i) := k_i$ and $\dim(\langle S_1 \cup \cdots \cup S_s \rangle) = k_1 + \cdots + k_s - 1$. Thus $\dim(\langle S_1 \cup \cdots \cup S_s \cup \{P\} \rangle) = k_1 + \cdots + k_s$, and for every $t$-dimensional linear space $D \subset \langle S_1 \cup \cdots \cup S_s \rangle$ the $(t+1)$-dimensional linear space $[D; \{P\}]$ contains a $(t+1)$-dimensional family of $t$-dimensional linear spaces not containing $P$ and mapped isomorphically onto $D$ by the linear projection from $P$. Fix any such $t$-dimensional linear space $D'$. There is a hyperplane $M$ of $\mathbb{P}^N$ such that $D' \subset M$ and $P \notin M$. Set

$$S_i^M := \bigcup_{Q \in S_i} \langle \{Q, P\} \rangle \subset M \subset X_i.$$ 

Thus $\dim(\langle S_i^M \cup \cdots \cup S_s^M \rangle) = k_1 + \cdots + k_s - 1$, $D' \subset \langle S_1^M \cup \cdots \cup S_s^M \rangle$ and hence $D' \in \Sec(\vec{X}; t, \vec{k})$. Thus if $\Sec(\vec{Y}; t, \vec{k}) = G(t + 1, N)$, then

$$\Sec(\vec{X}; t, \vec{k}) = G(t + 1, N + 1),$$

while if $\Sec(\vec{Y}; t, \vec{k}) \neq G(t + 1, N)$, then $\delta(\vec{X}; t, \vec{k}) = \delta(\vec{Y}; t, \vec{k}) + k_1 + \cdots + k_s - 1 - t$. Thus in the former case $(\vec{X}, \vec{k})$ is not $(t, \vec{k})$-defective. In the latter case if $t < k_1 + \cdots + k_s - 1$, then $\vec{X}$ is not $(t, \vec{k})$-defective. The fact that in the definition of defectivity we have to take the cut-off function min implies that very natural constructions do not always (but only almost always) give degenerate $s$-ples.
Remark 1. Fix $P \in \mathbb{P}^N$ and also fix a locally closed and irreducible subset $T$ of $G(t+1,N+1)$ such that $P \not\in A$ for every $A \in T$. Let $T \ast P$ be the closure in $G(t+1,N+1)$ of the set of all $B \in G(t+1,N+1)$ contained in some $(t+1)$-dimensional linear space $(A \cup \{P\})$ for some $A \in T$. Fix integers $s \geq 1$, $k_i > 0$, $1 \leq i \leq s$, and integral varieties $X_i \subset \mathbb{P}^N$, $1 \leq i \leq s$. Assume that $X_s$ is a positive-dimensional cone with vertex containing $P \in \mathbb{P}^N$, say $X_s = [D; \{P\}]$ with $\dim(D) = \dim(X_s) - 1$ and $P \notin D$. Set $\tilde{X} := (X_1, \ldots, X_{s-1}, X_s)$ and $\tilde{Y} := (X_1, \ldots, X_{s-1}, D)$. Obviously, $\text{Sec}(\tilde{X}; \tilde{k})$ is the cone with $P$ as vertex and $\text{Sec}(\tilde{Y}; \tilde{k})$ as a basis. Hence $\tilde{X}$ is $\tilde{k}$-defective if $k_s \geq 2$ and $\dim(\text{Sec}(\tilde{Y}; \tilde{k})) \leq N - 2$. Now assume $t > 0$. Let $\text{Sec}(\tilde{Y}; t, \tilde{k})'$ be the open subset of $\text{Sec}(\tilde{Y}; t, \tilde{k})$ formed by the $t$-planes not containing $P$. It is easy to check that $\text{Sec}(\tilde{X}; t, \tilde{k}) = \text{Sec}(\tilde{Y}; t, \tilde{k})' + P$. Since $\dim(G(t+1,t+2)) = t+1$, we obtain $\dim(\text{Sec}(\tilde{X}; t, \tilde{k})) \leq \dim(\text{Sec}(\tilde{Y}; t, \tilde{k})) + t+1$. Hence $\text{Sec}(\tilde{X}; t, \tilde{k})$ is $(t, \tilde{k})$-defective if $k_s \geq t+2$ and $\dim(\text{Sec}(\tilde{Y}; t, \tilde{k})) + t+1 < (t+1)(N-t)$.

Remark 2. Let $X_i \subset \mathbb{P}^m$, $1 \leq i \leq 3$, $m \geq 3$, be integral surfaces such that $\langle X_1 \cup X_2 \cup X_3 \rangle = \mathbb{P}^m$ and either $X_1 \neq X_2$ or $X_1$ is not a plane. It is easy to check that for a general $(A_1, A_2, A_3) \in X_1 \times X_2 \times X_3$ and a general $(B_1, B_2, B_3, B_4) \in X_1 \times X_2 \times X_3 \times X_4$ we have $\dim(\langle\{A_1, A_2, A_3\}\rangle) = 2$ and $\dim(\langle\{B_1, B_2, B_3, B_4\}\rangle) = 3$.

