SUPERCONGRUENCES FOR TRUNCATED $\frac{n+1}{n} F_n$
HYPERGEOMETRIC SERIES WITH APPLICATIONS TO CERTAIN WEIGHT THREE NEWFORMS

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Abstract. We prove general results on supercongruences between values of truncated $\frac{n+1}{n} F_n$ hypergeometric functions and their character analogs. As a consequence of the main results of this paper, we prove Beukers-type supercongruences for certain weight three newforms.

1. Introduction

In [RV1], Fernando Rodriguez-Villegas discovered numerically a number of Beukers-type supercongruences for hypergeometric Calabi-Yau manifolds of dimension $d \leq 3$. Specifically, he observed supercongruences between the truncated fundamental period of the Picard-Fuchs differential equation of the manifold and an expression derived from the number of its $F_p$-points. This had been motivated by his joint work with Candelas and de la Ossa [COV]. Here we prove general results on supercongruences between values of truncated $\frac{n+1}{n} F_n$ hypergeometric functions and their character analogs. As a consequence of these results, we prove some of the observed supercongruences for manifolds of dimension $d = 2$. Supercongruences of this type were first observed by Beukers [B] in connection with the Apéry numbers used in the proof of the irrationality of $\zeta(3)$. Ahlgren and Ono [AO] proved Beukers’ supercongruence conjecture relating Apéry numbers to the coefficients of a certain weight four newform.

In [RV1] and [RV2], Rodriguez-Villegas identified four modular K3 surfaces with potential supercongruences. We define Dedekind’s eta function by the infinite product:

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \quad q := e^{2\pi iz}. \tag{1.1}$$

We then define the integers $a(n), b(n),$ and $c(n)$ by

$$\sum_{n=1}^{\infty} a(n)q^n := \eta^6(4z) \in S_3(\Gamma_0(16), (\frac{-3}{d})), \tag{1.2}$$

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\begin{align}
\sum_{n=1}^{\infty} b(n)q^n &:= \eta^3(6z)\eta^3(2z) \in S_3(\Gamma_0(12), (\frac{2z}{q})), \\
\sum_{n=1}^{\infty} c(n)q^n &:= \eta^2(8z)\eta(4z)\eta(2z)\eta^2(z) \in S_3(\Gamma_0(8), (\frac{2z}{q})).
\end{align}

These weight three newforms are related to modular K3 surfaces. They are extensively studied in \[SB\], where, among other results, the authors prove several modulo \(p\) congruences. From \[RV1\] and \[RV2\] we are able to formulate the following:

**Conjecture.** If \(p \geq 5\) is a prime, then
\begin{align}
\sum_{n=0}^{p-1} \frac{(2n)!}{n!^3} 64^{-n} &\equiv a(p) \pmod{p^2}, \\
\sum_{n=0}^{p-1} \frac{(3n)!}{n!^3} 108^{-n} &\equiv b(p) \pmod{p^2}, \\
\sum_{n=0}^{p-1} \frac{(4n)!}{n!^3} 256^{-n} &\equiv c(p) \pmod{p^2}, \\
\sum_{n=0}^{p-1} \frac{(5n)!}{n!^3} 1728^{-n} &\equiv \gamma(p)a(p) \pmod{p^2},
\end{align}

where \(\gamma(p) := -1\) if \(p \equiv 5 \pmod{12}\) and \(\gamma(p) := 1\) otherwise.

It should be noted that (1.5) has already been proved by several individuals including Ishikawa \[I\], Van Hamme \[VH\], and Ahlgren \[A\]. The numbers 64, 108, 256, 1728 are called the conifold points (see \[RV1\]). Here we prove several cases of these conjectures.

