SUPERCONGRUENCES FOR TRUNCATED $\,_{n+1}F_n$
HYPERGEOMETRIC SERIES WITH APPLICATIONS
TO CERTAIN WEIGHT THREE NEWFORMS

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Abstract. We prove general results on supercongruences between values of truncated $\,_{n+1}F_n$ hypergeometric functions and their character analogs. As a consequence of the main results of this paper, we prove Beukers-type supercongruences for certain weight three newforms.

1. Introduction

In [RV1], Fernando Rodriguez-Villegas discovered numerically a number of Beukers-type supercongruences for hypergeometric Calabi-Yau manifolds of dimension $d \leq 3$. Specifically, he observed supercongruences between the truncated fundamental period of the Picard-Fuchs differential equation of the manifold and an expression derived from the number of its $\mathbb{F}_p$-points. This had been motivated by his joint work with Candelas and de la Ossa [COV]. Here we prove general results on supercongruences between values of truncated $\,_{n+1}F_n$ hypergeometric functions and their character analogs. As a consequence of these results, we prove some of the observed supercongruences for manifolds of dimension $d = 2$. Supercongruences of this type were first observed by Beukers [B] in connection with the Apéry numbers used in the proof of the irrationality of $\zeta(3)$. Ahlgren and Ono [AO] proved Beukers’ supercongruence conjecture relating Apéry numbers to the coefficients of a certain weight four newform.

In [RV1] and [RV2], Rodriguez-Villegas identified four modular K3 surfaces with potential supercongruences. We define Dedekind’s eta function by the infinite product:

\begin{equation}
\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q := e^{2\pi i z}.
\end{equation}

We then define the integers $a(n)$, $b(n)$, and $c(n)$ by

\begin{equation}
\sum_{n=1}^{\infty} a(n) q^n := \eta^6(4z) \in S_3(\Gamma_0(16), \left(\frac{-4}{d}\right)),
\end{equation}

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(1.3) \[ \sum_{n=1}^{\infty} b(n)q^n := \eta^3(6z)\eta^3(2z) \in S_3(\Gamma_0(12), (\frac{-3}{q})), \]

(1.4) \[ \sum_{n=1}^{\infty} c(n)q^n := \eta^2(8z)\eta(4z)\eta(2z)\eta^2(z) \in S_3(\Gamma_0(8), (\frac{-2}{q})). \]

These weight three newforms are related to modular K3 surfaces. They are extensively studied in [SB], where, among other results, the authors prove several modulo \( p \) congruences. From [RV1] and [RV2] we are able to formulate the following:

**Conjecture.** If \( p \geq 5 \) is a prime, then

(1.5) \[ \sum_{n=0}^{p-1} \frac{(2n)!^3}{n!^6} 64^{-n} \equiv a(p) \pmod{p^2}, \]

(1.6) \[ \sum_{n=0}^{p-1} \frac{(3n)!(2n)!}{n!^5} 108^{-n} \equiv b(p) \pmod{p^2}, \]

(1.7) \[ \sum_{n=0}^{p-1} \frac{(4n)!}{n!^4} 256^{-n} \equiv c(p) \pmod{p^2}, \]

(1.8) \[ \sum_{n=0}^{p-1} \frac{(6n)!}{(3n)!n!^5} 1728^{-n} \equiv \gamma(p)a(p) \pmod{p^2}, \]

where \( \gamma(p) := -1 \) if \( p \equiv 5 \pmod{12} \) and \( \gamma(p) := 1 \) otherwise.

It should be noted that (1.5) has already been proved by several individuals including Ishikawa [I], Van Hamme [VH], and Ahlgren [A]. The numbers 64, 108, 256, 1728 are called the conifold points (see [RV1]). Here we prove several cases of these conjectures.

