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A NOTE ON PRINCIPAL PARTS ON PROJECTIVE SPACE AND LINEAR REPRESENTATIONS

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Abstract. Let H be a closed subgroup of a linear algebraic group G defined over a field of characteristic zero. There is an equivalence of categories between the category of linear finite-dimensional representations of H, and the category of finite rank G-homogeneous vector bundles on G/H. In this paper we will study this correspondence for the sheaves of principal parts on projective space, and we describe the representation corresponding to the principal parts of a line bundle on projective space.

1. Introduction

In this note we will study the vector bundles of principal parts $\mathcal{P}^k(\mathcal{O}(n))$ of a line bundle on projective space over a field F of characteristic zero from a representation theoretic point of view. We consider projective N-space as a quotient $SL(V)/P$, where V is an $(N+1)$-dimensional vector space over F, and P is the subgroup of $SL(V)$ stabilizing a line L in V. There is an equivalence of categories between the category of finite rank $SL(V)$-homogeneous vector bundles on $SL(V)/P = \mathbf{P}(V^*)$ and the category of linear finite-dimensional representations of P. The principal parts $\mathcal{P}^k(\mathcal{O}(n))$ are $SL(V)$-homogeneous vector bundles on $\mathbf{P}(V^*)$, and the novelty of this note is that we describe the P-representation corresponding to the principal parts. The main result is Theorem 2.4, which says the following: Let L^* be the dual of the P-module L. Then for all $1 \leq k < n$, the P-representation corresponding to $\mathcal{P}^k(\mathcal{O}(n))$ is $S^{n-k}(L^*) \otimes S^k(V^*)$. As a corollary, we obtain the splitting type of $\mathcal{P}^k(\mathcal{O}(n))$ on $\mathbf{P}(V^*)$ for all $1 \leq k < n$, and recover results obtained in [6], [7], [8] and [9].

2. Principal parts on projective space

In this section we give the representation corresponding to $\mathcal{P}^k(\mathcal{O}(n))$ on $\mathbf{P}(V^*)$ for all $1 \leq k < n$, where V is an F-vector space of dimension $N+1$ and F is a field of characteristic 0. A variety is an integral scheme of finite type over F.

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We will consider closed points when we talk about points of a scheme. Let V be a finite-dimensional vector space over F. We let $GL(V)$ denote the group of all invertible linear transformations of V. It is an algebraic group in the sense of [2], Chapt. 1. A *linear algebraic group* is a closed subgroup of $GL(V)$. Let $SL(V)$ be the linear algebraic group of linear transformations of V with determinant 1. Let L in V be a line, and P the closed subgroup of $SL(V)$ stabilizing L. Then the quotient $SL(V)/P$ (which exists by [2], Theorem 6.8) is isomorphic to $\mathbf{P}(V^*)$, the projective space of lines in V (see [1], Section 4.2). This works over any field, not only the complex numbers. There exists a natural left $SL(V)$-action on $\mathbf{P}(V^*)$, making it into a *homogeneous space* for $SL(V)$. Also by [1], Chapt. 4, there exists an equivalence of categories between the category of finite rank homogeneous vector bundles on $SL(V)/P$ and the category of linear finite-dimensional representations of P, and under this correspondence the dimension of the representation gives the rank of the corresponding vector bundle. Hence any character of P gives a homogeneous line bundle on $SL(V)/P$. The line L corresponds to a character of P, and the bundle corresponding to the dual line L^* is the line bundle $\mathcal{O}(1)$ on $\mathbf{P}(V^*)$ (see [1], Section 4.2). It is also a standard fact that any linear finite-dimensional representation ρ of P lifting to a representation $\tilde{\rho}$ of $SL(V)$ corresponds to a trivial abstract vector bundle on $\mathbf{P}(V^*)$. There exists on any scheme an equivalence of categories between the category of locally free finite rank sheaves and the category of finite rank vector bundles; hence we will use these two notions interchangeably.