To state our results, we recall basic facts about characters and Jacobi sums and introduce some notation. We denote by \(\mathbb{F}_q\) the finite field with \(q = p^r\) elements, where \(p\) is a prime. We extend all multiplicative characters \(\chi : \mathbb{F}_q^* \to \mathbb{C}\), including the trivial character \(\varepsilon_q\), to \(\mathbb{F}_q\) by setting \(\chi(0) := 0\). If \(A\) and \(B\) are two characters on \(\mathbb{F}_q\), then we define \(\binom{A}{B}\) in terms of the Jacobi sum by
\begin{equation}
\binom{A}{B} := \frac{B(-1)}{q} J_r(A, B) = \frac{B(-1)}{q} \sum_{x \in \mathbb{F}_q} A(x) B(1-x),
\end{equation}
where \(J_r(\cdot, \cdot)\) is a Jacobi sum over \(\mathbb{F}_{p^r}\). We recall some useful properties of binomial coefficients (\[G\], (2.6)-(2.7)):
\begin{equation}
\binom{A}{B} = \binom{A}{AB} \quad \text{and} \quad \binom{A}{B} = \binom{BA}{B} B(-1).
\end{equation}

If \(A_0, A_1, \ldots, A_n, B_1, B_2, \ldots, B_n\) are characters on \(\mathbb{F}_q\) and if \(x \in \mathbb{F}_q\), then Greene \[G\] defines \(n+1F_n\) Gaussian hypergeometric series by
\begin{equation}
\binom{A_0, A_1, \ldots, A_n}{B_1, B_2, \ldots, B_n | x}_q := \frac{q}{q-1} \sum_{\chi} \binom{A_0\chi}{B_1\chi} \cdots \binom{A_n\chi}{B_n\chi} \chi(x),
\end{equation}
where the sum runs over all characters \(\chi\) on \(\mathbb{F}_q\). We note that this definition lives in some extension of \(\mathbb{Q}_p\). For certain choices of characters, the right-hand side actually is in \(\mathbb{Z}_p\).
If $m$ is a positive integer, then we define the truncated hypergeometric series by

$$(1.12) \quad _{n+1}F_n\left(\frac{a_0}{b_1}, \frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n} \mid x \right)_{\text{tr}(p)} := \sum_{k=0}^{m-1} \frac{(a_0)(a_1)_k \cdots (a_n)_k}{k!(b_1)_k \cdots (b_n)_k} x^k,$$

where $(a)_k := a(a+1) \cdots (a+k-1)$.

If $n \in \mathbb{N}$, we define the $p$-adic $\Gamma$-function on the ring $\mathbb{Z}_p$ of $p$-adic integers by

$$(1.13) \quad \Gamma_p(n) := (-1)^n \prod_{j<n, pj} j \text{ and } \Gamma_p(x) := \lim_{n \to x} \Gamma_p(n), \quad x \in \mathbb{Z}_p,$$

where in the limit we take any sequence of positive integers that approaches $x$ in the $p$-adic sense. Recall three basic properties of the $p$-adic $\Gamma$-function. If $p \geq 5$ is a prime and $x, y \in \mathbb{Z}_p$, then the following are true. We have

$$(1.14) \quad \Gamma_p(x+1) = \begin{cases} -x \Gamma_p(x) & \text{if } x \in \mathbb{Z}_p^* \\ -\Gamma_p(x) & \text{if } x \in p\mathbb{Z}_p. \end{cases}$$

If $n \geq 1$, then

$$(1.15) \quad x \equiv y \pmod{p^n} \Rightarrow \Gamma_p(x) \equiv \Gamma_p(y) \pmod{p^n}.$$ 

If $R(x)$ denotes the reduction of $x$ modulo $p$ to the range $\{1, \ldots, p\}$, then

$$(1.16) \quad \Gamma_p(x)\Gamma_p(1-x) = (-1)^{R(x)}.$$ 

We are now able to state the results of this paper. Let $\phi_q$ denote the character of order 2 on $\mathbb{F}_q$, and let $\epsilon_q$ denote the trivial character on $\mathbb{F}_q$. In the sequel we shall drop the subscript $q$ since it will be obvious from the context.