To state our results, we recall basic facts about characters and Jacobi sums and introduce some notation. We denote by \( F_q \) the finite field with \( q = p^r \) elements, where \( p \) is a prime. We extend all multiplicative characters \( \chi : F_q^\times \to \mathbb{C}_p \), including the trivial character \( \epsilon_q \), to \( F_{q^r} \) by setting \( \chi(0) := 0 \). If \( A \) and \( B \) are two characters on \( F_q \), then we define \( \binom{A}{B} \) in terms of the Jacobi sum by

(1.9) \[ \binom{A}{B} := \frac{B(-1)}{q} \sum_{x \in F_q} A(x)B(1-x), \]

where \( J_r(\cdot, \cdot) \) is a Jacobi sum over \( F_{q^r} \). We recall some useful properties of binomial coefficients \( \binom{A}{B} \), (2.6)-(2.7):

(1.10) \[ \binom{A}{B} = \binom{A}{\overline{AB}} \quad \text{and} \quad \binom{A}{B} = \binom{BA}{B}(-1). \]

If \( A_0, A_1, \ldots, A_n \) and \( B_1, B_2, \ldots, B_n \) are characters on \( F_q \) and if \( x \in F_q \), then Greene [G] defines \( F_{n+1}F_n \) Gaussian hypergeometric series by

(1.11) \[ n+1F_n \left( A_0, \ldots, A_n \mid x \right) := \frac{q}{q-1} \sum_{\chi} \binom{A_0\chi}{\chi} \binom{A_1\chi}{B_1\chi} \cdots \binom{A_n\chi}{B_n\chi} \chi(x), \]

where the sum runs over all characters \( \chi \) on \( F_q \). We note that this definition lives in some extension of \( \mathbb{Q}_p \). For certain choices of characters, the right-hand side actually is in \( \mathbb{Z}_p \).
If \( m \) is a positive integer, then we define the truncated hypergeometric series by

\[
(1.12) \quad _{n+1}F_n \left( \frac{a_0}{b_1}, \frac{a_1}{b_2}, \ldots, \frac{a_n}{b_{n+1}} \right) := \sum_{k=0}^{m-1} \frac{(a_0)_k(a_1)_k \cdots (a_n)_k}{k!(b_1)_k \cdots (b_{n+1})_k} x^k,
\]

where \((a)_k := a(a+1) \cdots (a+k-1)\).

If \( n \in \mathbb{N} \), we define the \( p \)-adic \( \Gamma \)-function on the ring \( \mathbb{Z}_p \) of \( p \)-adic integers by

\[
(1.13) \quad \Gamma_p(n) := (1-n)^n \prod_{j<n, p \nmid j} j \quad \text{and} \quad \Gamma_p(x) := \lim_{n \to x} \Gamma_p(n), \quad x \in \mathbb{Z}_p,
\]

where in the limit we take any sequence of positive integers that approaches \( x \) in the \( p \)-adic sense. We recall three basic properties of the \( p \)-adic \( \Gamma \)-function. If \( p \geq 5 \) is a prime and \( x, y \in \mathbb{Z}_p \), then the following are true. We have

\[
(1.14) \quad \Gamma_p(x+1) = \begin{cases} 
-x \Gamma_p(x) & \text{if } x \in \mathbb{Z}_p^*, \\
-\Gamma_p(x) & \text{if } x \in p\mathbb{Z}_p.
\end{cases}
\]

If \( n \geq 1 \), then

\[
(1.15) \quad x \equiv y \pmod{p^n} \Rightarrow \Gamma_p(x) \equiv \Gamma_p(y) \pmod{p^n}.
\]

If \( R(x) \) denotes the reduction of \( x \) modulo \( p \) to the range \( \{1, \ldots, p\} \), then

\[
(1.16) \quad \Gamma_p(x)\Gamma_p(1-x) = (-1)^{R(x)}.
\]

We are now able to state the results of this paper. Let \( \phi_q \) denote the character of order 2 on \( \mathbb{F}_q \), and let \( \epsilon_q \) denote the trivial character on \( \mathbb{F}_q \). In the sequel we shall drop the subscript \( q \) since it will be obvious from the context.