**Lemma 1.** Let $X_i \subset \mathbb{P}^4$, $1 \leq i \leq 2$, be integral non-degenerate surfaces. Then for a general $(A_1, A_2) \in X_1 \times X_2$ we have $\langle\{A_1, A_2\}\rangle \cap (X_1 \cup X_2) = \{A_1, A_2\}$.

*Proof.* If $X_1 = X_2$, then the lemma is [3], Cor. 1.3, for the invariants $r = 4$, $n = 2$ and $k = 1$. Assume $X_1 \neq X_2$. First assume that $\langle\{A_1, A_2\}\rangle \cap (X_1 \cup X_2)$ contains $B \subset X_1 \\backslash \{A_1\}$. Then for a general $A_2 \subset X_2$ the restriction to $X_1$ of the linear projection $\mathbb{P}^5 \\backslash \{A_2\} \to \mathbb{P}^3$ from $A_2$ is not birational. Since $X_2 \neq X_1$, this implies that $X_2$ is contained in the so-called Segre locus $\Sigma(X_1)$ of $X_1$, contradicting the inequality $\dim(\Sigma(X_1)) \leq 1$ proved in [3], Th. 1. Similarly, if $\langle\{A_1, A_2\}\rangle \cap (X_1 \cup X_2)$ contains $B \subset X_2 \\backslash \{A_2\}$, we see that $X_1$ is contained in the Segre locus of $X_2$, contradicting [3], Th. 1.

**Lemma 2.** Let $X_i \subset \mathbb{P}^5$, $1 \leq i \leq 3$, be integral surfaces such that $X_i \neq X_j$ and $\langle X_i \cup X_j \rangle = \mathbb{P}^5$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$. Then for a general $(A_1, A_2, A_3) \in X_1 \times X_2 \times X_3$ we have $\langle\{A_1, A_2, A_3\}\rangle \cap (X_1 \cup X_2 \cup X_3) = \{A_1, A_2, A_3\}$.

*Proof.* Assume that for a general $(A_1, A_2, A_3) \in X_1 \times X_2 \times X_3$ we have $\langle\{A_1, A_2, A_3\}\rangle \cap (X_1 \cup X_2 \cup X_3) \neq \{A_1, A_2, A_3\}$. Just to fix the notation assume that $\langle\{A_1, A_2, A_3\}\rangle \cap (X_1 \cup X_2 \cup X_3)$ contains $B \subset X_1 \\backslash \{A_3\}$. Let $f : \mathbb{P}^5 \\backslash \{A_3\} \to \mathbb{P}^3$ be the linear projection from the point $A_3$ and $g : \mathbb{P}^4 \\backslash \{f(A_2)\} \to \mathbb{P}^3$ the linear projection from the point $f(A_2)$. Using that $X_3 \neq X_2$, $X_3 \neq X_1$, $X_3$ is not contained in $\langle X_i \rangle$ ($i = 1, 2$) if $X_i$ is degenerate and $A_3$ is general in $X_3$, we obtain $A_3 \notin X_1 \cup X_2$, $\langle f(X_1) \cup f(X_2) \rangle = \mathbb{P}^4$ and $\dim(\langle f(X_i) \rangle) = \min\{4, \dim(\langle X_i \rangle)\}$ for $i = 1, 2$. Hence either $\langle f(X_1) \rangle = \mathbb{P}^4$ or $f(X_2) \not\subset \langle f(X_1) \rangle$. If $\langle f(X_1) \rangle \neq \mathbb{P}^4$, by the generality of $f(A_2)$ in $f(X_2)$ we obtain that $g(f(X_1))$ is injective (here we just take any $f(A_2) \notin \langle f(X_1) \rangle$), contradicting the existence of $A_1$ and $B$ (even if $A_1$ is not assumed to be general in $X_1$) such that $A_1 \neq B$ and $B \in \langle\{A_1, A_2, A_3\}\rangle \cap (X_1 \cup X_2 \cup X_3)$. Hence we may assume $\langle f(X_1) \rangle = \mathbb{P}^4$. To obtain a contradiction it is sufficient to show that $g(f(X_1))$ is birational. Assume that $g(f(X_1))$ is not birational. Since $f(A_2)$ is a general point of $f(X_2)$ and $f(X_1) \neq f(X_2)$, we obtain that a general point of $f(X_2) \backslash f(X_1)$ is in the Segre locus $\Sigma\langle f(X_1) \rangle$ of $f(X_1)$, contradicting [4], Th. 1.
Lemma 3. Let $X_i \subset \mathbb{P}^5$, $1 \leq i \leq 2$, be integral surfaces such that $X_1 \neq X_2$, $\langle X_1 \cup X_2 \rangle = \mathbb{P}^5$ and $\dim(\langle X_i \rangle) \geq 4$. Then for a general $(A_1, A_2, B) \in X_1 \times X_1 \times X_2$ we have $\langle \{A_1, A_2, B\} \rangle \cap (X_1 \cup X_2) = \{A_1, A_2, A_3\}$.