**Theorem 1.** If $p$ is a prime, $p \equiv 1 \pmod{d_i}$ with $1 \leq m_i < d_i$, $\rho_i$ is a character of order $d_i$ on $\mathbb{F}_p$, and $\sum_{i=1}^{n+1} \frac{m_i}{d_i} \geq n-1$, then

$$_{n+1}F_n\left(\frac{m_1}{d_1}, \frac{m_2}{d_2}, \ldots, \frac{m_{n+1}}{d_{n+1}} \mid 1 \right)_{\text{tr}(p)} = (-1)^n p^{n \cdot _{n+1}F_n\left(\rho_1^{m_1}, \rho_2^{m_2}, \ldots, \rho_{n+1}^{m_{n+1}} \mid 1 \right)}_{p} - \delta \cdot p \pmod{p^2},$$

where $\delta := \begin{cases} 0 & \text{if } \sum_{i=1}^{n+1} \frac{m_i}{d_i} > n-1, \\ \prod_{i=1}^{n+1} \Gamma_p(1 - \frac{m_i}{d_i}) & \text{if } \sum_{i=1}^{n+1} \frac{m_i}{d_i} = n-1. \end{cases}$

**Corollary 1.** If $p$ is a prime, $p \equiv 1 \pmod{d_i}$, $1 \leq m_i < d_i$, and $\rho_i$ is a character of order $d_i$ on $\mathbb{F}_p$, then

$$_4F_3\left(\frac{m_1}{d_1}, \frac{m_2}{d_2}, \frac{1 - m_2}{1}, \frac{1}{1} \mid 1 \right)_{\text{tr}(p)} = -p^3 \cdot _4F_3\left(\rho_1^{m_1}, \rho_1^{m_1}, \rho_2^{m_2}, \rho_2^{m_2} \mid 1 \right)_{p} - (-1)^{\frac{m_1}{d_1}(p-1) + \frac{m_2}{d_2}(p-1)}p \pmod{p^2}.$$ 

**Corollary 2.** If $p$ is a prime, $p \equiv 1 \pmod{d}$, $1 \leq m < d$, and $\rho$ is a character of order $d$ on $\mathbb{F}_p$, then

$$_3F_2\left(\frac{1}{2}, \frac{m}{d}, \frac{1 - m}{1} \mid 1 \right)_{\text{tr}(p)} = p^{2} \cdot _3F_2\left(\phi_q, \rho^{m} \mid 1 \right)_{p} \pmod{p^2}.$$
Theorem 2. If \( p \) is a prime, \( p \equiv -1 \pmod{d_i} \), \( 1 \leq m_i < d_i \), and \( \rho_i \) is a character of order \( d_i \) on \( \mathbb{F}_{p^2} \), then
\[
4F3 \left( \frac{m_1}{\overline{m}_1}, 1 - \frac{m_1}{\overline{m}_1}, \frac{m_2}{\overline{m}_2}, 1 - \frac{m_2}{\overline{m}_2} \mid 1 \right)^2_{tr(p)}
\equiv -p^6 \cdot 4F3 \left( \rho_1^{m_1}, \overline{\rho}_1^{m_1}, \rho_2^{m_2}, \overline{\rho}_2^{m_2} \mid 1 \right)_{p^2} \quad (\text{mod } p^2).
\]

Theorem 3. If \( p \) is a prime, \( p \equiv -1 \pmod{d} \), \( 1 \leq m < d \), and \( \rho \) is a character of order \( d \) on \( \mathbb{F}_{p^2} \), then
\[
3F2 \left( \frac{1}{2}, \frac{m}{\overline{m}}, 1 - \frac{m}{\overline{m}} \mid 1 \right)^2_{tr(p)} \equiv p^4 \cdot 3F2 \left( \phi_q, \rho^m, \overline{\rho}^m \mid 1 \right)_{p^2} \quad (\text{mod } p^2).
\]

For (1.5) – (1.8), we are able to prove the following.

Theorem 4. Let \( p \geq 5 \) be a prime.

1. We have \( \sum_{n=0}^{p-1} \frac{(2n)!}{n!} 64^{-n} = a(p) \pmod{p^2} \).
2. If \( p \equiv 1 \pmod{3} \), then \( \sum_{n=0}^{p-1} \frac{(3n)!}{n!} 108^{-n} = b(p) \pmod{p^2} \).
   
   If \( p \equiv 2 \pmod{3} \), then \( \left( \sum_{n=0}^{p-1} \frac{(3n)!}{n!} 108^{-n} \right)^2 \equiv b(p)^2 \pmod{p^2} \).
3. If \( p \equiv 1 \pmod{4} \), then \( \sum_{n=0}^{p-1} \frac{(4n)!}{n!} 256^{-n} = c(p) \pmod{p^2} \).
   
