A NOTE ON PRINCIPAL PARTS ON PROJECTIVE SPACE AND LINEAR REPRESENTATIONS

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Abstract. Let $H$ be a closed subgroup of a linear algebraic group $G$ defined over a field of characteristic zero. There is an equivalence of categories between the category of linear finite-dimensional representations of $H$, and the category of finite rank $G$-homogeneous vector bundles on $G/H$. In this paper we will study this correspondence for the sheaves of principal parts on projective space, and we describe the representation corresponding to the principal parts of a line bundle on projective space.

1. Introduction

In this note we will study the vector bundles of principal parts $P^k(O(n))$ of a line bundle on projective space over a field $F$ of characteristic zero from a representation theoretic point of view. We consider projective $N$-space as a quotient $SL(V)/P$, where $V$ is an $(N + 1)$-dimensional vector space over $F$, and $P$ is the subgroup of $SL(V)$ stabilizing a line $L$ in $V$. There is an equivalence of categories between the category of finite rank $SL(V)$-homogeneous vector bundles on $SL(V)/P = \mathbb{P}(V^*)$ and the category of linear finite-dimensional representations of $P$. The principal parts $P^k(O(n))$ are $SL(V)$-homogeneous vector bundles on $\mathbb{P}(V^*)$, and the novelty of this note is that we describe the $P$-representation corresponding to the principal parts. The main result is Theorem 2.4, which says the following: Let $L^*$ be the dual of the $P$-module $L$. Then for all $1 \leq k < n$, the $P$-representation corresponding to $P^k(O(n))$ is $S^{n-k}(L^*) \otimes S^k(V^*)$. As a corollary, we obtain the splitting type of $P^k(O(n))$ on $\mathbb{P}(V^*)$ for all $1 \leq k < n$, and recover results obtained in [6], [7], [8] and [9].

2. Principal parts on projective space

In this section we give the representation corresponding to $P^k(O(n))$ on $\mathbb{P}(V^*)$ for all $1 \leq k < n$, where $V$ is an $F$-vector space of dimension $N + 1$ and $F$ is a field of characteristic $0$. A variety is an integral scheme of finite type over $F$.

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We will consider closed points when we talk about points of a scheme. Let $V$ be a finite-dimensional vector space over $F$. We let $GL(V)$ denote the group of all invertible linear transformations of $V$. It is an algebraic group in the sense of [2], Chapt. 1. A linear algebraic group is a closed subgroup of $GL(V)$. Let $SL(V)$ be the linear algebraic group of linear transformations of $V$ with determinant 1. Let $L$ in $V$ be a line, and $P$ the closed subgroup of $SL(V)$ stabilizing $L$. Then the quotient $SL(V)/P$ (which exists by [2], Theorem 6.8) is isomorphic to $P(V^*)$, the projective space of lines in $V$ (see [1], Section 4.2). This works over any field, not only the complex numbers. There exists a natural left $SL(V)$-action on $P(V^*)$, making it into a homogeneous space for $SL(V)$. Also by [1], Chapt. 4, there exists an equivalence of categories between the category of finite rank homogeneous vector bundles on $SL(V)/P$ and the category of linear finite-dimensional representations of $P$, and under this correspondence the dimension of the representation gives the rank of the corresponding vector bundle. Hence any character of $P$ gives a homogeneous line bundle on $SL(V)/P$. The line $L$ corresponds to a character of $P$, and the bundle corresponding to the dual line $L^*$ is the line bundle $O(1)$ on $P(V^*)$ (see [1], Section 4.2). It is also a standard fact that any linear finite-dimensional representation $\rho$ of $P$ lifting to a representation $\tilde{\rho}$ of $SL(V)$ corresponds to a trivial abstract vector bundle on $P(V^*)$. There exists on any scheme an equivalence of categories between the category of locally free finite rank sheaves and the category of finite rank vector bundles; hence we will use these two notions interchangeably.

