A NOTE ON MEROMORPHIC OPERATORS

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Abstract. Let $X$ be a complex Banach space and $T$ a bounded linear operator on $X$. $T$ is called meromorphic if the spectrum $\sigma(T)$ of $T$ is a countable set, with 0 the only possible point of accumulation, such that all the nonzero points of $\sigma(T)$ are poles of $(\lambda I - T)^{-1}$. By means of the analytical core $K(T)$ we give a spectral theory of meromorphic operators. Our results are a generalization of some results obtained by Gong and Wang (2003).

1. Introduction and terminology

Throughout this paper, $X$ will denote an infinite-dimensional complex Banach space. By $\mathcal{L}(X)$ we denote the Banach algebra of all bounded linear operators on $X$. Let $T \in \mathcal{L}(X)$. The kernel and the range of $T$ will be denoted by $N(T)$ and $T(X)$, respectively. The spectrum, the set of eigenvalues, and the resolvent set of $T$ are denoted by $\sigma(T)$, $\sigma_p(T)$ and $\rho(T)$, respectively. For the resolvent $(\lambda I - T)^{-1}$ we write $R_\lambda(T)$ ($\lambda \in \rho(T)$).

The nullity $\alpha(T)$ of $T$ is the dimension of $N(T)$. The defect $\beta(T)$ of $T$ is the codimension of $T(X)$. The ascent $p(T)$ and the descent $q(T)$ are the extended integers given by

$$p(T) = \inf\{n \geq 0 : N(T^n) = N(T^{n+1})\},$$

$$q(T) = \inf\{n \geq 0 : T^n(X) = T^{n+1}(X)\}.$$

The infimum over the empty set is taken to be $\infty$. It follows from Satz 72.3 that if $p(T)$ and $q(T)$ are both finite, then they are equal. If $\lambda_0$ is an isolated point in $\sigma(T)$, the spectral projection corresponding to $\lambda_0$ will denoted by $P_{\lambda_0}$. We have $X = P_{\lambda_0}(X) \oplus N(P_{\lambda_0})$. From Satz 101.2 we have the following characterization of the poles of $R_\lambda(T)$:

**Theorem 1.** The complex number $\lambda_0$ is a pole of $R_\lambda(T)$ if and only if $0 < p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$. In this case we have

$$P_{\lambda_0}(X) = N((\lambda_0 I - T)^p) \text{ and } N(P_{\lambda_0}) = (\lambda_0 I - T)^p(X),$$

where $p = p(\lambda_0 I - T)$ is the order of the pole $\lambda_0$, and $\lambda_0 \in \sigma_p(T)$. 

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We now list various classes of operators, which will be discussed in this note:

\[
\mathcal{F}(X) = \{ T \in \mathcal{L}(X) : \dim T(X) < \infty \};
\]

\[
\mathcal{K}(X) = \{ T \in \mathcal{L}(X) : T \text{ is compact} \};
\]

\[
\Phi(X) = \{ T \in \mathcal{L}(X) : \alpha(T) < \infty, \beta(T) < \infty \}.
\]

Operators in \( \Phi(X) \) are called \textit{Fredholm operators}.

Let \( T \in \mathcal{L}(X) \) and \( \lambda \in \mathbb{C} \). \( \lambda \) is called a \textit{Riesz point} of \( T \) if

\[
\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty \quad \text{and} \quad p(\lambda I - T) = q(\lambda I - T) < \infty.
\]

\( T \in \mathcal{L}(X) \) is called a \textit{Riesz operator} if each \( \lambda \neq 0 \) is a Riesz point of \( T \). We denote by \( \mathcal{R}(X) \) the class of all Riesz operators in \( \mathcal{L}(X) \).

We have the following characterization of Riesz operators (see [4, §105]):

**Theorem 2.** Let \( T \in \mathcal{L}(X) \). Then:

\( T \in \mathcal{R}(X) \iff \) each \( \lambda_0 \in \sigma(T) \setminus \{0\} \) is an isolated point of \( \sigma(T) \) and \( P_{\lambda_0} \in \mathcal{F}(X) \).