   If \( p \equiv 3 \pmod{4} \), then \( \left( \sum_{n=0}^{p-1} \frac{(4n)!}{n!} 256^{-n} \right)^2 \equiv c(p)^2 \pmod{p^2} \).
4. If \( p \equiv 1 \pmod{6} \), then \( \sum_{n=0}^{p-1} \frac{(6n)!}{(3n)!n!} 1728^{-n} = a(p) \pmod{p^2} \).
   
   If \( p \equiv 5 \pmod{6} \), then \( \left( \sum_{n=0}^{p-1} \frac{(6n)!}{(3n)!n!} 1728^{-n} \right)^2 \equiv a(p)^2 \pmod{p^2} \).

In sections 2 and 3, we prove Theorems 1-3 using the method of proof in [M2] (i.e. we use basic character theory, the Gross-Koblitz formula [GK], and properties of the \( p \)-adic \( \Gamma \)-function). For these proofs, the arguments are similar enough to those in [M2] that we only point out the changes made in the strategy. The key change is in dealing with the strange combinatorial expressions involving harmonic numbers that we encounter. In [M2], using Wilf-Zeilberger theory, the author evaluated two families of expressions explicitly in terms of \( p \) (see (5.28), (6.21)). Here, by writing the expressions in a different way and by using new techniques, we avoid WZ-theory, Corollary 2 is immediate and Corollary 1 uses (1.16).

In section 4, we prove Theorem 4 using Corollary 2 and Theorem 3. In addition, we need to evaluate the Gaussian hypergeometric series in terms of the trace of Frobenius. To accomplish this we borrow an idea from Ono [O] and use a character analog of Whipple’s theorem for classical \( 3F2 \) hypergeometric series. This analog was found by Greene [G], and it yields an expression in terms of Jacobi sums. Using several theorems of Berndt, Evans, and Williams ([BE], [BEW]), and a theorem of Beukers and Stienstra [SB], we evaluate these Jacobi sums in terms of the coefficients of the respective weight three modular forms.

2. Proof of Theorem 1

We begin this section with a lemma and a proposition. The proof of the lemma is trivial.
Lemma 2.1. If \( p \geq 5 \) is a prime and \( n \geq 1 \), then
\[
\sum_{k=1}^{p-1} k^n \equiv \begin{cases} 
0 & \text{mod } p \\
-1 & \text{mod } p
\end{cases}
\] if \( p-1 \nmid n \),
and
\[
\sum_{k=1}^{p-1} k^n \equiv \begin{cases} 
0 & \text{mod } p \\
-1 & \text{mod } p
\end{cases}
\] if \( p-1 \mid n \).

Proposition 2.2. Let \( m \) and \( d \) be integers such that \( 1 \leq m < d \). If \( p \equiv 1 \pmod{d} \) is a prime, then define \( r \) such that \( p = dr + 1 \).

1. If \( 0 \leq j \leq mr \), then \( \left( \frac{m}{d} \right)_j \equiv \Gamma_p(1 - \frac{m}{d})(d-m)r + j)! \pmod{p} \).
2. If \( mr + 1 \leq j \leq p - 2 \), then \( \left( \frac{m}{d} \right)_j \left( \frac{d}{mp} \right) \equiv \Gamma_p(1 - \frac{m}{d})(d-m)r + j)! \pmod{p} \).

Proof of Proposition 2.2. We first prove (1). From Proposition (1),
\[
\Gamma_p(\frac{m}{d} + j) = (-1)^j(\frac{m}{d})_j\Gamma_p(\frac{m}{d}).
\]

Using (1.15) and (1.13), we obtain
\[
\Gamma_p(\frac{m}{d} + j) \equiv \Gamma_p((d-m)r + 1 + j) \equiv (-1)^{(d-m)r + 1 + j}((d-m)r + j)! \pmod{p}.
\]

We then equate the two expressions and use Proposition (1.16).