Pick a basis $e_0, \ldots, e_N$ for $V$. Let $x_0, \ldots, x_N$ be the dual basis, and let $L$ be the line spanned by $e_0$. Having chosen a basis for $V$, it follows that $SL(V)$ may be identified with the group of square rank $N + 1$ matrices with determinant equal to 1. The group $P$ may be identified with the subgroup of $SL(V)$ consisting of matrices $g$ of the form

$$g = \begin{pmatrix} a & * & \cdots & * \\ 0 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n1} & \cdots & a_{nn} \end{pmatrix}. $$

The one-dimensional representation $\chi : P \to GL(S^n(L^*))$ corresponding to the line bundle $O(n)$ is given by $\chi(g) = a^{-n}$.

Let $X$ be a smooth variety of dimension $d$ and consider the diagonal $\Delta$ in $X \times X$. Let $I$ be the sheaf of ideals of $O_{X \times X}$ defining the diagonal $\Delta$, and define $O_{\Delta^k}$ to be $O_{X \times X}/I^{k+1}$.

**Definition 2.1.** Let $p, q$ be the projection maps from $X \times X$ to $X$, and let $E$ be an $O_X$-module. Define $P^k(E) = p_*(O_{\Delta^k} \otimes q^*E)$ to be the $k$th order principal parts of the module $E$. We put $P^k(O_X) = P^k$.

Note that by [6], if the rank of $E$ is $e$, $P^k(E)$ is a vector bundle of rank $e^{(d+k)}$ on $X$. Assume that $G$ is an algebraic group, and that $X$ is a homogeneous space for $G$. Assume furthermore that $E$ is a $G$-homogeneous vector bundle on $X$; then it follows that $P^k(E)$ is again a $G$-homogeneous vector bundle on $X$. Consider the line bundle $O(n)$ on $P(V^*) = SL(V)/P$; then $O(n)$ is an $SL(V)$-homogeneous line bundle on $P(V^*)$ for all $n$, and we may consider the $SL(V)$-homogeneous vector bundle $P^k(O(n))$. We want to compute the representation $\rho$ of $P$ corresponding to the homogeneous vector bundle $P^k(O(n))$ for all $1 \leq k < n$ on $P(V^*)$. Let, in the following, $X = P(V^*)$ and consider the projection maps $p, q$ from $X \times X$ to
Let $I$ in $\mathcal{O}_{X \times X}$ be the ideal of the diagonal. We have an exact sequence of $SL(V)$-homogeneous vector bundles on $X \times X$:

\begin{equation}
(2.1.1) \quad 0 \to I^{k+1} \to \mathcal{O}_{X \times X} \to \mathcal{O}_{\Delta_k} \to 0.
\end{equation}

Apply the functor $p_*(- \otimes q^*\mathcal{O}(n))$ to the sequence $(2.1.1)$ to get a long exact sequence

\begin{equation}
(2.1.2) \quad 0 \to p_*(I^{k+1} \otimes q^*\mathcal{O}(n)) \to p_*q^*\mathcal{O}(n) \to \mathcal{P}^k(\mathcal{O}(n))
\end{equation}

of vector bundles. The sequence $(2.1.2)$ is a sequence of vector bundles because all sheaves in the sequence are coherent, and it is a standard fact that a homogeneous coherent sheaf on a homogeneous space is locally free. Since the sequence $(2.1.2)$ is a sequence of vector bundles, we get an exact sequence of $P$-representations when we pass to the fiber at $\pi$.

Consider the diagram

$$
\begin{array}{ccc}
\text{Spec}(\kappa(\pi)) \times X & \xrightarrow{\pi} & X \times X \\
\downarrow & & \downarrow \pi \\
\text{Spec}(\kappa(\pi)) & \xrightarrow{i} & X
\end{array}
$$