The class \( \mathcal{M}(X) \) of \textit{meromorphic operators} is defined as follows:

\[
\mathcal{M}(X) = \{ T \in \mathcal{L}(X) : \text{ each } \lambda_0 \in \sigma(T) \setminus \{0\} \text{ is a pole of } R_{\lambda_0}(T) \}.
\]

We have the following inclusions:

\[
\mathcal{F}(X) \subseteq \mathcal{K}(X) \subseteq \mathcal{R}(X) \subseteq \mathcal{M}(X).
\]

Two subclasses of \( \mathcal{M}(X) \) are also considered in this note:

\[
\mathcal{Q}(X) = \{ T \in \mathcal{L}(X) : \sigma(T) = \{0\} \}
\]

and

\[
\mathcal{M}_0(X) = \{ T \in \mathcal{M}(X) : \sigma(T) \text{ is finite} \}.
\]

An operator in \( \mathcal{Q}(X) \) is called \textit{quasinilpotent}.

In [5] Mbekhta introduced two important subspaces for \( T \in \mathcal{L}(X) \): the \textit{analytical core} \( K(T) \) of \( T \) is defined by

\[
K(T) = \{ x \in X : \text{ there exist } c > 0 \text{ and a sequence } (x_n)_{n \geq 1} \text{ in } X \text{ such that } Tx_n = x, \quad Tx_{n+1} = x_n \quad \text{and} \quad ||x_n|| \leq c^n ||x|| \text{ for all } n \in \mathbb{N} \}.
\]

Observe that if \( Y \) is a closed subspace of \( X \) such that \( T(Y) = Y \), then \( Y \subseteq K(T) \) ([8, Proposition 2]).

The subspace \( H_0(T) \), defined by

\[
H_0(T) = \{ x \in X : \lim_{n \to \infty} ||T^nx||^{1/n} = 0 \},
\]

is called the \textit{quasinilpotent part} of \( T \).

We close the section with the following definition: an operator \( T \in \mathcal{L}(X) \) is said to have the \textit{single-valued extension property} (SVEP) in \( \lambda_0 \in \mathbb{C} \) if for any holomorphic function \( f : U \to X \), where \( U \) is a neighbourhood of \( \lambda_0 \), with \( (\lambda I - T)f(\lambda) \equiv 0 \) for all \( \lambda \in U \), the result is \( f(\lambda) \equiv 0 \). We say that \( T \) has the SVEP if \( T \) has the SVEP in each \( \lambda \in \mathbb{C} \).

It is clear that each \( T \in \mathcal{M}(X) \) has the SVEP. Furthermore, we have \( \sigma(T) \setminus \{0\} \subseteq \sigma_p(T) \) if \( T \in \mathcal{M}(X) \) (see Theorem 1).
2. Preliminary results

In this section we collect some results which we need in the sequel.

Proposition 1. Let $T, S \in \mathcal{L}(X)$.
\begin{enumerate}
    
    
    \item $T(K(T)) = K(T)$ and $T(H_0(T)) \subseteq H_0(T)$.
    \item $K(T) \subseteq T^n(X)$ and $N(T^n) \subseteq H_0(T)$ for all $n \in \mathbb{N}$.
    \item $N(\lambda I - T) \subseteq K(T)$ for all $\lambda \in \mathbb{C}\setminus\{0\}$.
    \item $H_0(T) \subseteq (\lambda I - T)(X)$ for all $\lambda \in \mathbb{C}\setminus\{0\}$.
    \item If $TS = ST$, then $H_0(T) \subseteq H_0(TS)$.
    \item $0 \in \rho(T) \iff K(T) = X$ and $H_0(T) = \{0\}$.
\end{enumerate}

Proof. (1) is shown in [6];
(2) is clear;
(3) if $x \in N(\lambda I - T)$, put $c = |\lambda|^{-1}$ and $x_n = |\lambda|^{-n}x$ if $n \in \mathbb{N}$;
(4) is shown in [8] Proposition 1;
(5) is clear;
(6) follows from (2) and (5). \qed

Proposition 2. Let $T \in \mathcal{L}(X)$, $\lambda_0 \in \sigma(T)$ and $K(\lambda_0 I - T) = \{0\}$. Then $\lambda_0$ is the only possible isolated point in $\sigma(T)$.