**Theorem 1.** If \( p \) is a prime, \( p \equiv 1 \pmod{d_i} \) with \( 1 \leq m_i < d_i \), \( \rho_i \) is a character of order \( d_i \) on \( \mathbb{F}_p \), and \( \sum_{i=1}^{n+1} \frac{m_i}{d_i} \geq n-1 \), then

\[
_{n+1}F_n \left( \frac{m_1}{d_1}, \frac{m_2}{d_2}, \ldots, \frac{m_{n+1}}{d_{n+1}} \right) \equiv (-1)^{n} p^n \cdot _{n+1}F_n \left( \frac{\rho_1^{m_1}}{\epsilon_p}, \frac{\rho_2^{m_2}}{\epsilon_p}, \ldots, \frac{\rho_{n+1}^{m_{n+1}}}{\epsilon_p} \right)_{tr(p)} \delta \cdot p \pmod{p^2},
\]

where \( \delta := \begin{cases} 
0 & \text{if } \sum_{i=1}^{n+1} \frac{m_i}{d_i} > n-1, \\
\prod_{i=1}^{n+1} \Gamma_p(1 - \frac{m_i}{d_i}) & \text{if } \sum_{i=1}^{n+1} \frac{m_i}{d_i} = n-1.
\end{cases} \)

**Corollary 1.** If \( p \) is a prime, \( p \equiv 1 \pmod{d_i} \), \( 1 \leq m_i < d_i \), and \( \rho_i \) is a character of order \( d_i \) on \( \mathbb{F}_p \), then

\[
^4F_3 \left( \frac{m_1}{d_1}, \frac{m_2}{d_2}, \frac{1-m_1}{1}, \frac{1-m_2}{1} \right)_{tr(p)} \equiv -p^3 \cdot ^4F_3 \left( \frac{\rho_1^{m_1}}{\epsilon_p}, \frac{\rho_2^{m_2}}{\epsilon_p}, \frac{\rho_3^{m_3}}{\epsilon_p} \right)_{tr(p)} \equiv (-1)^{\frac{m_1}{d_1}(p-1) + \frac{m_2}{d_2}(p-1)} p \pmod{p^2}.
\]

**Corollary 2.** If \( p \) is a prime, \( p \equiv 1 \pmod{d} \), \( 1 \leq m < d \), and \( \rho \) is a character of order \( d \) on \( \mathbb{F}_p \), then

\[
^3F_2 \left( \frac{1}{2}, \frac{m}{d}, \frac{1-m}{1}, \frac{1}{1} \right)_{tr(p)} \equiv p^2 \cdot ^3F_2 \left( \frac{\phi_q}{\epsilon_p}, \frac{\rho^m}{\epsilon_p}, \frac{\overline{\rho}^m}{\epsilon_p} \right)_{tr(p)} \pmod{p^2}.
\]
**Theorem 2.** If $p$ is a prime, $p \equiv -1 \pmod{d}$, $1 \leq m_i < d$, and $\rho_i$ is a character of order $d_i$ on $\mathbb{F}_{p^2}$, then
\begin{equation}
\begin{aligned}
4F_3\left(\frac{m_1}{p^2}, \frac{1-m_1}{1}; \frac{m_2}{p^2}, \frac{1-m_2}{1}; 1\right)_{tr(p)}^2 \\
\equiv -p^6 \cdot 4F_3\left(\rho_1m_1, \rho_1m_1; \rho_2m_2, \rho_2m_2; \epsilon_{p^2}, \epsilon_{p^2}; 1\right)_{p^2} \quad (\text{mod } p^2).
\end{aligned}
\end{equation}

**Theorem 3.** If $p$ is a prime, $p \equiv -1 \pmod{d}$, $1 \leq m < d$, and $\rho$ is a character of order $d$ on $\mathbb{F}_{p^2}$, then
\begin{equation}
\begin{aligned}
3F_2\left(\frac{1}{2}, \frac{m}{p^2}; 1 - \frac{m}{p^2} \mid 1\right)_{tr(p)}^2 \\
\equiv p^4 \cdot 3F_2\left(\phi_q, \rho^m, \rho^m; \epsilon_{p^2}, \epsilon_{p^2}; 1\right)_{p^2} \quad (\text{mod } p^2).
\end{aligned}
\end{equation}

For (1.5) – (1.8), we are able to prove the following.