Then by [5], Chapt. III, Sect. 12, we get maps

$$
\phi^i : R^i p_*(I^{k+1} \otimes q^*\mathcal{O}(n))(\pi) \to R^i \pi_*(j^*(I^{k+1} \otimes q^*\mathcal{O}(n)))
$$

of $\mathcal{O}_X$-modules. Put, for any $\mathcal{O}_{X \times X}$-module $\mathcal{E}$,

$$
h^i(y, E) = \dim_{\kappa(y)} H^i(X_y, \mathcal{E}_y),
$$

where $X_y$ is the fiber $p^{-1}(y)$ and $\mathcal{E}_y$ is the restriction of $\mathcal{E}$ to $X_y$. We see that

$$
h^i(y, I^{k+1} \otimes q^*\mathcal{O}(n)) = \dim_{\kappa(y)} H^i(X, m_y^{k+1} \otimes \mathcal{O}(n))
$$

is a constant function of $y$ for $i = 0, 1, \ldots$ for the following reason: Consider the commutative diagram

$$
\begin{array}{ccc}
\text{Spec}(\kappa(y)) \times X & \xrightarrow{\bar{g}} & \text{Spec}(\kappa(gv)) \times X \\
\downarrow f & & \downarrow k \\
X \times X & \xrightarrow{g} & X \times X
\end{array}
$$

where the action of $SL(V)$ on $X \times X$ is given by $g(x, y) = (gx, gy)$. In general, if $G \times Y \to Y$ is an algebraic group acting on a scheme $Y$, and $\mathcal{E}$ is a $G$-linearized sheaf on $Y$, then there exists an isomorphism $I : \sigma^* \mathcal{E} \to p^* \mathcal{E}$, where $p : G \times Y \to Y$ is the projection map. It follows that for all $g \in G$ we get an isomorphism $g^* \mathcal{E} \cong \mathcal{E}$ of sheaves. Then, since $\mathcal{O}(n)$ and $I^{k+1} \otimes q^*\mathcal{O}(n)$ are $SL(V)$-homogeneous sheaves, we have an isomorphism

$$
\bar{g}^* (m_y^{k+1} \otimes \mathcal{O}(n)) = j^* g^* (I^{k+1} \otimes q^*\mathcal{O}(n)) = m_y^{k+1} \otimes \mathcal{O}(n);
$$

hence since $\bar{g}$ is an isomorphism, we see that we have an isomorphism

$$
m_y^{k+1} \otimes \mathcal{O}(n) \cong m_y^{k+1} \otimes \mathcal{O}(n).$$
of sheaves for all $g$ in $SL(V)$. It follows that
\[
\dim_{\kappa(y)} H^i(X, m_y^{k+1} \otimes \mathcal{O}(n)) = \dim_{\kappa(\varphi y)} H^i(X, m_y^{k+1}, \mathcal{O}(n))
\]
for all $g$ in $SL(V)$; hence by [5], Chapt. III, Corr. 12.9, it follows that the maps $\phi^i$ are isomorphisms for $i = 0, 1, \ldots$. Here $m_y$ is the sheaf of ideals corresponding to the point $y$ in $X$. We get an exact sequence
\[
0 \twoheadrightarrow H^0(X, \mathcal{O}(n)) \otimes m_y^{k+1} \rightarrow H^0(X, \mathcal{O}(n)) \rightarrow \mathcal{P}^k(\mathcal{O}(n))(\varphi)
\]
\[
\rightarrow H^1(X, \mathcal{O}(n)) \otimes m_y^{k+1} \rightarrow H^1(X, \mathcal{O}(n)) \rightarrow \cdots
\]
of $P$-representations.

**Lemma 2.2.** For all $1 \leq k < n$ we have that $H^1(X, \mathcal{O}(n)) \otimes m_y^{k+1} = 0$.

**Proof.** Consider the exact sequence (2.1.3). We prove that
\[
\dim_F H^0(X, \mathcal{O}(n)) - \dim_F H^0(X, \mathcal{O}(n)) \otimes m_y^{k+1} = \dim_F \mathcal{P}^k(\mathcal{O}(n))(\varphi),
\]
and then the result follows by counting dimensions. We have that $H^0(X, \mathcal{O}(n))$ equals $S^n(V^*)$, where $V^*$ is the $F$-vector space on the basis $x_0, \ldots, x_N$. We also see that $H^0(X, \mathcal{O}(n) \otimes m_y^{k+1})$ equals $m_y^{k+1} S^{n-(k+1)}(V^*)$ considered as a subvector-space of $S^n(V^*)$. Here $m_y$ is the $F$-vector space on the basis $x_1, \ldots, x_N$ and $m_y^{k+1} S^{n-(k+1)}(V^*)$ is the image of the natural map
\[
S^{k+1}(m) \otimes S^{n-(k+1)}(V^*) \rightarrow S^n(V^*).
\]
Write $V^*$ as the direct sum $Fx_0 \oplus m$. Then it follows that
\[
m_y^{k+1} S^{n-(k+1)}(V^*) = x_0^{n-(k+1)} m_y^{k+1} \oplus \cdots \oplus x_0 m_y^{n-1} \oplus m;
\]