Proof. Assume that for a general $(A_1, A_2, B) \in X_1 \times X_1 \times X_2$ we have $\langle \{A_1, A_2, B\} \rangle \cap (X_1 \cup X_2) \neq \{A_1, A_2, B\}$. Let $f : \mathbb{P}_2 \setminus \{B\} \rightarrow \mathbb{P}^1$ be the linear projection from $B$. First assume that $\langle \{A_1, A_2, B\} \rangle \cap (X_1 \cup X_2)$ contains $D \in X_1 \setminus \{A_1, A_2\}$. Since $B$ is general in $X_2$, $f(X_1)$ spans $\mathbb{P}^1$. Hence a general secant line of $f(X_1)$ is not a trisecant line ([3], Cor. 1.3). Since $f(D) \in \langle f(A_1), f(A_2) \rangle$, we obtain that either $f(D) = f(A_1)$ or $f(D) = f(A_2)$. Just to fix the notation we assume $f(D) = f(A_1)$. Since $A_1$ is general in $X_1$, we obtain that $f|X_1$ is not birational. Hence a general $B \in X_2$ is in the Segre locus $\Sigma(X_1)$ of $X_1$, contradicting [4], Th. 1. Now assume that $\langle \{A_1, A_2, B\} \rangle \cap (X_1 \cup X_2)$ contains $C \in X_2 \setminus \{B\}$. We obtain that any secant line to $f(X_1)$ intersects $f(X_2 \setminus \{B\})$. Hence $f(X_2) \subseteq \Sigma(f(X_1))$, a contradiction. □

Lemma 4. Let $C, D \subset \mathbb{P}^m$, $m \geq 3$, be integral non-degenerate curves. Assume $C \neq D$. Then a general secant line to $C$ is not secant to $D$.

Proof. Assume that this is not true and fix a general $P \in C$. By assumption for a general $Q \in C$ the line $\langle \{P, Q\} \rangle$ is secant to $D$. Hence the linear projection from $P$ is not birational. Thus a general $P \in C$ is contained in the Segre locus $\Sigma(D)$ of $D$, contradicting [4], Th. 1. □

Lemma 5. Let $X_i \subset \mathbb{P}^5$, $1 \leq i \leq 2$, be integral surfaces such that $\langle X_1 \cup X_2 \rangle = \mathbb{P}^5$ and neither $X_1$ nor $X_2$ is a plane. Then for a general $(A_1, A_2, B_1, B_2) \in X_1 \times X_1 \times X_2 \times X_2$ the set $\langle \{A_1, A_2, B_1, B_2\} \rangle \cap (X_1 \cup X_2)$ is finite.

Proof. Assume that the lemma is false and that for instance $\langle \{A_1, A_2, B_1, B_2\} \rangle \cap (X_1 \cup X_2)$ contains an integral curve $C \subset X_2$. Let $f : \mathbb{P}_2 \setminus \{A_1, A_2\} \rightarrow \mathbb{P}^3$ be the linear projection from the line $\langle \{A_1, A_2\} \rangle$. First assume that $C$ is not contained in a plane containing $\langle \{A_1, A_2\} \rangle$. Thus $f(C \setminus \langle \{A_1, A_2\} \rangle)$ is a curve. Since $f(C \setminus \langle \{A_1, A_2\} \rangle)$ is contained in the line $\langle \{f(B_1), f(B_2)\} \rangle$, we obtain $\langle \{f(B_1), f(B_2)\} \rangle \subseteq f(X_2 \setminus \langle \{A_1, A_2\} \rangle) \cap \{f(B_1), f(B_2)\}$ is general in $f(X_2 \setminus \langle \{A_1, A_2\} \rangle)$ or $f(X_2)$ is a plane. Thus $\dim(\langle X_2 \cup \{A_1, A_2\} \rangle) = 4$. By the generality of $A_1$ and $A_2$ and the assumption $\langle X_1 \cup X_2 \rangle = \mathbb{P}^5$, we obtain that $X_2$ is a plane, a contradiction. Now assume that $C$ is contained in a plane $M$ containing $\langle \{A_1, A_2\} \rangle$. Varying $A_1$ and $A_2$ we obtain that $X_2$ contains at least a two-dimensional family of plane curves. If $\dim(\langle X_2 \rangle) \geq 3$, we obtain that $C$ is a plane conic and $X_2$ is either the Veronese surface or a projection of the Veronese surface ([6], Segre’s lemma at p. 623). We have $M = \langle C \rangle$, $\{A_1, A_2\} \subset M$ and the scheme-theoretic intersection of $\langle \{A_1, A_2\} \rangle$ with $C$ has length two. Hence any secant line to $X_1$ is secant to $X_2$. Take a general hyperplane $H$ of $\mathbb{P}^5$ and apply Lemma 4 to a general projection of the curves $X_1 \cap H$ and $X_2 \cap H$ in $\mathbb{P}^3$ to obtain a contradiction. Now assume $\dim(\langle X_2 \rangle) \leq 3$. Hence $\dim(\langle X_2 \rangle) = 3$. Since $\langle X_1 \cup X_2 \rangle = \mathbb{P}^5$, for general $(A_1, A_2) \in X_1 \times X_1$ we have $\langle \{A_1, A_2\} \rangle \cap X_1 = 0$, and hence $\langle \{A_1, A_2, B_1, B_2\} \rangle \cap X_2$ is contained in $\langle \{B_1, B_2\} \rangle \cap X_2$ and hence it is finite, a contradiction. □

Look at the set-up of Lemma 4. If $X_2$ is a plane, then $\langle \{A_1, A_2, B_1, B_2\} \rangle \cap X_2$ is a line.