**Theorem 4.** Let $p \geq 5$ be a prime.

1. We have $\sum_{n=0}^{p-1} \frac{(2n)!}{n!n!} 64^{-n} = a(p)$ (mod $p^2$).
2. If $p \equiv 1 \pmod{3}$, then $\sum_{n=0}^{p-1} \frac{(3n)!}{n!n!} 108^{-n} = b(p)$ (mod $p^2$).
3. If $p \equiv 2 \pmod{3}$, then $\left(\sum_{n=0}^{p-1} \frac{(3n)!}{n!n!} 108^{-n}\right)^2 = b(p)^2$ (mod $p^2$).
4. If $p \equiv 1 \pmod{4}$, then $\sum_{n=0}^{p-1} \frac{(4n)!}{n!n!} 256^{-n} = c(p)$ (mod $p^2$).
5. If $p \equiv 3 \pmod{4}$, then $\left(\sum_{n=0}^{p-1} \frac{(4n)!}{n!n!} 256^{-n}\right)^2 = c(p)^2$ (mod $p^2$).
6. If $p \equiv 1 \pmod{6}$, then $\sum_{n=0}^{p-1} \frac{(6n)!}{n!n!} 1728^{-n} = a(p)$ (mod $p^2$).
7. If $p \equiv 5 \pmod{6}$, then $\left(\sum_{n=0}^{p-1} \frac{(6n)!}{n!n!} 1728^{-n}\right)^2 = a(p)^2$ (mod $p^2$).

In sections 2 and 3, we prove Theorems 1-3 using the method of proof in [M2] (i.e. we use basic character theory, the Gross-Koblitz formula [GK], and properties of the $p$-adic $\Gamma$-function). For these proofs, the arguments are similar enough to those in [M2] that we only point out the changes made in the strategy. The key change is in dealing with the strange combinatorial expressions involving harmonic numbers that we encounter. In [M2], using Wilf-Zeilberger theory, the author evaluated two families of expressions explicitly in terms of $p$ (see (5.28), (6.21)). Here, by writing the expressions in a different way and by using new techniques, we avoid WZ-theory. Corollary 2 is immediate and Corollary 1 uses (1.16).

In section 4, we prove Theorem 4 using Corollary 2 and Theorem 3. In addition, we need to evaluate the Gaussian hypergeometric series in terms of the trace of Frobenius. To accomplish this we borrow an idea from Ono [O] and use a character analog of Whipple’s theorem for classical $3F_2$ hypergeometric series. This analog was found by Greene [G], and it yields an expression in terms of Jacobi sums. Using several theorems of Berndt, Evans, and Williams ([BE], [BEW]), and a theorem of Beukers and Stienstra [SB], we evaluate these Jacobi sums in terms of the coefficients of the respective weight three modular forms.

2. **Proof of Theorem 1**

We begin this section with a lemma and a proposition. The proof of the lemma is trivial.
Lemma 2.1. If \( p \geq 5 \) is a prime and \( n \geq 1 \), then
\[
\sum_{k=1}^{p-1} k^n \equiv \begin{cases} 
0 & \text{mod } p \text{ if } p - 1 \nmid n, \\
-1 & \text{mod } p \text{ if } p - 1 \mid n.
\end{cases}
\]

Proposition 2.2. Let \( m \) and \( d \) be integers such that \( 1 \leq m < d \). If \( p \equiv 1 \pmod{d} \) is a prime, then define \( r \) such that \( p = dr + 1 \).

(1) If \( 0 \leq j \leq mr \), then \( \left( \frac{m}{d} \right)_j \equiv \Gamma_p(1 - \frac{m}{d})(d - m)r + j \pmod{p} \).

(2) If \( mr + 1 \leq j \leq p - 2 \), then \( \left( \frac{m}{d} \right)_j \left( \frac{d}{mp} \right) \equiv \Gamma_p(1 - \frac{m}{d})(d - m)r + j \pmod{p} \).

Proof of Proposition 2.2. We first prove (1). From Proposition (1.14), we have
\[
\Gamma_p \left( \frac{m}{d} + j \right) = (-1)^j \left( \frac{m}{d} \right)_j \Gamma_p \left( \frac{m}{d} \right).
\]
Using (1.15) and (1.13), we obtain
\[
(2.2) \Gamma_p \left( \frac{m}{d} + j \right) \equiv \Gamma_p((d - m)r + 1 + j) \equiv (-1)^{(d - m)r + j}((d - m)r + j) \pmod{p}.
\]
We then equate the two expressions and use Proposition (1.16).