Pick a basis $e_0, \ldots, e_N$ for V. Let $x_0, \ldots, x_N$ be the dual basis, and let L be the line spanned by e_0. Having chosen a basis for V, it follows that $SL(V)$ may be identified with the group of square rank $N+1$ matrices with determinant equal to 1. The group P may be identified with the subgroup of $SL(V)$ consisting of matrices g of the form

$$g = \begin{pmatrix} a & * & \cdots & * \\ 0 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

The one-dimensional representation $\chi : P \to GL(S^n(L^*))$ corresponding to the line bundle $\mathcal{O}(n)$ is given by $\chi(g) = a^{-n}$.

Let X be a smooth variety of dimension d and consider the diagonal Δ in $X \times X$. Let $\mathcal{I}$ be the sheaf of ideals of $\mathcal{O}_{X\times X}$ defining the diagonal Δ, and define $\mathcal{O}_{\Delta^k}$ to be $\mathcal{O}_{X\times X}/\mathcal{I}^{k+1}$.

Definition 2.1. Let p, q be the projection maps from $X \times X$ to X, and let $\mathcal{E}$ be an $\mathcal{O}_X$-module. Define $\mathcal{P}^k(\mathcal{E}) = p_*(\mathcal{O}_{\Delta^k} \otimes q^*\mathcal{E})$ to be the *kth order principal parts* of the module $\mathcal{E}$. We put $\mathcal{P}^k(\mathcal{O}_X) = \mathcal{P}^k$.

Note that by [6], if the rank of $\mathcal{E}$ is e, $\mathcal{P}^k(\mathcal{E})$ is a vector bundle of rank $e\binom{d+k}{d}$ on X. Assume that G is an algebraic group, and that X is a homogeneous space for G. Assume furthermore that $\mathcal{E}$ is a G-homogeneous vector bundle on X; then it follows that $\mathcal{P}^k(\mathcal{E})$ is again a G-homogeneous vector bundle on X. Consider the line bundle $\mathcal{O}(n)$ on $\mathbf{P}(V^*) = SL(V)/P$; then $\mathcal{O}(n)$ is an $SL(V)$-homogeneous line bundle on $\mathbf{P}(V^*)$ for all n, and we may consider the $SL(V)$-homogeneous vector bundle $\mathcal{P}^k(\mathcal{O}(n))$. We want to compute the representation ρ of P corresponding to the homogeneous vector bundle $\mathcal{P}^k(\mathcal{O}(n))$ for all $1 \leq k < n$ on $\mathbf{P}(V^*)$. Let, in the following, $X = \mathbf{P}(V^*)$ and consider the projection maps p, q from $X \times X$ to

X. Let $\mathcal{I}$ in $\mathcal{O}_{X\times X}$ be the ideal of the diagonal. We have an exact sequence of $SL(V)$-homogeneous vector bundles on $X \times X$:

$$0 \to \mathcal{I}^{k+1} \to \mathcal{O}_{X\times X} \to \mathcal{O}_{\Delta^k} \to 0. \tag{2.1.1}$$

Apply the functor $p_*(- \otimes q^*\mathcal{O}(n))$ to the sequence (2.1.1) to get a long exact sequence

$$0 \to p_*(\mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n)) \to p_*q^*\mathcal{O}(n) \to \mathcal{P}^k(\mathcal{O}(n)) \tag{2.1.2}$$

$$\to R^1p_*(\mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n)) \to R^1p_*q^*\mathcal{O}(n) \to R^1p_*(\mathcal{O}_{\Delta^k} \otimes q^*\mathcal{O}(n)) \to \cdots$$

of vector bundles. The sequence (2.1.2) is a sequence of vector bundles because all sheaves in the sequence are coherent, and it is a standard fact that a homogeneous coherent sheaf on a homogeneous space is locally free. Since the sequence (2.1.2) is a sequence of vector bundles, we get an exact sequence of P-representations when we pass to the fiber at $\overline{e}$.