Proof. Corollary 1.3 in [7]. \qed

Proposition 3. Suppose that $T \in \mathcal{L}(X)$ has the SVEP in $\lambda_0 = 0$.
\begin{enumerate}
    \item If $q(T) < \infty$, then $p(T) = q(T)$.
    \item $0$ is a pole of $R_\lambda(T)$ if and only if $0 < q(T) < \infty$.
\end{enumerate}

Proof. (1) is shown in [8] Proposition 3.
(2) If $0$ is a pole of $R_\lambda(T)$, then $0 < q(T) < \infty$ by Theorem 2. If $0 < q(T) < \infty$,

it follows from (1) that $0 < p(T) = q(T) < \infty$: Theorem 2 shows now that $0$ is a pole of $R_\lambda(T)$. \qed

Proposition 4. Let $T \in \mathcal{L}(X)$. $0$ is an isolated point of $\sigma(T)$ if and only if $K(T)$ is closed, $X = K(T) + H_0(T)$ and $K(T) \cap H_0(T) = \{0\}$. In this case,

\[ P_0(X) = H_0(T) \text{ and } N(P_0) = K(T). \]

Proof. Proposition 4 and Theorem 4 in [8]. \qed

Notation. If $T \in \mathcal{L}(X)$ and if $Y$ is a $T$-invariant subspace of $X$, then $T|_Y$ means the restriction of $T$ to $Y$.

Proposition 5. Let $T \in \mathcal{L}(X)$ and $\lambda_0 \in \mathbb{C}\setminus\{0\}$. If $\lambda_0$ is an isolated point of $\sigma(T)$, then

\[ H_0(\lambda_0 I - T) \text{ is a closed } T\text{-invariant subspace and } \sigma(T|_{H_0(\lambda_0 I - T)}) = \{\lambda_0\}. \]

Proof. By Proposition 1(1) and Proposition 4, $T(H_0(\lambda_0 I - T)) \subseteq H_0(\lambda_0 I - T)$ and $H_0(\lambda_0 I - T) = P_{\lambda_0}(X)$, thus $H_0(\lambda_0 I - T)$ is closed and $T$-invariant. From [4] Satz 100.1 we get $\sigma(T|_{H_0(\lambda_0 I - T)}) = \{\lambda_0\}$. \qed

The next result generalizes Proposition 2.4 in [7].
**Proposition 6.** Suppose that $T \in \mathcal{L}(X)$ has the SVEP, $\lambda_0 \in \mathbb{C}\setminus\{0\}$, $\lambda_0 \in \rho(T)$ or $\lambda_0$ is an isolated point of $\sigma(T)$ and that

$$H_0(\lambda_0 I - T) + H_0(T) = X.$$ 

Then $0 \in \rho(T)$ or $0$ is an isolated point of $\sigma(T)$.

**Proof.** Put $Y = H_0(\lambda_0 I - T)$. If $\lambda_0 \in \rho(T)$, then $Y = \{0\}$ (by Proposition 1(6)); thus,

$$(\lambda I - T)(Y) = Y \quad \text{for all } \lambda \in \mathbb{C}.\$$

If $\lambda_0 \in \sigma(T)$, then, by Proposition 5, there exists $\rho > 0$ such that

$$(\lambda I - T)(Y) = Y \quad \text{for } |\lambda| < \rho.$$ 

Therefore we have in both cases that there is some $\rho > 0$ with $(\lambda I - T)(Y) = X$ for $|\lambda| < \rho$.

Now take $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \rho$. Then

$$H_0(\lambda I - T) = Y \subseteq (\lambda I - T)(X),$$

thus $X = H_0(\lambda_0 I - T) + H_0(T) \subseteq (\lambda I - T)(X) + H_0(T)$.

Since $H_0(T) \subseteq (\lambda I - T)(X)$ (by Proposition 1(4)),

$$X = (\lambda I - T)(X),$$

therefore $q(\lambda I - T) = 0$ for $0 < |\lambda| < \rho$. Since $T$ has the SVEP, we get from Proposition 3(1) that $p(\lambda I - T) = q(\lambda I - T) = 0$ for $0 < |\lambda| < \rho$. Hence $\{\lambda \in \mathbb{C} : 0 < |\lambda| < \rho\} \subseteq \rho(T)$.

**Corollary 1.** Suppose that $T \in \mathcal{L}(X)$ has the SVEP, $\lambda_0 \in \mathbb{C}\setminus\{0\}$, $\lambda_0 \in \rho(T)$ or $\lambda_0$ is an isolated point of $\sigma(T)$ and that

$$H_0(T) = K(\lambda_0 I - T).$$

Then $0 \in \rho(T)$ or $0$ is an isolated point of $\sigma(T)$.