For (2), the argument is similar. We use Proposition (1.14) to obtain
\[
(2.3) \Gamma_p \left( \frac{m}{d} + j \right) = (-1)^j \cdot \frac{d}{mp} \cdot \left( \frac{m}{d} \right)_j \Gamma_p \left( \frac{m}{d} \right),
\]
and we use Proposition (1.15) and (1.13) to obtain
\[
(2.4) \Gamma_p \left( \frac{m}{d} + j \right) \equiv \Gamma_p((d - m)r + 1 + j) \equiv (-1)^{(d - m)r + j} \cdot \frac{d}{p} \cdot ((d - m)r + j) \pmod{p}.
\]

We note that the expressions in (2.3) and (2.4) are \( p \)-integral. The terms with the \( p \)'s in their denominator are only present to cancel out their reciprocals.

Proof of Theorem 1. Recalling the notation of Theorem 1, we define \( r_i := \frac{p - 1}{d_i} \). We also define the harmonic number \( H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} \). Without loss of generality, we can assume \( m_1r_1 \leq m_2r_2 \leq \cdots \leq m_{n+1}r_{n+1} \). Using basic character theory, the Gross-Koblitz formula, and \( p \)-adic \( \Gamma \)-function properties, we follow the method of proof in [M2, section 5] to obtain
\[
(2.5) \quad (-1)^np^n \cdot n + 1 F_n \left( \rho_1^{m_1}, \rho_2^{m_2}, \ldots, \rho_{n+1}^{m_{n+1}} \mid 1 \right)
= p \left( \sum_{m_1r_1+1}^{m_2r_2} \sum_{i=1}^{n+1} \left( \frac{m_i}{d_i} \right)_j \right)
+ \sum_{j=0}^{m_1r_1} \left( \prod_{i=1}^{n+1} \left( \frac{m_i}{d_i} \right)_j \right) \left[ 1 + j \cdot \left( \sum_{i=1}^{n+1} \left( H(d_i, m_i)r_i + j - H_j \right) \right) \right]
+ \sum_{j=0}^{m_2r_2} \left( \prod_{i=1}^{n+1} \left( \frac{m_i}{d_i} \right)_j \right) \pmod{p^2}.
\]

This is analogous to (5.25) in [M2]. If \( m_1r_1 = m_2r_2 \), then only the second sum in the braces is present. Using Proposition 2.2 and arguing as we did for (5.27) in
Theorem 2 yields 

\[ (-1)^n p^n \cdot n+1 F_{n+1} \left( \frac{m_1}{p}, \frac{m_2}{p}, \ldots, \frac{m_{n+1}}{p} \mid 1 \right) \]

\[ \equiv \sum_{j=0}^{m_2 r_2} \prod_{i=1}^{n+1} \frac{(d_i - m_i) r_i + j}{j!} + p \cdot \left( \prod_{i=1}^{n+1} \Gamma_p(1 - \frac{m_i}{p}) \right) \cdot A \pmod{p^2}, \]

where

\[ A := \sum_{j=0}^{m_2 r_2} \prod_{i=1}^{n+1} \frac{(d_i - m_i) r_i + j}{j!} \cdot \left[ 1 + j \cdot \sum_{i=1}^{n+1} (H(d_i - m_i)r_i) \right]. \]

We determine when \( A \equiv 0 \pmod{p} \) and when \( A \equiv 1 \pmod{p} \). We can extend the sum in \( A \) to \( p - 1 \) to obtain

\[ A \equiv \sum_{j=0}^{p-1} \prod_{i=1}^{n+1} \frac{(d_i - m_i) r_i + j}{j!} \cdot \left[ 1 + j \cdot \sum_{i=1}^{n+1} (H(d_i - m_i)r_i) \right] \pmod{p}. \]

In other words, for \( j \geq m_2 r_2 + 1 \) the factorials for \( i = 1 \) and \( i = 2 \) each contain a factor of \( p \); moreover, at most one \( p \) is cancelled by a term of the harmonic number. Noting that

\[ ((d_i - m_i) r_i + j)! / j! = (j + 1)(d_i - m_i)r_i, \]

we can rewrite the right-hand side:

\[ A \equiv \sum_{j=0}^{p-1} \frac{d}{dj} \left[ j \prod_{i=1}^{n+1} (j + 1)(d_i - m_i)r_i \right] \pmod{p}. \]