Consider the diagram

$$\begin{array}{ccc} \operatorname{Spec}(\kappa(\overline{e})) \times X & \xrightarrow{j} & X \times X \\ \downarrow \pi & & \downarrow p \\ \operatorname{Spec}(\kappa(\overline{e})) & \xrightarrow{i} & X \end{array}$$

Then by [5], Chapt. III, Sect. 12, we get maps

$$\phi^i : R^ip_*(\mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n))(\overline{e}) \to R^i\pi_*(j^*(\mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n)))$$

of $\mathcal{O}_X$-modules. Put, for any $\mathcal{O}_{X\times X}$-module $\mathcal{E}$,

$$h^i(y, E) = \dim_{\kappa(y)} \mathrm{H}^i(X_y, \mathcal{E}_y),$$

where X_y is the fiber $p^{-1}(y)$ and $\mathcal{E}_y$ is the restriction of $\mathcal{E}$ to X_y. We see that

$$h^i(y, \mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n)) = \dim_{\kappa(y)} \mathrm{H}^i(X, \mathfrak{m}_y^{k+1} \otimes \mathcal{O}(n))$$

is a constant function of y for $i = 0, 1, \ldots$ for the following reason: Consider the commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(\kappa(y)) \times X & \xrightarrow{\tilde{g}} & \operatorname{Spec}(\kappa(gv)) \times X \\ \downarrow j & & \downarrow k \\ X \times X & \xrightarrow{g} & X \times X \end{array}$$

where the action of $SL(V)$ on $X \times X$ is given by $g(x, y) = (gx, gy)$. In general, if $G \times Y \to^{\sigma} Y$ is an algebraic group acting on a scheme Y, and $\mathcal{E}$ is a G-linearized sheaf on Y, then there exists an isomorphism $I : \sigma^*\mathcal{E} \to p^*\mathcal{E}$, where $p : G \times Y \to Y$ is the projection map. It follows that for all $g \in G$ we get an isomorphism $g^*\mathcal{E} \cong \mathcal{E}$ of sheaves. Then, since $\mathcal{O}(n)$ and $\mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n)$ are $SL(V)$-homogeneous sheaves, we have an isomorphism

$$\tilde{g}^*(\mathfrak{m}_{gy}^{k+1} \otimes \mathcal{O}(n)) = j^*g^*(\mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n)) = \mathfrak{m}_y^{k+1} \otimes \mathcal{O}(n);$$

hence since $\tilde{g}$ is an isomorphism, we see that we have an isomorphism

$$\mathfrak{m}_y^{k+1} \otimes \mathcal{O}(n) \cong \mathfrak{m}_{gy}^{k+1} \otimes \mathcal{O}(n)$$

of sheaves for all g in $SL(V)$. It follows that

$$\dim_{\kappa(y)} \mathrm{H}^i(X, \mathfrak{m}_y^{k+1} \otimes \mathcal{O}(n)) = \dim_{\kappa(gy)} \mathrm{H}^i(X, \mathfrak{m}_{gy}^{k+1}, \mathcal{O}(n))$$

for all g in $SL(V)$; hence by [5], Chapt. III, Corr. 12.9, it follows that the maps ϕ^i are isomorphisms for $i = 0, 1, \ldots$. Here $\mathfrak{m}_y$ is the sheaf of ideals corresponding to the point y in X. We get an exact sequence

$$\begin{aligned} 0 \to \mathrm{H}^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\overline{e}}^{k+1}) \to \mathrm{H}^0(X, \mathcal{O}(n)) \to \mathcal{P}^k(\mathcal{O}(n))(\overline{e}) \\ \to \mathrm{H}^1(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\overline{e}}^{k+1}) \to \mathrm{H}^1(X, \mathcal{O}(n)) \to \cdots \end{aligned} \tag{2.1.3}$$

of P-representations.

Lemma 2.2. *For all* $1 \leq k < n$ *we have that* $\mathrm{H}^1(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\overline{e}}^{k+1}) = 0$.