For (2), the argument is similar. We use Proposition (1.14) to obtain
\[
\Gamma_p(\frac{m}{d} + j) = (-1)^j \cdot \frac{d}{mp} \cdot (\frac{m}{d})_j \Gamma_p(\frac{m}{d}),
\]
and we use Proposition (1.15) and (1.13) to obtain
\[
\Gamma_p(\frac{m}{d} + j) \equiv \Gamma_p((d-m)r + 1 + j) \equiv (-1)^{(d-m)r + 1 + j} \cdot \frac{d}{p} \cdot ((d-m)r + j)! \pmod{p}.
\]

We note that the expressions in (2.3) and (2.4) are \( p \)-integral. The terms with the \( p \)'s in their denominator are only present to cancel out their reciprocals.

Proof of Theorem 1. Recalling the notation of Theorem 1, we define \( r_i := \frac{p-1}{d_i} \). We also define the harmonic number \( H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \). Without loss of generality, we can assume \( m_1 r_1 \leq m_2 r_2 \leq \cdots \leq m_{n+1} r_{n+1} \). Using basic character theory, the Gross-Koblitz formula, and \( p \)-adic \( \Gamma \)-function properties, we follow the method of proof in [M2, section 5] to obtain
\[
(-1)^{n} p^{n \cdot n+1} F_n \left( \frac{m_1}{r_1}, \frac{m_2}{r_2}, \ldots, \frac{m_{n+1}}{r_{n+1}} \mid 1 \right) \equiv p \left\{ \sum_{m_1 r_1 = 1}^{m_2 r_2} \left( \prod_{i=1}^{n+1} \frac{m_{r_i}}{d_i} \right) \left( \frac{j d_1}{m_1 p} \right) \right\}
\]
\[
+ \sum_{j=0}^{m_1 r_1} \left[ \prod_{i=1}^{n+1} \frac{m_{r_i}}{d_i} \right] \left[ 1 + j \cdot \left( \sum_{i=1}^{n+1} \left( H_{(d_i, m_i)r_i, j} - H_j \right) \right) \right]
\]
\[
+ \sum_{j=0}^{m_2 r_2} \left( \prod_{i=1}^{n+1} \frac{m_{r_i}}{d_i} \right) \pmod{p^2}.
\]

This is analogous to (5.25) in [M2]. If \( m_1 r_1 = m_2 r_2 \), then only the second sum in the braces is present. Using Proposition 2.2 and arguing as we did for (5.27) in
\[ (-1)^n p^n \cdot n+1 F_n \left( \rho_1^{m_1}, \rho_2^{m_2}, \ldots, \rho_{n+1}^{m_{n+1}} | \epsilon_p \right) \]

\[ = \sum_{j=0}^{m_2 r_2} \prod_{i=1}^{n+1} \left( \frac{(m_i - m_i) r_i + j!}{j!} \right) \cdot \left[ 1 + j \cdot \sum_{i=1}^{n+1} (H_{(d_i - m_i) r_i + j} - H_j) \right]. \]

We determine when \( A \equiv 0 \pmod{p} \) and when \( A \equiv 1 \pmod{p} \). We can extend the sum in \( A \) to \( p-1 \) to obtain

\[ A := \sum_{j=0}^{p-1} \prod_{i=1}^{n+1} \left( \frac{(d_i - m_i) r_i + j!}{j!} \right) \cdot \left[ 1 + j \cdot \sum_{i=1}^{n+1} (H_{(d_i - m_i) r_i + j} - H_j) \right] \pmod{p}. \]

In other words, for \( j \geq m_2 r_2 + 1 \) the factorials for \( i = 1 \) and \( i = 2 \) each contain a factor of \( p \); moreover, at most one \( p \) is cancelled by a term of the harmonic number. Noting that

\[ ((d_i - m_i) r_i + j!) / j! = (j + 1)_{(d_i - m_i) r_i}, \]

we can rewrite the right-hand side:

\[ A := \sum_{j=0}^{p-1} \frac{d}{d j} \left[ \prod_{i=1}^{n+1} (j + 1)_{(d_i - m_i) r_i} \right] \pmod{p}. \]