Define the polynomial \( p(j) \in \mathbb{Z}[j] \) by

\[ p(j) := \frac{d}{dj} \left[ j \prod_{i=1}^{n+1} (j + 1)(d_i - m_i)r_i \right] = \sum_{k=0}^{D} a_k j^k, \]

to obtain

\[ A \equiv \sum_{j=0}^{p-1} \left( a_0 + \sum_{k=1}^{D} a_k j^k \right) \equiv \sum_{k=1}^{D} a_k \sum_{j=1}^{p-1} j^k \pmod{p}. \]

We consider the case \( D < 2(p - 1) \). By Lemma 2.1, the only \( k \) we need to be concerned with in the \( j \) summation is \( k = p - 1 \). Since \( p(j) \) is a derivative, the coefficient \( a_{p-1} \) will contain a factor of \( p \). Hence in this case, \( A \equiv 0 \pmod{p} \). We consider the case \( D = 2(p - 1) \). Using the above information, the only \( k \) we need to concern ourselves with is \( k = 2(p - 1) \). Since \( p(j) \) is a derivative of a monic polynomial, it follows that \( \alpha_{2(p-1)} = (2p - 1) \). Using Lemma 2.1, we find that \( A \equiv 1 \pmod{p} \). Since we can extend the first sum in (2.6) from \( m_2 r_2 \) to \( p - 1 \), the theorem follows. \( \square \)
3. Proofs of Theorems 2 and 3

**Proof of Theorem 3.** We recall the notation of Theorem 3. We define \( n \) such that \( p = d(n + 1) - 1 \) and define \( N_1 := n(n + 1) - 1, N_2 := (d - m)(n + 1) - 1 \). Using the method of proof in [M2, section 6], and using the appropriate analog of Proposition 2.2, we obtain

\[
\begin{align*}
\phi, \rho^m, \varepsilon_{p^2}, & \quad \frac{\phi}{\varepsilon_{p^2}} \Rightarrow \left( \sum_{k=0}^{p-1} \phi_k \right)^2 \\
2, & \quad \phi_k \left( m \right) \frac{1}{k^{13}} (1 - \frac{m}{k}) \\
+ 2 \cdot p \cdot \left( \sum_{k=0}^{N_1} k \cdot \phi_k \left( m \right) \frac{1}{k^{13}} (1 - \frac{m}{k}) \right) \cdot \left( \Gamma_p \left( \frac{d}{p} \right) \Gamma_p \left( \frac{40}{p} \right) \Gamma_p \left( 1 - \frac{m}{p} \right) \right) B \pmod{p^2},
\end{align*}
\]

where

\[
B := \left( \sum_{j=0}^{p-1} \frac{(j+1)j!}{j!} \left( \frac{N_1+j}{j} \right) \left( \frac{N_2+j}{j} \right) \right) \left( H_{N_1+j} + H_{N_2+j} - 3H_j \right).
\]

We point out that line (3.1) is similar to (6.21) of [M2]. We note that the case where \( m = 1, d = p + 1 \) is handled like it is in (6.15) of [M2]. Arguing as in section 2, we obtain

\[
B \equiv \sum_{j=0}^{p-1} \frac{d}{j} \left[ (j + 1) (j + 1)_{N_1} (j + 1)_{N_2} \right] \pmod{p}.
\]

If we let \( d \) be the degree of the polynomial in terms of \( j \), we see that \( d < 2(p-1) \).

Arguing as in section 2, we have that \( B \equiv 0 \pmod{p} \).

**Proof of Theorem 2.** We recall the notation of Theorem 2. We define \( n_i \) such that \( p = d_i(n_i + 1) - 1 \). Without loss of generality we assume \( m_i/d_i \leq m_2/d_2 \leq 1/2 \). We define \( R_i := m_i(n_i + 1) - 1 \) and \( S_i := (d_i - m_i)(n_i + 1) - 1 \). We note \( R_1 \leq R_2 \). Using the method of proof in [M2, section 6], and using the appropriate analog of Proposition 2.2, we obtain

\[
- p^6 \cdot 4 F_3 \left( \rho_1^{m_1}, \frac{\rho_1^{m_1}}{\varepsilon_{p^2}}, \frac{\rho_2^{m_2}}{\varepsilon_{p^2}}, \frac{\rho_2^{m_2}}{\varepsilon_{p^2}} \right) \pmod{p^2}
\]