**Proof.** If $\lambda_0 \in \rho(T)$, then $K(\lambda_0 I - T) = X$ and $H_0(\lambda_0 I - T) = \{0\}$ (Proposition 1(6)). Thus

$$X = K(\lambda_0 I - T) + H(\lambda_0 I - T),$$

hence

$$X = H_0(T) + H_0(\lambda_0 I - T).$$

If $\lambda_0 \in \sigma(T)$, then, by Proposition 4,

$$X = K(\lambda_0 I - T) + H(\lambda_0 I - T),$$

therefore

$$X = H_0(T) + H_0(\lambda_0 I - T).$$

Thus we have in both cases that $X = H_0(T) + H_0(\lambda_0 I - T)$. Now use Proposition 6.

**Remark.** Corollary 1 generalizes [7, Corollary 2.5].
3. Meromorphic operators

In this section we present the main results of this paper. The first result deals with Riesz operators and generalizes Theorem 2.6 in [7].

**Theorem 3.** Let $T \in \mathcal{R}(X)$. The following assertions are equivalent:

1. $0$ is a pole of $R_\lambda(T)$;
2. there exists $q \in \mathbb{N}$ such that $T^q \in \mathcal{F}(X)$;
3. there exists $n \in \mathbb{N}$ with $K(T) = T^n(X)$;
4. $q(T) < \infty$.

**Proof.** $(1) \Leftrightarrow (2)$: [Aufgabe 105.2].

$(2) \Rightarrow (3)$: Since $T^{q+k}(X) \subseteq T^q(X)$ for $k \geq 0$ and $\dim T^q(X) < \infty$, we get $q \leq q(T) < \infty$. Put $n = q(T)$. Then $\dim T^n(X) < \infty$, hence $T^n(X)$ is closed. Furthermore $T(T^n(X)) = T^{n+1}(X) = T^n(X)$. Proposition 2 in [8] implies now that $T^n(X) \subseteq K(T)$. Therefore $K(T) = T^n(X)$, by Proposition 1(2).

$(3) \Rightarrow (4)$: From $T^{n+1}(X) = T(T^n(X)) = T(K(T))$ and $T(K(T)) = K(T)$ (Proposition 1(1)) we derive $T^{n+1}(X) = T^n(X)$, thus $q(T) \leq n < \infty$.

$(4) \Rightarrow (1)$: Since $T$ has the SVEP, it follows from Proposition 3(2) that $0$ is a pole of $R_\lambda(T)$.

**Remark.** The above proof shows that if $T \in \mathcal{L}(X)$ has the SVEP in $\lambda_0 = 0$ and if $0 \in \sigma(T)$, then the assertions (1), (3) and (4) in Theorem 3 are equivalent (for the implication $(1) \Rightarrow (3)$ use Theorem 1 and Proposition 4).

Our next result generalizes Theorem 2.1 in [7].

**Theorem 4.** Let $T \in \mathcal{M}(X)$. Then:

$0 \in \rho(T)$ or $0$ is an isolated point of $\sigma(T) \Leftrightarrow K(T)$ is closed.

**Proof.** $\Rightarrow$: Proposition 1(6) and Proposition 4 show that $K(T)$ is closed if $0 \in \rho(T)$ or $0$ is an isolated point of $\sigma(T)$.

$\Leftarrow$: Case 1: $K(T) = \{0\}$. Proposition 1(6) shows that $0 \in \sigma(T)$. Proposition 2 implies then that $0$ is the only possible isolated point of $\sigma(T)$. Since $T \in \mathcal{M}(X)$ we get $\sigma(T) = \{0\}$ (hence $T \in \mathcal{Q}(X)$).

Case 2: $K(T) \neq \{0\}$. Since $K(T)$ is closed, $K(T)$ is a Banach space. Put $T_0 := T|_{K(T)}$ and $I_0 = I|_{K(T)}$. Use Proposition 1(1) to get $T_0 \in \mathcal{L}(K(T))$ and $q(T_0) = 0$.

Since $T$ has the SVEP, $T_0$ has the SVEP.

From Proposition 3(1) we therefore derive $\rho(T_0) = q(T_0) = 0$, hence $0 \in \rho(T_0)$. Thus there is $\rho > 0$ such that $\{\lambda \in \mathbb{C} : |\lambda| < \rho\} \subseteq \rho(T_0)$. Now take $\lambda \in \mathbb{C}$ with $0 < |\lambda| < \rho$. Then $N(\lambda I - T) \subseteq K(T)$ (Proposition 1(3)), thus $N(\lambda I - T) = N(\lambda I_0 - T_0) = \{0\}$, hence $\lambda \notin \sigma_p(T)$. Since $\lambda \neq 0$ and $T \in \mathcal{M}(X)$, $\lambda \notin \sigma(T)$. Therefore $\lambda \in \mathbb{C} : 0 < |\lambda| < \rho \subseteq \rho(T)$.