Define the polynomial \( p(j) \in \mathbb{Z}[j] \) by

\[ p(j) := \frac{d}{d j} \left[ \prod_{i=1}^{n+1} (j + 1)_{(d_i - m_i) r_i} \right] = \sum_{k=0}^{D} a_k j^k, \text{ where } D := \sum_{i=1}^{n+1} (d_i - m_i) r_i, \]

to obtain

\[ A := \sum_{j=0}^{p-1} \left( a_0 + \sum_{k=1}^{D} a_k j^k \right) \equiv \sum_{k=1}^{D} a_k \sum_{j=1}^{p-1} j^k \pmod{p}. \]

We consider the case \( D < 2(p-1) \). By Lemma 2.1, the only \( k \) we need to be concerned with in the \( j \) summation is \( k = p-1 \). Since \( p(j) \) is a derivative, the coefficient \( a_{p-1} \) will contain a factor of \( p \). Hence in this case, \( A \equiv 0 \pmod{p} \). We consider the case \( D = 2(p-1) \). Using the above information, the only \( k \) we need to concern ourselves with is \( k = 2(p-1) \). Since \( p(j) \) is a derivative of a monic polynomial, it follows that \( a_{2(p-1)} = (2p - 1) \). Using Lemma 2.1, we find that \( A \equiv 1 \pmod{p} \). Since we can extend the first sum in (2.6) from \( m_2 r_2 \) to \( p-1 \), the theorem follows. \( \square \)
3. Proofs of Theorems 2 and 3

Proof of Theorem 3. We recall the notation of Theorem 3. We define \( n \) such that \( p = d(n + 1) - 1 \) and define \( N_1 := m(n + 1) - 1 \), \( N_2 := (d - m)(n + 1) - 1 \). Using the method of proof in \([M2\) section 6], and using the appropriate analog of Proposition 2.2, we obtain

\[
(3.1) 
\]

\[
p^4 \cdot {}_3 F_2 \left( \phi, \rho^m, \frac{\rho^m}{p^2}; \epsilon, \frac{\epsilon}{p^2} \mid 1 \right)_{p^2} = \left( \sum_{k=0}^{p-1} \frac{(\frac{1}{k})k(\frac{m}{k})k(1 - \frac{m}{k})k}{k!^3} \right)^2
\]

\[
+ 2 \cdot p \cdot \left( \sum_{k=0}^{N_1} k \cdot \frac{(\frac{1}{k})k(\frac{m}{k})k(1 - \frac{m}{k})k}{k!^3} \right) \cdot \left( \Gamma_p(\frac{1}{2}) \Gamma_p(\frac{m}{\sigma}) \Gamma_p(1 - \frac{m}{\sigma}) \right) \cdot B \pmod{p^2},
\]

where

\[
(3.2) 
B := \left( \sum_{j=0}^{p-1} \frac{(\frac{1}{j+1})! \left( N_1 + j \right)! \left( N_2 + j \right)!}{j!^3} \right) \left[ H_{\frac{p-1}{2} + j} + H_{N_1 + j} + H_{N_2 + j} - 3 H_j \right].
\]

We point out that line (3.1) is similar to (6.21) of \([M2] \). We note that the case where \( m = 1 \), \( d = p + 1 \) is handled like it is in (6.15) of \([M2] \). Arguing as in section 2, we obtain

\[
(3.3) 
B \equiv \sum_{j=0}^{p-1} \frac{d}{d_j} \left( j + 1 \right)_{p-1} (j + 1)_{N_1} (j + 1)_{N_2} \pmod{p}.
\]

If we let \( d \) be the degree of the polynomial in terms of \( j \), we see that \( d < 2(p - 1) \). Arguing as in section 2, we have that \( B \equiv 0 \pmod{p} \). \( \square \)

Proof of Theorem 2. We recall the notation of Theorem 2. We define \( n_1 \) such that \( p = d(n_1 + 1) - 1 \). Without loss of generality we assume \( m_1/d_1 \leq m_2/d_2 \leq 1/2 \).

We define \( R_1 := m_1(n_1 + 1) - 1 \) and \( S_i := (d_i - m_i)(n_i + 1) - 1 \). We note \( R_1 \leq R_2 \).