Proof of Theorem 7 We divide the proof into 9 steps. Steps 1 to 7 are just the translation in our set-up of the corresponding steps in the proof of the Theorem in
section 2 of \cite{3}. The degree 3 curve arising in Step 10 of \cite{3} does not appear in our proof of Theorem\cite{3} because the integer 3 is now distributed between \(X_1, X_2\) and \(X_3\). Instead, in our proof of Theorem\cite{3} we obtain a one-dimensional family \(\Phi\) of lines contained in \(X_3\). Furthermore, in Step 9 we will use again that \(X_2 \neq X_3\). Therefore the proof of Theorem\cite{3} is shorter and easier than the proof of the Theorem in \cite{6}, \S2.

**Step 1.** Taking a general linear projection into \(\mathbb{P}^5\) we reduce to the case \(N = 5\). By assumption we have \(\dim([X_1; X_2]) = 5\), i.e. \([X_1; X_2] = \mathbb{P}^5\). Let \(J := \{[I; Q] : Q \in \Pi \} \subset \text{Sec}((\overline{X}; \overline{k}) \times \mathbb{P}^5)\) be the incidence variety and \(q : J \to \text{Sec}(\overline{X}; \overline{k}), p : J \to \mathbb{P}^5\) the projections. We have \(\dim(J) = 8\) (see e.g. the proof of \cite{3}, Prop. 1.1). Since \([X_1; X_2] = \mathbb{P}^5\), \(p\) is surjective. Thus for a general \(P \in \mathbb{P}^5\) every irreducible component of \(p^{-1}(P)\) has dimension 3. Fix a general \(P \in \mathbb{P}^5\) and choose one irreducible component \(L_P\) of \(p^{-1}(P)\).

**Step 2.** Let \(W_P := p(q^{-1}(q(L_P)))\) be the union of all planes belonging to \(L_P\). In this step we will check the existence of a choice of the component \(L_P\) of \(p^{-1}(P)\) such that \(W_P\) is an irreducible variety containing \(X_3\). \(W_P\) is irreducible because \(L_P\) is irreducible and \(q\) is equidimensional and with irreducible fibers. Since \([X_1; X_2] = \mathbb{P}^5\) and \(P\) is general, there are \(A \in X_1\) and \(B \in X_2\) such that \(P \in \{[A, B]\}\). Hence for a general \(Q \subset X_3\) the plane \(\langle A, B, Q \rangle\) belongs to \(\text{Sec}(\overline{X}; \overline{k})\) and contains \(P\), i.e. \(\langle A, B, Q \rangle \in p^{-1}(P)\). Thus \(X_3\) is contained in \(p(q^{-1}(p^{-1}(P)))\). Since \(X_3\) is irreducible, there is at least one irreducible component \(L_P\) of \(p^{-1}(P)\) such that \(X_3 \subseteq p(q^{-1}(q(L_P)))\).

**Step 3.** In order to obtain a contradiction, from now on we assume that \(\overline{X}\) is \((1, \overline{k})\)-defective. Here we will check that \(\dim(W_P) = 4\). Assume \(\dim(W_P) = 5\). Then for a general \(Q \in \mathbb{P}^5\) there is \(\Pi \in L_P\) such that \(Q \in \Pi\). Thus the line \(\langle P, Q \rangle\) is contained in \(W_P\). By the generality of \(P\) and \(Q\) we obtain \(G(2, 6) = \text{Sec}(\overline{X}; 1, \overline{k})\), a contradiction. Now assume \(\dim(W_P) \leq 3\). Since \(W_P\) is irreducible and contains \(X_3\) (Step 2) and \(P \notin X_3\), \(\dim(W_P) = 3\). For a general \(Q \subset X_3\) there is \(\Pi \in W_P\) such that \(Q \in \Pi\). Thus \(\langle P, Q \rangle \subset \Pi\). Hence \(W_P\) is the cone \([X_3; \{P\}]\). Since \(W_P\) contains a 3-dimensional family of planes, the projection of \(X_3\) from \(P\) is a surface \(Y\) containing a 3-dimensional family of lines. No such surface \(Y\) exists because any two general points of it would be contained in a line contained in \(Y\); hence \(Y\) would be a plane, while a plane does not contain a 3-dimensional family of lines.

**Step 4.** Choose \(A \in X_1\) and \(B \in X_2\) such that \(P \in \{[A, B]\}\). Since \(P\) is general, the pair \(\langle A, B \rangle\) is general in \(X_1 \times X_2\). From now on we fix a general \(\langle A, B \rangle \in X_1 \times X_2\) and a general \(P \in \{[A, B]\}\). Let \(\Psi\) be the rational map from \(X_3\) into \(G(3, 6)\) that sends a general \(C \subset X_3\) into the plane \(\langle A, B, C \rangle \in G(3, 6)\). Call \(L_P\) the closure of \(\text{Im}(\Psi)\). Clearly, \(L_P\) is irreducible and by construction it lies in \(q(p^{-1}(P))\). We choose as \(L_P\) a component of \(q(p^{-1}(P))\) containing \(L_P\).