We proceed with a corollary that generalizes Corollary 2.2 in [7].

**Corollary 2.** Let $T \in \mathcal{M}(X)$. Then:

1. $K(T) = \{0\} \Leftrightarrow T \in \mathcal{Q}(X)$;
2. $K(T)$ is closed and $K(T) \neq \{0\} \Leftrightarrow T \in \mathcal{M}_0(X) \setminus \mathcal{Q}(X)$;
3. $K(T)$ is not closed $\Leftrightarrow T \notin \mathcal{M}_0(X)$. 

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Proof. (1) We have seen in the proof of Theorem 4 that \( K(T) = \{0\} \) implies \( T \in Q(X) \).

Now let \( T \in Q(X) \). Remarque 1.1 in [3] shows that \( H_0(T) = X \). Since \( H_0(T) \cap K(T) = \{0\} \), by Proposition 4, we derive \( K(T) = \{0\} \).

(2) “\( \Rightarrow \)” : (1) gives \( T \notin Q(X) \). Theorem 4 shows that \( 0 \in \rho(T) \) or \( 0 \) is an isolated point of \( \sigma(T) \). Therefore, since \( T \in M(X) \), \( \sigma(T) \) is finite, hence \( T \in M_0(X) \).

“\( \Leftarrow \)” : From (1) we get \( K(T) \neq \{0\} \). Since \( \sigma(T) \) is finite, we see that \( 0 \in \rho(T) \) or \( 0 \) is an isolated point of \( \sigma(T) \). Theorem 4 implies then that \( K(T) \) is closed.

(3) “\( \Rightarrow \)” : By Theorem 4, \( 0 \in \sigma(T) \) and \( 0 \) is not an isolated point of \( \sigma(T) \), thus \( T \notin M_0(X) \).

“\( \Leftarrow \)” : Since \( T \in M(X) \setminus M_0(X) \), \( 0 \) is a point of accumulation of \( \sigma(T) \), thus \( K(T) \) is not closed by Theorem 4. \( \square \)

We denote by \( X^* \) the dual space of \( X \) and by \( T^* \) the adjoint of \( T \in L(X) \).

**Proposition 7.** Let \( T \in L(X) \), and suppose that \( T \) and \( T^* \) have the SVEP in 0. Then:

\[
T \in \Phi(X) \iff 0 \text{ is a Riesz point of } T.
\]

Proof. The implication “\( \Leftarrow \)” follows from the definition of a Riesz point.

Now suppose that \( T \in \Phi(X) \). Since \( T \) has the SVEP in 0, it follows from [3] Theorem 15 that \( p(T) < \infty \). Satz 82.1 in [4] gives \( T^* \in \Phi(X^*) \). Since \( T^* \) has the SVEP in 0, we have \( q(T^*) < \infty \) by [3] Corollary 16. Hence \( p(T) = q(T) < \infty \). Satz 72.5 in [4] implies now that \( \alpha(T) = \beta(T) \). \( \square \)

**Corollary 3.** For \( T \in M(X) \) the following assertions are equivalent:

1. \( K(T) \) is closed and \( \text{codim} K(T) < \infty \);
2. \( K(T) \) is closed and \( \dim H_0(T) < \infty \);
3. \( 0 \) is a Riesz point of \( T \);
4. \( T \in \Phi(X) \).

Proof. Since \( T \in M(X) \) and \( \sigma(T^*) = \sigma(T) \), \( T \) and \( T^* \) have the SVEP. Proposition 7 shows then that (3) and (4) are equivalent.

Now suppose that \( K(T) \) is closed. By Theorem 4, \( 0 \in \rho(T) \) or \( 0 \) is an isolated point of \( \sigma(T) \). Now use Proposition 1(6) and Proposition 4 to derive

\[ X = K(T) + H_0(T) \text{ and } K(T) \cap H_0(T) = \{0\}. \]

Hence \( \dim H_0(T) = \text{codim} K(T) \). Therefore (1) and (2) are equivalent.

Now we show that (2) implies (3): By Theorem 4, \( 0 \in \rho(T) \) or \( 0 \) is an isolated point of \( \sigma(T) \). If \( 0 \notin \sigma(T) \), then \( 0 \) is a Riesz point of \( T \). Hence suppose that \( 0 \in \sigma(T) \). By Proposition 4, \( P_0(X) = H_0(T) \), thus \( P_0 \in \mathcal{F}(X) \). [4] Satz 105.2 shows now that \( 0 \) is a Riesz point of \( T \).