\[
= \left( \sum_{k=0}^{R_2} \left( \frac{m_1}{d_1} \right) \frac{1}{d_1} \left( \frac{m_2}{d_2} \right) \frac{1}{d_2} \right)^2 \\
+ 2 \cdot p \cdot \left( \sum_{k=0}^{R_1} k \cdot \left( \frac{m_1}{d_1} \right) \frac{1}{d_1} \left( \frac{m_2}{d_2} \right) \frac{1}{d_2} \right) \cdot \left( \Gamma_p \left( \frac{m_1}{d_1} \right) \Gamma_p \left( \frac{d_1 - m_1}{d_1} \right) \Gamma_p \left( \frac{d_2 - m_2}{d_2} \right) \right) \cdot C \pmod{p^2},
\]

where

\[
C := \sum_{j=0}^{R_2} \frac{(R_1+j)!}{j!} \frac{(S_1+j)!}{j!} \frac{(S_2+j)!}{j!} \left[ H_{R_1+j} + H_{S_1+j} + H_{R_2+j} + H_{S_2+j} - 4H_j \right].
\]
4. Proof of Theorem 4

We begin with a theorem of Beukers and Stienstra that describes the coefficients of the three modular forms in question. We recall the modular forms (1.2)-(1.4).

**Theorem (SB 14.2).** If we define $\Phi_1(p) := a(p)$, $\Phi_3(p) := b(p)$, $\Phi_2(p) := c(p)$, then the $p$-th coefficients of the modular forms are given by

$$
\Phi_M(p) = \begin{cases} 
0 & \text{if } (\frac{M}{p}) = -1, \\
4a^2 - 2p & \text{if } (\frac{M}{p}) = 1, p = a^2 + Mb^2.
\end{cases}
$$

We rewrite the conjecture to motivate the use of Corollary 2 and Theorem 3.

**Conjecture.** If $p \geq 5$ is a prime and $\gamma(p)$ is as before, then

$$
3F_2\left( \begin{array}{c} \frac{2}{3}, \frac{1}{3}, \frac{1}{3} \\ 1, 1 
\end{array} \right)_{tr(p)} \equiv a(p) \pmod{p^2},
$$

$$
3F_2\left( \begin{array}{c} \frac{1}{3}, \frac{1}{3}, \frac{3}{3} \\ 1, 1 
\end{array} \right)_{tr(p)} \equiv b(p) \pmod{p^2},
$$

$$
3F_2\left( \begin{array}{c} \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \\ 1, 1 
\end{array} \right)_{tr(p)} \equiv c(p) \pmod{p^2},
$$

$$
3F_2\left( \begin{array}{c} \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \\ 1, 1 
\end{array} \right)_{tr(p)} \equiv \gamma(p)a(p) \pmod{p^2}.
$$

From the new formulation, we see that we need to use Corollary 2 and Theorem 3 where $m = 1$ and $d = 2, 3, 4$ or 6. For primes $p$ where $p \equiv 1 \pmod{d}$ we use Corollary 2, for primes $p$ where $p \equiv 1 \pmod{d}$ we use Theorem 3. The proof of Theorem 4 is thus reduced to evaluating the $3F_2$ Gaussian hypergeometric series. First, we state a theorem which is a special case of Greene ([G], 4.38(ii)). The corollary follows from the two binomial coefficient properties (1.10).

**Theorem (G).** If $B$ is a nontrivial character on $\mathbb{F}_q$, then

$$
3F_2\left( \begin{array}{c} \phi, B, \overline{B} \\ \epsilon_q, \epsilon_q \end{array} \left| 1 \right. \right)_{q} = B(-1) \left\{ \begin{array}{l} 
0 \\
\binom{\chi}{\phi\chi} + \binom{\phi\chi}{\phi\chi} 
\end{array} \right\} \frac{1}{\phi} \text{ if } B \neq \square,
$$

$$
q^2 \cdot 3F_2\left( \begin{array}{c} \phi, B, \overline{B} \\ \epsilon_q, \epsilon_q \end{array} \left| 1 \right. \right)_{q} = B(-1) \left\{ \begin{array}{l} 
0 \\
J_r(\chi, \phi)^2 + J_r(\overline{\chi}, \phi)^2 
\end{array} \right\} \frac{1}{\phi} \text{ if } B = \chi^2.
$$