**First Claim:** We have \(\dim(L_P) = 2\). Let \(p(q^{-1}(L_P')) = W_P = \{[A]; [B]; X_3]\). With this choice of \(L_P\) we have \(X_3 \subseteq W_P\), i.e., the statement of Step 2 holds for this component of \(q(p^{-1}(P))\).

**Proof of the First Claim** It is easy to check (see Lemma \cite{2} or Lemma \cite{3} for stronger statements) that \(\Psi\) has finite fibers. Hence \(\dim(L_P') = 2\). Since \(L_P' \subseteq L_P\), we have \(p(q^{-1}(L_P')) \subseteq W_P\). By the very definition of the rational map \(\Psi\) we have \(p(q^{-1}(L_P')) = \{[A]; [B]; X_3]\). Hence to prove the First Claim it is sufficient to prove that the cone \(\{[A]; [B]; X_3\}\) has dimension 4, i.e. that \(X_3\) is not a cone with vertex containing \(B\) and that the vertex of the cone \(\{[B]; X_3\}\) does not contain
neither a plane nor a smooth quadric surface, is the only positive-dimensional

First Claim: \( \Lambda \) is a 3-dimensional linear space contained in \( W \), and \( W \) is the closure of the union of the spaces \( \Lambda \) as \( \Pi \) varies in \( L_P \setminus L_P' \). For a general \( \Pi \in L_P \) the scheme \( \Lambda \cap X_3 \) contains a curve.

Proof of the First Claim Since \( P \in \Gamma(\{A, B\}) \cap \Pi \) and \( \Pi \notin L_P \), we have \( \dim(\Lambda) = 3 \). By the First Claim for a general \( \Pi \in L_P \) and a general \( Q \in \Pi \) there is \( C \in X_3 \) such that \( Q \in \Gamma(\{A, B, C\}) \). Thus \( \Gamma(\{A, B, Q\}) \subseteq W \) and hence \( \Lambda \subseteq W \). Since \( \Lambda \neq \Lambda' \) for a general pair \((\Pi, \Pi') \in L_P \times L_P \) and \( \dim(W) = 4 \), \( W \) is the closure of the union of the spaces \( \Lambda \) as \( \Pi \) varies in \( L_P \setminus L_P' \). Since \( X_3 \subseteq W \), for a general \( \Pi \) the set \( \Lambda \cap X_3 \) contains a curve, proving the First Claim.

Step 5. Here we will check that \( \Lambda \cap \Lambda' = \Gamma(\{A, B\}) \). Assume on the contrary that \( \Lambda \cap \Lambda' \neq \Gamma(\{A, B\}) \). By the linear Lemma in [6], \( \dim(L_P) = 4 \), this implies that either all 3-spaces \( \Lambda \) are contained in a 4-dimensional linear space \( M \) or for every \( R \in L_P \setminus L_P' \), the 3-space \( \Lambda_R \) contains \( V \). The first possibility cannot occur because \( X_3 \) is non-degenerate and contained in \( W \) (Step 2) and \( W \) is the closure of the union of all \( \Lambda_R \) (First Claim). Assume that for every \( R \in L_P \setminus L_P' \), the 3-space \( \Lambda_R \) contains \( V \). The linear projection \( \alpha : X_3 \setminus X_3 \cap V \to \mathbb{P}^2 \) is dominant because the last assertion of the First Claim implies that \( \alpha \) does not contract infinitely many lines. Hence the linear projection of \( X_3 \) from the line \( \langle\{A, B\}\rangle \) into \( \mathbb{P}^3 \) is a cone. By the Lemma proved in [6], Step 5 at p. 625, \( X_3 \) is a cone, a contradiction.

Step 7. Here we will check that \( \langle\{A, B\}\rangle \) is the only line containing \( P \) and intersecting \( X_1 \setminus X_1 \cap X_2 \) and \( X_2 \setminus X_1 \cap X_2 \). Since the tangent developable of \( X_3 \) has dimension 4 and \( P \) is general, \( P \) is not contained in any line tangent to \( X_3 \) at one of its smooth points. Since \( \dim(L_P) = 4 \), the set \( D \) of all lines containing \( P \) and intersecting both \( X_1 \setminus X_1 \cap X_2 \) and \( X_2 \setminus X_1 \cap X_2 \) is finite. Now we will check that \( D = \{\langle\{A, B\}\rangle\} \). Take any \( D \in D \). By the finiteness of \( D \), \( D \) must be fixed as \( \Pi \) varies. Hence \( D \subseteq \Lambda \cap \Lambda' = \Gamma(\{A, B\}) \) (Step 6).