Proof. Consider the exact sequence (2.1.3). We prove that

$$\dim_F \mathrm{H}^0(X, \mathcal{O}(n)) - \dim_F \mathrm{H}^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\overline{e}}^{k+1}) = \dim_F \mathcal{P}^k(\mathcal{O}(n))(\overline{e}),$$

and then the result follows by counting dimensions. We have that $\mathrm{H}^0(X, \mathcal{O}(n))$ equals $S^n(V^*)$, where V^* is the F-vector space on the basis $x_0, \ldots, x_N$. We also see that $\mathrm{H}^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\overline{e}}^{k+1})$ equals $m^{k+1}S^{n-(k+1)}(V^*)$ considered as a subvectorspace of $S^n(V^*)$. Here m is the F-vector space on the basis $x_1, \ldots, x_N$ and $m^{k+1}S^{n-(k+1)}(V^*)$ is the image of the natural map

$$S^{k+1}(m) \otimes S^{n-(k+1)}(V^*) \to S^n(V^*).$$

Write V^* as the direct sum $Fx_0 \oplus m$. Then it follows that

$$m^{k+1}S^{n-(k+1)}(V^*) = x_0^{n-(k+1)}m^{k+1} \oplus \cdots \oplus x_0 m^{n-1} \oplus m^n;$$

hence we see that the dimension of $m^{k+1}S^{n-(k+1)}(V^*)$ equals $\sum_{i=k+1}^{n} \binom{i+N-1}{N-1}$. We also see that the dimension of $S^n(V^*)$ equals $\sum_{i=0}^{n} \binom{i+N-1}{N-1}$, and it follows that

$$\dim_F S^n(V^*) - \dim_F m^{k+1}S^{n-(k+1)}(V^*) = \sum_{i=0}^{k} \binom{i+N-1}{N-1}.$$

It follows that

$$\sum_{i=0}^{k} \binom{i+N-1}{N-1} = \binom{k+N}{N} = \dim_F \mathcal{P}^k(\mathcal{O}(n))(\overline{e}),$$

and we have proved that

$$\dim_F \mathrm{H}^0(X, \mathcal{O}(n)) - \dim_F \mathrm{H}^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\overline{e}}^{k+1}) = \dim_F \mathcal{P}^k(\mathcal{O}(n))(\overline{e}).$$

The result follows from the fact that the sequence (2.1.3) is exact and that $\mathrm{H}^1(X, \mathcal{O}(n)) = 0$ for $n \geq 1$. □

Note that by Lemma 2.2 and the sequence (2.1.3), there exists for all $1 \leq k < n$ an exact sequence of P-representations

$$0 \to \mathrm{H}^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\overline{e}}^{k+1}) \to \mathrm{H}^0(X, \mathcal{O}(n)) \to \mathcal{P}^k(\mathcal{O}(n))(\overline{e}) \to 0.$$

Since the representation $\mathrm{H}^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\overline{e}}^{k+1})$ equals $m^{k+1}S^{n-(k+1)}(V^*)$ as a subrepresentation of $\mathrm{H}^0(X, \mathcal{O}(n)) = S^n(V^*)$, it follows that we have an exact sequence of P-representations

$$0 \to m^{k+1}S^{n-(k+1)}(V^*) \to S^n(V^*) \to \mathcal{P}^k(\mathcal{O}(n))(\overline{e}) \to 0. \tag{2.2.1}$$

From the exact sequence

$$0 \to m \to V^* \to V^*/m \to 0,$$

where m is the F-vector space on $x_1, \dots, x_N$, we see that the representation V^*/m is the representation corresponding to the module L^* of P, giving the line bundle $\mathcal{O}(1)$ on $X = SL(V)/P$.

Lemma 2.3. *For all $1 \leq k < n$ there exists a surjective map of P-representations*

$$\phi : S^n(V^*) \to S^{n-k}(L^*) \otimes S^k(V^*).$$

Proof. Recall that we have chosen a basis $e_0, \dots, e_N$ for V, with the property that $\overline{x}_0$ is a basis for L^*. The P-representation m with basis $x_1, \dots, x_N$ gives an exact sequence

$$0 \to m \to V^* \to L^* \to 0$$

of P-representations. Define a map

$$\phi : S^n(V^*) \to S^{n-k}(L^*) \otimes S^k(V^*)$$

as follows: $\phi(f) = \overline{x}_0^{n-k} \otimes \partial_0^{n-k}(f)$, where ∂_0^{n-k} is the $n-k$ times partial derivative with respect to the x_0-variable. Let g be an element of P. Then by induction on the degree of the differential operator ∂_0^{n-k} and applying the chain rule for derivation, it follows that

$$\begin{aligned} \phi(gf) &= \overline{x}_0^{n-k} \otimes \partial_0^{n-k}(gf) = \overline{x}_0^{n-k} \otimes a^{-(n-k)} g(\partial_0^{n-k} f) \\ &= a^{-(n-k)} \overline{x}_0^{n-k} \otimes g(\partial_0^{n-k} f) = g(\overline{x}_0^{-(n-k)} \otimes \partial_0^{n-k} f) = g\phi(f), \end{aligned}$$

and we see that ϕ is P-linear. It is clearly surjective, and the lemma follows. $\square$