Using the method of proof in \([M2\) section 6], and using the appropriate analog of Proposition 2.2, we obtain

\[
-p^6 \cdot {}_4 F_3 \left( \rho_1^{m_1}, \rho_1^{m_1}, \rho_2^{m_2}, \rho_2^{m_2}; \epsilon, \frac{\epsilon}{p^2} \mid 1 \right)_{p^2}
\]

\[
\equiv \left( \sum_{k=0}^{R_2} \frac{(\frac{m}{d})_k(\frac{d_1 - m}{d_1})_k(\frac{d_2 - m}{d_2})_k}{k!^3} \right)^2
\]

\[
+ 2 \cdot p \cdot \left( \sum_{k=0}^{R_1} k \cdot \frac{(\frac{m}{d})_k(\frac{d_1 - m}{d_1})_k(\frac{d_2 - m}{d_2})_k}{k!^3} \right)
\]

\[
\cdot \left( \Gamma_p(\frac{m}{d_1}) \Gamma_p(\frac{d_1 - m}{d_1}) \Gamma_p(\frac{d_2 - m}{d_2}) \right) \cdot C \pmod{p^2},
\]

where

\[
(3.5) 
C := \sum_{j=0}^{R_2} \frac{(R_1 + j)! \left( S_1 + j \right)! \left( R_2 + j \right)! \left( S_2 + j \right)!}{j!^3} \left[ H_{R_1 + j} + H_{S_1 + j} + H_{R_2 + j} + H_{S_2 + j} - 4 H_j \right].
\]
The $m_1 = 1, d_1 = p+1$ case and the $m_1 = m_2 = 1, d_1 = d_2 = p+1$ case are handled like they are in [M2]. Arguing as we did in the proof of Theorem 3, we find that $C \equiv 0 \pmod{p}$.

\[ \square \]

4. Proof of Theorem 4

We begin with a theorem of Beukers and Stienstra that describes the coefficients of the three modular forms in question. We recall the modular forms (1.2)-(1.4).

**Theorem** ([SB, 14.2]). If we define $\Phi_1(p) := a(p), \Phi_3(p) := b(p), \Phi_2(p) := c(p)$, then the $p$-th coefficients of the modular forms are given by

\[
\Phi_M(p) = \begin{cases} 
0 & \text{if } (\frac{-M}{p}) = -1, \\
4a^2 - 2p & \text{if } (\frac{-M}{p}) = 1, p = a^2 + Mb^2.
\end{cases}
\]

We rewrite the conjecture to motivate the use of Corollary 2 and Theorem 3.

**Conjecture.** If $p \geq 5$ is a prime and $\gamma(p)$ is as before, then

\[
\begin{align*}
(4.1) \quad 3F_2 \left( \frac{1}{2}, \frac{1}{2}, 1 | 1 \right)_{tr(p)} & \equiv a(p) \pmod{p^2}, \\
(4.2) \quad 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2} | 1 \right)_{tr(p)} & \equiv b(p) \pmod{p^2}, \\
(4.3) \quad 3F_2 \left( \frac{1}{2}, 1, \frac{3}{2} | 1 \right)_{tr(p)} & \equiv c(p) \pmod{p^2}, \\
(4.4) \quad 3F_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{5}{2} | 1 \right)_{tr(p)} & \equiv \gamma(p)a(p) \pmod{p^2}.
\end{align*}
\]

From the new formulation, we see that we need to use Corollary 2 and Theorem 3 where $m = 1$ and $d = 2, 3, 4$ or 6. For primes $p$ where $p \equiv 1 \pmod{d}$ we use Corollary 2, for primes $p$ where $p \equiv -1 \pmod{d}$ we use Theorem 3. The proof of Theorem 4 is thus reduced to evaluating the $3F_2$ Gaussian hypergeometric series. First, we state a theorem which is a special case of Greene ([G], 4.38(ii)). The corollary follows from the two binomial coefficient properties (1.10).