hence we see that the dimension of $m_y^{k+1} S^{n-(k+1)}(V^*)$ equals $\sum_{i=k+1}^n \binom{i+N-1}{N-1}$. We also see that the dimension of $S^n(V^*)$ equals $\sum_{i=0}^n \binom{i+N-1}{N-1}$, and it follows that
\[
\dim_F S^n(V^*) - \dim_F m_y^{k+1} S^{n-(k+1)}(V^*) = \sum_{i=0}^k \binom{i+N-1}{N-1},
\]
It follows that
\[
\sum_{i=0}^k \binom{i+N-1}{N-1} = \binom{k+N}{N} = \dim_F \mathcal{P}^k(\mathcal{O}(n))(\varphi),
\]
and we have proved that
\[
\dim_F H^0(X, \mathcal{O}(n)) - \dim_F H^0(X, \mathcal{O}(n)) \otimes m_y^{k+1} = \dim_F \mathcal{P}^k(\mathcal{O}(n))(\varphi).
\]
The result follows from the fact that the sequence (2.1.3) is exact and that $H^1(X, \mathcal{O}(n)) = 0$ for $n \geq 1$. \qed

Note that by Lemma 222 and the sequence (2.1.3), there exists for all $1 \leq k < n$ an exact sequence of $P$-representations
\[
0 \rightarrow H^0(X, \mathcal{O}(n)) \otimes m_y^{k+1} \rightarrow H^0(X, \mathcal{O}(n)) \rightarrow \mathcal{P}^k(\mathcal{O}(n))(\varphi) \rightarrow 0.
\]
Since the representation $H^0(X, \mathcal{O}(n)) \otimes m_y^{k+1}$ equals $m_y^{k+1} S^{n-(k+1)}(V^*)$ as a subrepresentation of $H^0(X, \mathcal{O}(n)) = S^n(V^*)$, it follows that we have an exact sequence of $P$-representations
\[
0 \rightarrow m_y^{k+1} S^{n-(k+1)}(V^*) \rightarrow S^n(V^*) \rightarrow \mathcal{P}^k(\mathcal{O}(n))(\varphi) \rightarrow 0.
\]
From the exact sequence
\[ 0 \to m \to V^* \to V^*/m \to 0, \]
where \( m \) is the \( F \)-vector space on \( x_1, \ldots, x_N \), we see that the representation \( V^*/m \) is the representation corresponding to the module \( L^* \) of \( P \), giving the line bundle \( \mathcal{O}(1) \) on \( X = SL(V)/P \).

**Lemma 2.3.** For all \( 1 \leq k < n \) there exists a surjective map of \( P \)-representations
\[ \phi : S^n(V^*) \to S^{n-k}(L^*) \otimes S^k(V^*). \]

**Proof.** Recall that we have chosen a basis \( e_0, \ldots, e_N \) for \( V \), with the property that \( \pi_0 \) is a basis for \( L^* \). The \( P \)-representation \( m \) with basis \( x_1, \ldots, x_N \) gives an exact sequence
\[ 0 \to m \to V^* \to L^* \to 0 \]
of \( P \)-representations. Define a map
\[ \phi : S^n(V^*) \to S^{n-k}(L^*) \otimes S^k(V^*) \]
as follows: \( \phi(f) = \pi_0^{n-k} \otimes \partial_0^{n-k}(f) \), where \( \partial^{n-k}_0 \) is the \( n-k \) times partial derivative with respect to the \( x_0 \)-variable. Let \( g \) be an element of \( P \). Then by induction on the degree of the differential operator \( \partial^{n-k}_0 \) and applying the chain rule for derivation, it follows that
\[
\phi(gf) = \pi_0^{n-k} \otimes \partial_0^{n-k}(gf) = \pi_0^{n-k} \otimes a^{-(n-k)}g(\partial_0^{n-k}f) = a^{-(n-k)}\pi_0^{n-k} \otimes \partial_0^{n-k}(gf) = g\phi(f),
\]
and we see that \( \phi \) is \( P \)-linear. It is clearly surjective, and the lemma follows. \( \square \)