Theorem 2.4. *For all $1 \leq k < n$, the representation corresponding to $\mathcal{P}^k(\mathcal{O}(n))$ is $S^{n-k}(L^*) \otimes S^k(V^*)$.*

Proof. By Lemma 2.3 there exists a surjective map of P-representations

$$\phi : S^n(V^*) \to S^{n-k}(L^*) \otimes S^k(V^*).$$

We claim that $m^{k+1} S^{n-(k+1)}(V^*)$ equals $\ker \phi$: We first prove the inclusion

$$m^{k+1} S^{n-(k+1)}(V^*) \subseteq \ker \phi.$$

Pick a monomial $x_0^{p_0} x_1^{p_1} \cdots x_N^{p_N}$ in $m^{k+1} S^{n-(k+1)}(V^*)$; then $p_0 + \cdots + p_N = n$ and $p_0 < n - k$. These monomials form a basis for $m^{k+1} S^{n-(k+1)}(V^*)$. We see that $\partial_0^{n-k}(x_0^{p_0} \cdots x_N^{p_N})$ is zero; hence, since ϕ is a linear map, it follows that we have an inclusion

$$m^{k+1} S^{n-(k+1)}(V^*) \subseteq \ker \phi$$

of vector spaces. The reverse inclusion follows from counting dimensions and the fact that ϕ is surjective: We have that

$$\begin{aligned} \dim_F \ker \phi &= \dim_F S^n(V^*) - \dim_F S^{n-k}(V^*) \otimes S^k(V^*) \\ &= \sum_{i=0}^{n} \binom{i+N-1}{N-1} - \sum_{i=0}^{k} \binom{i+N-1}{N-1} = \sum_{i=k+1}^{n} \binom{i+N-1}{N-1}, \end{aligned}$$

and we see that $\dim_F \ker \phi = \dim_F m^{k+1} S^{n-(k+1)}(V^*)$. It follows that

$$m^{k+1} S^{n-(k+1)}(V^*) = \ker \phi;$$

hence we have an exact sequence of P-representations

$$0 \to m^{k+1} S^{n-(k+1)}(V^*) \to S^n(V^*) \to S^{n-k}(L^*) \otimes S^k(V^*) \to 0.$$

Using sequence (2.2.1) we get isomorphisms

$$\begin{aligned} \mathcal{P}^k(\mathcal{O}(n))(\overline{e}) &\cong \mathrm{H}^0(X, \mathcal{O}(n)) / \mathrm{H}^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\overline{e}}^{k+1}) \\ &\cong S^n(V^*)/m^{k+1} S^{n-(k+1)}(V^*) \cong S^{n-k}(V^*) \otimes S^n(V^*), \end{aligned}$$

and it follows that $\mathcal{P}^k(\mathcal{O}(n))(\overline{e})$ is isomorphic to $S^{n-k}(L^*) \otimes S^k(V^*)$ as a representation. □

Note that the result in Theorem 2.4 is true if $\mathrm{char}(F) > n$.

Corollary 2.5. *For all $1 \leq k < n$, $\mathcal{P}^k(\mathcal{O}(n))$ splits as an abstract vector bundle as $\bigoplus^{\binom{N+k}{N}} \mathcal{O}(n-k)$.*

Proof. Since $S^k(V^*)$ corresponds to the trivial rank $\binom{N+k}{N}$ abstract vector bundle on $\mathbf{P}(V^*)$, and $S^{n-k}(L^*)$ corresponds to the line bundle $\mathcal{O}(n-k)$, the assertion is proved. □

We see that we recover results on the splitting type of the principal parts obtained in [6], [7], [8] and [9].

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