Step 8. Call \( \Gamma_0 \) the union of the one-dimensional components of \( \Lambda \cap X_3 \). Here we will check that for general \( \Pi \) the curve \( \Gamma_0 \) is a line. Recall that \( W \setminus \Pi \) is the closure of the union of all spaces \( \Lambda \) with \( \Pi \in L_P \). Let \( Y \subseteq \mathbb{P}^m \) be an irreducible \( m \)-dimensional variety, \( m \geq 2 \), containing a two-dimensional family of \((m-1)\)-dimensional linear spaces. By [6], Lemma in Step 9 of §2, \( Y \) is a linear space. Thus \( W \setminus \Pi \) contains only a one-dimensional family of distinct 3-spaces \( \Lambda \). Since \( \dim(L_P) = 3 \) and each plane of \( L_P \) belongs to some 3-space \( \Lambda \) contained in \( W \), it follows that the general plane \( U \) of \( \Lambda \) containing \( P \) intersects \( X_1 \), \( X_2 \) and \( X_3 \) and that, for general \( P, \Pi \) and \( U \), it intersects each \( X_i \) exactly at one point (see [6], Cor. 1.3). Hence \( \Gamma_0 \) is a line. Hence the variety \( X_3 \) contains an irreducible family \( \Phi \) of lines \( \Gamma_0, \Pi \) general in \( L_P \), with \( \Gamma_0 \subseteq \Lambda \). Since \( \Lambda \cap \Lambda' = \Gamma(\{A, B\}) \) for a general pair \((\Pi, \Pi') \) (Step 6), we have \( \dim(\Phi) > 0 \). Since \( X_3 \) is not a plane, we have \( \dim(\Phi) = 1 \). If all lines \( \Gamma_0 \) pass through a common point \( Q \), then \( X_3 \) is a cone with vertex \( Q \), contradicting our assumptions. Since not all lines \( \Gamma_0 \) pass through a common point and \( X_3 \) is not a plane, we have \( \Gamma_0 \cap \Gamma_0' = \emptyset \) for a general pair \((\Pi, \Pi') \) (Linear Lemma in [6], ¶1). We now give a side remark. Since \( X_3 \) is neither a plane nor a smooth quadric surface, \( \Phi \) is the only positive-dimensional
irreducible family of lines contained in $X_3$. Hence $\Phi$ does not depend on the choice of $P$, $A$ and $B$. For a general $B_3 \in X_3$ there is a unique line $D(B_3)$ such that $B_3 \in D(B_3) \subset X_4$. Since $\Phi$ covers $X_3$, we have $D(B_3) \in \Phi$. Notice that $D(B_3)$ depends only on $B_3$ and $X_3$, not on $X_1$, $X_2$ and the choices of $A$, $B$ and $P$ that we made to construct $\Phi$.

Step 9. Take a general triple $(A_1, A_2, A_3) \in X_1 \times X_2 \times X_3$. Hence $\langle \{A_1, A_2, A_3\} \rangle$ is a plane and a general element of $\text{Sec}(X; 2, k)$, and hence $\langle \{A_1, A_2, A_3\} \rangle \cap X_1 = \{A_i\}$ (Lemma 2). A general $P \in \langle \{A_1, A_2, A_3\} \rangle$ may be considered as a general element of $\mathbb{P}^5$ because $[X_1; [X_2; X_3]] = \mathbb{P}^5$. Since $[X_1; X_2] = \mathbb{P}^5$, there is $(B_1, B_2) \in X_1 \times X_2$ such that $B_1 \neq B_2$ and $P \in \langle \{B_1, B_2\} \rangle$: furthermore, there are only finitely many such pairs $(B_1, B_2)$. Conversely, given a general quadruple $(A'_1, A'_2, B'_1, B'_2) \in X_1 \times X_1 \times X_2 \times X_2$, the 3-dimensional linear space $\langle \{A'_1, A'_2, B'_1, B'_2\} \rangle$ intersects $X_3$ in a non-empty set and one point of this set, for fixed $A'_1$, $A'_2$ but for a general pair $(B'_1, B'_2)$, may be considered as a general point $A'_3$ of $X_3$; furthermore, for general $A'_1$ and $A'_2$ we may find $A'_3$ not collinear with $A'_1$ and $A'_2$. Hence $\langle \{A'_1, A'_2, A'_3\} \rangle$ is a plane of $\langle \{A'_1, A'_2, B'_1, B'_2\} \rangle$, and thus it intersects the line $\langle \{B'_1, B'_2\} \rangle$. Since $\dim(\langle \{A'_1, A'_2, B'_1, B'_2\} \rangle) = 3$, for general $B'_1$ and $B'_2$ we have $\langle \{A'_1, A'_2\} \rangle \cap \langle \{B'_1, B'_2\} \rangle = \emptyset$. Thus we may do the construction of Step 4 starting from $B'_1$, $B'_2$ and $P$ instead of $A$, $B$ and $P$.

By construction $\langle \{A'_1, A'_2, A'_3\} \rangle \in \Lambda_P \setminus \Lambda_{P'}$. Thus by Step 8 the 3-dimensional linear space $G$ spanned by $B'_1$, $B'_2$ and $\langle \{A'_1, A'_2, A'_3\} \rangle$ intersects $X_3$ in a line $D(A'_1, B'_1, A'_2, B'_2) \in \Phi$. However, $G = \langle \{A'_1, A'_2, B'_1, B'_2\} \rangle$. Thus for a general quadruple $(A'_1, A'_2, B'_1, B'_2) \in X_1 \times X_1 \times X_2 \times X_2$ the set $X_3 \cap \langle \{A'_1, A'_2, B'_1, B'_2\} \rangle$ contains a line. Furthermore, $\langle \{A'_1, A'_2, B'_1, B'_2\} \rangle$ is spanned by $D(A'_1, B'_1, A'_2, B'_2)$, $A'_1$ and $A'_2$. Since $\dim(\Phi) = 1$, we may find a one-dimensional irreducible family, $\Delta$, of pairs $(B''_1, B''_2) \in X_1 \times X_2$ such that $X_3 \cap \langle \{A'_1, A'_2, B'_1, B'_2\} \rangle = D(A'_1, B'_1, A'_2, B'_2)$ for every $(B''_1, B''_2) \in \Delta$. Hence $\langle \{A'_1, A'_2, B'_1, B'_2\} \rangle$ contains all such pairs $(B''_1, B''_2)$, contradicting Lemma 5 and hence concluding the proof.