**Theorem** ([G]). If $B$ is a nontrivial character on $F_q$, then

\[
3F_2 \left( \phi, B, \epsilon_q | \frac{1}{q} \right) = B(-1) \left\{ \begin{array}{ll}
0 & \text{if } B \neq \Box, \\
\left( \frac{\gamma}{q} \right)(\phi \chi) + (\phi \chi) \frac{\gamma}{q} & \text{if } B = \chi^2.
\end{array} \right.
\]

**Corollary 4.1.** If $B$ is a nontrivial character on $F_q$, then

\[
q^2 \cdot 3F_2 \left( \phi, B, \epsilon_q | \frac{1}{q} \right) = B(-1) \left\{ \begin{array}{ll}
0 & \text{if } B \neq \Box, \\
J_r(\chi, \phi)^2 + J_r(\chi, \phi) & \text{if } B = \chi^2.
\end{array} \right.
\]

The following two propositions evaluate the Gaussian hypergeometric series in Corollary 2 and Theorem 3, respectively. Theorem 4 is then immediate. We recall (1.2)-(1.4) and define $\alpha_2(p) := a(p), \alpha_3(p) := b(p), \alpha_4(p) := c(p)$, and $\alpha_6(p) := a(p)$.
Proposition 4.2. Fix a $d$, $d \in \{2, 3, 4, 6\}$. Let $p$ be a prime, $p \equiv 1 \pmod{d}$. If $\rho_d$ is a character of order $d$ on $\mathbb{F}_p$, then

$$p^2 \cdot 3F_2\left(\phi, \rho_d, \frac{\rho_d}{\epsilon_p} \mid 1\right)_{p^2} = \alpha_d(p).$$

Proof of Proposition 4.2. This method comes from Ono [O], where he does the case $d = 2$. We have two cases. For the first case we consider $p, p \equiv d + 1 \pmod{2d}$. Here $\rho_d$ is not a square, so the Gaussian hypergeometric series evaluates to zero. Using the theorem of [SB] and basic Legendre symbol properties, we have that $\alpha_d(p) = 0$. For $d = 3$ this case is vacuous.

For the second case we consider $p, p \equiv 1 \pmod{2d}$. Here $\rho_d = \chi^2$ for some character $\chi$. We consider $d = 4$. By [BEW] Theorem 3.3.1,

$$J_1(\chi, \phi)^2 + J_1(\bar{\chi}, \phi)^2 = (a + ib\sqrt{2})^2 + (a - ib\sqrt{2})^2 = 4a^2 - 2p,$$

where $p = a^2 + 2b^2$. For $d = 2, 3$ and $6$ we use [BEW] Theorems 3.2.1, 3.1.1, and 3.5.2, respectively. For each $d$, we use [SB] and see that this equals $\alpha_d(p)$.

Proposition 4.3. Fix a $d$, $d \in \{3, 4, 6\}$. Let $p$ be a prime, $p \equiv -1 \pmod{d}$. If $\rho_d$ is a character of order $d$ on $\mathbb{F}_p$, then

$$p^4 \cdot 3F_2\left(\phi, \rho_d, \frac{\rho_d}{\epsilon_p^2} \mid 1\right)_{p^2} = \alpha_d(p)^2 - (-1)^{p-(d-1)} \frac{p}{d} 2p^2.$$

Proof of Proposition 4.3. We note that $\rho_d$ is always a square. We have two cases to consider. For the first case, we consider $p$ with $p \equiv -1 \pmod{2d}$. Using the theorem of [SB] we have that $\alpha_d(p) = 0$. By [BE] Theorem 2.14,

$$J_2(\chi, \phi)^2 + J_2(\bar{\chi}, \phi)^2 = 2p^2.$$

For the second case, we consider $p$ with $p \equiv d - 1 \pmod{2d}$. We consider $d = 4$. By [BEW] Theorem 4.6,

$$J_2(\chi, \phi)^2 + J_2(\bar{\chi}, \phi)^2 = (a + ib\sqrt{2})^4 + (a - ib\sqrt{2})^4 = (4a^2 - 2p)^2 + 2p^2 = c(p)^2 - 2p^2,$$

where $p = a^2 + 2b^2$, and the last equality follows from [SB]. For $d = 3$ this case is vacuous. For $d = 6$ we use [BEW] Theorem 4.10.

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References


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