It remains to show that (3) implies (2):

Case 1: \( 0 \in \rho(T) \). By Proposition 1(6), \( K(T) = X \) and \( H_0(T) = \{0\} \). Hence \( K(T) \) is closed and \( \dim H_0(T) < \infty \).

Case 2: \( 0 \in \sigma(T) \). Since \( 0 \) is a Riesz point of \( T \), \( 0 \) is an isolated point of \( \sigma(T) \) and \( P_0 \in \mathcal{F}(X) \), by Satz 105.2 in [4]. From Proposition 4 and Theorem 4 it follows that \( \dim H_0(T) = \dim P_0(X) < \infty \) and that \( K(T) \) is closed. \( \square \)

**Corollary 4.** For \( T \in M(X) \) the following assertions are equivalent:

1. \( \dim K(T) < \infty \);
2. \( T \in R(X) \cap M_0(X) \).
Proof. (1) ⇒ (2): Since \( \dim X = \infty \) and \( \dim K(T) < \infty \), it follows from Proposition 1(6) that \( 0 \in \sigma(T) \). Corollary 2 shows that \( T \in \mathcal{M}_0(X) \).

Now take \( \lambda \in \mathbb{C} \setminus \{0\} \). Since \( T \in \mathcal{M}(X) \), \( \lambda \in \rho(T) \) or \( \lambda \) is a pole of \( R_\lambda(T) \), thus \( m(\lambda I - T) = q(\lambda I - T) < \infty \). By Proposition 1(3), \( N(\lambda I - T) \subseteq K(T) \), thus \( \alpha(\lambda I - T) < \infty \). Satz 72.5 in [4] implies now that
\[
\beta(\lambda I - T) = \alpha(\lambda I - T) < \infty.
\]
Therefore \( \lambda \) is a Riesz point of \( T \). Since \( \lambda \in \mathbb{C} \setminus \{0\} \) was arbitrary, \( T \in \mathcal{R}(X) \).

(2) ⇒ (1): We can assume that \( K(T) \neq \{0\} \). Since \( T \in \mathcal{M}_0(X) \) and \( 0 \in \sigma(T) \) (see [4] Aufgabe 105.2]), \( 0 \) is an isolated point of \( \sigma(T) \). Hence \( K(T) \) is closed (Theorem 4). Put \( T_0 = T|_{K(T)} \). By Proposition 1(1), \( T(K(T)) = K(T) \), thus \( T_0 \in \mathcal{L}(K(T)) \). From Proposition 4 we get \( K(T) = N(P_0) \). Now use Satz 100.1 in [4] to derive
\[
\sigma(T_0) = \sigma(T) \setminus \{0\},
\]
thus \( 0 \notin \sigma(T_0) \). Since \( T \in \mathcal{R}(X) \) it follows from [4] Satz 105.6] that \( T_0 \in \mathcal{R}(K(T)) \).

Now assume that \( \dim K(T) = \infty \). Thus, by [4] Aufgabe 105.2] \( 0 \notin \sigma(T_0) \), a contradiction. Hence \( \dim K(T) < \infty \).

4. Final remarks

1. The proof of Theorem 4 shows that the following result is valid.

Theorem 5. Suppose that \( T \in \mathcal{L}(X) \) has the SVEP in \( 0 \) and that there is a sequence \((\lambda_n) \) in \( \sigma_p(T) \) with \( \lambda_n \neq 0 \) for all \( n \in \mathbb{N} \) and \( \lambda_n \to 0 \) as \( n \to \infty \). Then \( K(T) \) is not closed.

That the condition “\( T \) has the SVEP in \( 0 \)” cannot be dropped in Theorem 5 shows the example of the unilateral left shift on \( l^2(\mathbb{N}) \):

Example. Let \( X = l^2(\mathbb{N}) \), and define the operator \( T \in \mathcal{L}(X) \) by
\[
T(\xi_1, \xi_2, \xi_3, \ldots) = (\xi_2, \xi_3, \ldots).
\]

It is well known that \( \sigma_p(T) = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \} \). Since \( T(X) = X \), we have \( K(T) = X \), by [3] Proposition 2]. Thus \( K(T) \) is closed. Example 1.7 in [2] shows that \( T \) does not have the SVEP in 0.

2. In [1] W. Bouamama proves independently some of the results of our paper.

References


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