Proof of Theorem 2 If $X \subset \mathbb{P}^m$, $m \geq 5$, is a non-degenerate surface such that $\dim([X; X]) = 4$, then $X$ is either a cone or a Veronese surface (2). Notice that the role of the surface $X_3$ in the proof of Theorem 1 was quite different from the roles of $X_1$ and $X_2$, while the roles of $X_1$ and $X_2$ were exactly the same. The same proof works in the case $X_1 = X_2$ (and proves Theorem 2), except that in Steps 4 and 9 we need to quote Lemma 8 instead of Lemma 2 and that in the proof of Theorem 1 we assumed and heavily used that $\dim([X_1; X_2]) = 5$. Hence to complete the proof of Theorem 2 it is sufficient to check it when $\dim([X_1; X_1]) \leq 4$ and $X_1$ is not a cone, i.e. when $X_1$ is the Veronese surface. Assume $N = 5$ and let $S \subset \mathbb{P}^5$ be the Veronese surface. Let $Y \subset \mathbb{P}^5$ be an integral non-degenerate surface with $Y \neq S$ and $Y$ not a cone. The pair $(S, Y)$ is not a $(1, 2, 1)$-defective pair if and only if a general line $D \subset \mathbb{P}^5$ is contained in a plane spanned by two points of $S$ and one point of $Y$. Let $Z \subset \mathbb{P}^5$ be the secant variety of $S$. Thus $S$ is the hypersurface of $\mathbb{P}^5$ union all planes spanned by the conics contained in $S$. Let $D \subset \mathbb{P}^5$ be a general line. Fix $P \in D \cap Z$ and call $E \subset Z$ the plane such that $P \in E$ and $E \cap S = C$, where $C$ is a smooth conic. Set $M := \langle E \cup D \rangle$. Thus $\dim(M) = 3$. Take $Q \in M \cap Y$ and set $F := E \cap \langle \{Q\} \cup D \rangle$. Hence $F$ is the intersection of two planes contained in $M$. The linear space $M$ moves if we move $D$ among the lines through $P$. For a general line $D$ containing $P$ the set $F$ is a line not tangent to $C$. 


Hence the general line $D$ containing $P$ is contained in the plane spanned by $Q$ and the two points of $F \cap C \subset S$, concluding the proof.

3. Further results on $(t,\tilde{k})$-defectivity

Let $\sigma : \mathbb{P}^t \times \mathbb{P}^N \to \mathbb{P}^{tN+tN}$ be the Segre embedding. The proof of [8], Th. 2.1 (i.e. a computation of a certain Jacobian matrix), gives the following result.

**Theorem 3.** $\tilde{X}$ is $(t,\tilde{k})$-defective if and only if $\sigma(\mathbb{P}^t \times \tilde{X}) := (\sigma(\mathbb{P}^t \times X_1), \ldots, \sigma(\mathbb{P}^t \times X_s))$ is $\tilde{k}$-defective and the total order of $(t,\tilde{k})$-defectivity of $\tilde{X}$ and the $\tilde{k}$-defectivity of $\sigma(\mathbb{P}^t \times \tilde{X})$ are the same.

To obtain results for the $(t,\tilde{k})$-defectivity it is essential to have results for the $\tilde{k}$-defectivity of other joins; Theorem 3 has just one reason. Since the role of the varieties $X_1, \ldots, X_s$ is not the same if $k_i \neq k_j$ for some $i, j$ or the geometric properties (and even the dimensions) of the varieties $X_1, \ldots, X_s$ may be quite different (as, for instance, in Question [4]), we introduce the following definition.

**Definition 1.** Fix $N, s, k_i, n_i, 1 \leq i \leq s$, and $t$ as above and assume that $\sum_{i=1}^s k_i(n_i + 1) \leq N - 1$. Fix an integer $i$ with $1 \leq i \leq s$ such that $k_i > 0$. Set $\tilde{k}(i) := (k'_1, \ldots, k'_s)$ with $k'_j = k_j$ if $j \neq i$ and $k'_i = k_i - 1$. Set $\delta(\tilde{X}; \tilde{k}; i) := \delta(\tilde{X}; \tilde{k}) - \delta(\tilde{X}; \tilde{k}(i))$. The integer $\delta(\tilde{X}; \tilde{k}; i)$ will be called the $\tilde{k}$-defect of $\tilde{X}$ for the factor $i$.

The proof of [2], Th. 1.1, gives the following result.