**Theorem 2.4.** For all \( 1 \leq k < n \), the representation corresponding to \( P^k(\mathcal{O}(n)) \) is \( S^{n-k}(L^*) \otimes S^k(V^*) \).

**Proof.** By Lemma 2.3 there exists a surjective map of \( P \)-representations
\[ \phi : S^n(V^*) \to S^{n-k}(L^*) \otimes S^k(V^*). \]

We claim that \( m^{k+1}S^{n-(k+1)}(V^*) \) equals \( \ker \phi \): We first prove the inclusion
\[ m^{k+1}S^{n-(k+1)}(V^*) \subseteq \ker \phi. \]

Pick a monomial \( x_0^{p_0}x_1^{p_1} \cdots x_N^{p_N} \) in \( m^{k+1}S^{n-(k+1)}(V^*) \); then \( p_0 + \cdots + p_N = n \) and \( p_0 < n-k \). These monomials form a basis for \( m^{k+1}S^{n-(k+1)}(V^*) \). We see that \( \partial^{n-k}_0(x_0^{p_0} \cdots x_N^{p_N}) \) is zero; hence, since \( \phi \) is a linear map, it follows that we have an inclusion
\[ m^{k+1}S^{n-(k+1)}(V^*) \subseteq \ker \phi \]
of vector spaces. The reverse inclusion follows from counting dimensions and the fact that \( \phi \) is surjective: We have that
\[
\dim_F \ker \phi = \dim_F S^n(V^*) - \dim_F S^{n-k}(V^*) \otimes S^k(V^*)
\]
\[
eq \sum_{i=0}^{n} \binom{i+N-1}{N-1} - \sum_{i=0}^{k} \binom{i+N-1}{N-1} = \sum_{i=k+1}^{n} \binom{i+N-1}{N-1},
\]
and we see that \( \dim_F \ker \phi = \dim_F m^{k+1} S^{n-(k+1)}(V^*) \). It follows that
\[
m^{k+1} S^{n-(k+1)}(V^*) = \ker \phi;
\]
hence we have an exact sequence of \( P \)-representations
\[
0 \to m^{k+1} S^{n-(k+1)}(V^*) \to S^n(V*) \to S^{n-k}(L^*) \otimes S^k(V^*) \to 0.
\]
Using sequence (2.2.1) we get isomorphisms
\[
P^k(\mathcal{O}(n))(\pi) \cong H^0(X, \mathcal{O}(n))/H^0(X, \mathcal{O}(n) \otimes m^{k+1})
\cong S^n(V^*)/m^{k+1} S^{n-(k+1)}(V^*) \cong S^{n-k}(V^*) \otimes S^k(V^*),
\]
and it follows that \( P^k(\mathcal{O}(n))(\pi) \) is isomorphic to \( S^{n-k}(L^*) \otimes S^k(V^*) \) as a representation. \( \square \)

Note that the result in Theorem 2.4 is true if \( \text{char}(F) > n \).

**Corollary 2.5.** For all \( 1 \leq k < n \), \( P^k(\mathcal{O}(n)) \) splits as an abstract vector bundle as \( \bigoplus_{N}^{N+k} \mathcal{O}(n-k) \).

**Proof.** Since \( S^k(V^*) \) corresponds to the trivial rank \( (n,k) \) abstract vector bundle on \( P(V^*) \), and \( S^{n-k}(L^*) \) corresponds to the line bundle \( \mathcal{O}(n-k) \), the assertion is proved. \( \square \)

We see that we recover results on the splitting type of the principal parts obtained in [6], [7], [8] and [9].

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