**Corollary 4.1.** If $B$ is a nontrivial character on $\mathbb{F}_q$, then

$$
q^2 \cdot 3F_2\left( \begin{array}{c} \phi, B, \overline{B} \\ \epsilon_q, \epsilon_q \end{array} \left| 1 \right. \right)_{q} = B(-1) \left\{ \begin{array}{l} 
0 \\
J_r(\chi, \phi)^2 + J_r(\overline{\chi}, \phi)^2 
\end{array} \right\} \frac{1}{\phi} \text{ if } B = \chi^2.
$$

The following two propositions evaluate the Gaussian hypergeometric series in Corollary 2 and Theorem 3, respectively. Theorem 4 is then immediate. We recall (1.2)-(1.4) and define $\alpha_2(p) := a(p)$, $\alpha_3(p) := b(p)$, $\alpha_4(p) := c(p)$, and $\alpha_6(p) := a(p)$. 

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Proposition 4.2. Fix a $d$, $d \in \{2, 3, 4, 6\}$. Let $p$ be a prime, $p \equiv 1 \pmod{d}$. If $\rho_d$ is a character of order $d$ on $\mathbb{F}_p$, then

$$p^2 \cdot 3F_2 \left( \phi, \rho_d, \frac{\rho_d}{\epsilon_p}, \frac{\rho_d}{\epsilon_p} | 1 \right) = \alpha_d(p).$$

Proof of Proposition 4.2. This method comes from Ono [O], where he does the case $d = 2$. We have two cases. For the first case we consider $p, p \equiv d + 1 \pmod{2d}$. Here $\rho_d$ is not a square, so the Gaussian hypergeometric series evaluates to zero. Using the theorem of [SB] and basic Legendre symbol properties, we have that $\alpha_d(p) = 0$. For $d = 3$ this case is vacuous.

For the second case we consider $p, p \equiv 1 \pmod{2d}$. Here $\rho_d$ is $\chi^2$ for some character $\chi$. We consider $d = 4$. By BEW Theorem 3.3.1,

$$J_1(\chi, \phi)^2 \cdot J_1(\bar{\chi}, \phi)^2 = (a + ib\sqrt{2})^2 + (a - ib\sqrt{2})^2 = 4a^2 - 2p,$$

where $p = a^2 + 2b^2$. For $d = 2, 3$ and 6 we use BEW Theorems 3.2.1, 3.1.1, and 3.5.2, respectively. For each $d$, we use [SB] and see that this equals $\alpha_d(p)$. 

Proposition 4.3. Fix a $d$, $d \in \{3, 4, 6\}$. Let $p$ be a prime, $p \equiv -1 \pmod{d}$. If $\rho_d$ is a character of order $d$ on $\mathbb{F}_p^*$, then

$$p^4 \cdot 3F_2 \left( \phi, \rho_d, \frac{\rho_d}{\epsilon_p^2}, \frac{\rho_d}{\epsilon_p^2} | 1 \right) = \alpha_d(p)^2 - (-1)^{\frac{p-(d-1)}{d}}2p^2.$$

Proof of Proposition 4.3. We note that $\rho_d$ is always a square. We have two cases to consider. For the first case, we consider $p$ with $p \equiv -1 \pmod{2d}$. Using the theorem of [SB] we have that $\alpha_d(p) = 0$. By BE Theorem 2.14,

$$J_2(\chi, \phi)^2 + J_2(\bar{\chi}, \phi)^2 = 2p^2.$$

For the second case, we consider $p$ with $p \equiv d - 1 \pmod{2d}$. We consider $d = 4$. By BE Theorem 4.6,

$$J_2(\chi, \phi)^2 + J_2(\bar{\chi}, \phi)^2 = (a + ib\sqrt{2})^4 + (a - ib\sqrt{2})^4 = (4a^2 - 2p)^2 - 2p^2 = c(p)^2 - 2p^2,$$

where $p = a^2 + 2b^2$, and the last equality follows from [SB]. For $d = 3$ this case is vacuous. For $d = 6$ we use [BE] Theorem 4.10. 

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References


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