**Lemma 6.** Fix $N, \tilde{k}, i$ and $\tilde{X}$ as above and assume that each variety $X_j$ is non-degenerate. Assume $\delta(\tilde{X}; \tilde{k}; i) > 0$. Fix a general $P_{u,v} \in X_u, 1 \leq u \leq s$ and $1 \leq v \leq k_u$, and let $A$ be the linear span of the tangent spaces $(TX_u)_{P_{u,v}}, 1 \leq u \leq s$ and $1 \leq v \leq k_u$. Then the general hyperplane $H$ of $\mathbb{P}^N$ containing $A$ is tangent to $X_i$ at least along an irreducible variety of dimension $\delta(\tilde{X}; \tilde{k}; i)$ containing one of the points $P_{i,j}$ with $1 \leq j \leq k_i$.

**Corollary 1.** Fix $N, \tilde{k}, i$ and $\tilde{X}$ as above and assume that each variety $X_j$ is non-degenerate. Assume $\delta(\tilde{X}; \tilde{k}; i) > 0$. Then $\dim(X_i) > \delta(\tilde{X}; \tilde{k}; i)$ and in particular $X_i$ is not a curve.

**Remark 3.** In Lemma 6 and Corollary 1 we may substitute the condition that each variety $X_j$ is non-degenerate with the condition that for each proper subspace $M$ of $\mathbb{P}^N$ containing some of the varieties $X_j$, say $X_j$ for $j \in S \subset \{1, \ldots, s\}$, we have $\sum_{j \in S} k_j(\dim(X_j) + 1) \leq \dim(M)$.

**Theorem 4.** Fix an integer $s \geq 2$, positive integers $k_1, \ldots, k_s$ such that $k_s = 1$ and integral subvarieties $X_i \subset \mathbb{P}^N$ such that $X_j$ is non-degenerate for every $j < s$ and $\dim(X_s) = 1$. Set $\tilde{k} := (k_1, \ldots, k_s)$, $\tilde{k}(s) := (k_1, \ldots, k_{s-1}), \tilde{X} := (X_1, \ldots, X_s)$ and $\tilde{X}(s) := (X_1, \ldots, X_s)$. Assume that $\tilde{X}(s)$ is neither $\tilde{k}(s)$-defective nor $(1, \tilde{k}(s))$-defective and that $\text{Sec}(\tilde{X}(s); \tilde{k}(s))$ is not a cone with vertex containing $X_s$. Then $\tilde{X}$ is not $(1, \tilde{k})$-defective.

**Proof.** Assume that $\tilde{X}$ is $(1, \tilde{k})$-defective. A general line $L \subset \text{Sec}(\tilde{X}; 1, \tilde{k})$ is obtained by taking general $P_{i,j} \in X_i, 1 \leq i \leq s, 1 \leq j \leq k_i$ and then taking a general line $L$ contained in the $|\tilde{k}|$-dimensional linear space spanned by the points $P_{i,j}$. Let
$A(s)$ be the join of all points $P_{i,j}$ with $1 \leq i \leq s - 1$ and $1 \leq j \leq k_i$. Set $Q(L) := A(s) \cap L$. Hence $Q(L) \in \text{Sec}(\tilde{X}(s); \tilde{k}(s))$. By the generality of the points $P_{i,j}$ and of the line $L$ the point $Q(L)$ may be considered as a general point of $\text{Sec}(\tilde{X}(s); \tilde{k}(s))$. By assumption there is a one-dimensional family of $[\tilde{k}]$-planes, say $\{\Pi_t\}_{t \in T}$ with $T$ irreducible curve such that each $\Pi_t$ intersects each $X_i$ at $k_i$ distinct points, say $P_{t,i}(t) \subset \Pi_t$ for every $t$, and there is $o \in T$ such that $P_{t,i}(o) = P_{i,j}$ for all $i,j$. Let $A(s,t)$ be the join of all points $P_{i,j}(t)$ with $1 \leq i \leq s - 1$ and $1 \leq j \leq k_i$. Set $Q(L,t) := A(s,t) \cap L$. Hence $Q(L,t) \in A(s,t)$. First assume $Q(L,t) = Q(L)$ for every $t$. Since $Q(L,t) := A(s,t) \cap L$, this implies that $Q(L)$ is contained in infinitely many $([\tilde{k}] - 1)$-dimensional linear spaces belonging to $\text{Sec}(\tilde{X}(s); [\tilde{k}(s)] - 1, \tilde{k}(s))$. Since $Q(L)$ is general in $\text{Sec}(\tilde{X}(s); \tilde{k}(s))$, this implies $\text{dim}(\text{Sec}(\tilde{X}(s); \tilde{k}(s)))) \leq \sum_{i=1}^{s-1} k_i(\text{dim}(X_i) + 1) - 2$, a contradiction. If $Q(L,t) \neq Q(L)$ for general $t \in T$, then $L$ is contained in $\text{Sec}(\tilde{X}(s); \tilde{k}(s))$. By the generality of $L$ we obtain $\text{Sec}(\tilde{X}; \tilde{k}) = \text{Sec}(\tilde{X}(s); \tilde{k}(s))$, i.e. $\text{Sec}(\tilde{X}(s); \tilde{k}(s))$ is a cone with vertex containing $X_s$, a contradiction. \hfill $\square$

REFERENCES


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