ON THE ALGEBRA OF FUNCTIONS $C^k$-EXTENDABLE FOR EACH $k$ FINITE

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Abstract. For each positive integer $l$ we construct a $C^l$-function of one real variable, the graph of which has the following property: there exists a real function on $\mathbb{R}$ which is $C^k$-extendable to $\mathbb{R}^2$ for each $k$ finite, but it is not $C^\infty$-extendable.

Introduction

Let $X$ be a locally closed subset of $\mathbb{R}^n$, i.e. closed in an open subset $G$ of $\mathbb{R}^n$. Consider the following $\mathbb{R}$-algebras of functions:

$$C^k(X) = \{ f : X \to \mathbb{R} | \exists \tilde{f} : G \to \mathbb{R} \text{ of class } C^k : \tilde{f}|X = f \},$$

where $k \in \mathbb{N} \cup \{\infty\}$, and the $\mathbb{R}$-algebra of functions which can be called almost $C^\infty$-functions on $X$:

$$C^{(\infty)}(X) = \bigcap_{k \in \mathbb{N}} C^k(X) = \lim_{k \to \infty} C^k(X).$$

Obviously, we have

$$C^\infty(X) \subset C^{(\infty)}(X) \subset C^k(X), \quad k \in \mathbb{N}.$$

A fundamental question concerning singularities of the set $X$ is the following:

When does $C^{(\infty)}(X) = C^\infty(X)$?

The answer is affirmative in the following cases:

1) Clearly, when $X$ is open, it means when $X = G$. More generally, if $X$ is a closed $C^\infty$-submanifold of $G$.

2) When $X = \overline{\text{int}X} \cap G$, because then $C^k(X)$ is naturally isomorphic to the algebra $\mathcal{E}^k(X)$ of $C^k$-Whitney fields on $X$ ($k \in \mathbb{N} \cup \{\infty\}$) (cf. [W]), and consequently,

$$C^{(\infty)}(X) = \lim_{k \to \infty} C^k(X) = \lim_{k \to \infty} \mathcal{E}^k(X) = \mathcal{E}^\infty(X) = C^\infty(X).$$

More generally, when $X \subset M$, $M$ is a closed $C^\infty$-submanifold of $G$ and $X$ is the closure of its interior in $M$.

3) When $n = 1$ (cf. [M]).
4) When $X$ is a closed semianalytic subset of $G$. Not all subanalytic subsets have this property, and this property distinguishes an important class of subanalytic sets (cf. \cite{BMP}).

In \cite{Pawlucki}, the author gave an example of a subset of $\mathbb{R}^2$ on which there are almost $C^\infty$-functions that are not $C^\infty$. Simplifying and clarifying the construction from \cite{Pawlucki}, here we will prove the following.

**Theorem.** For each positive integer $l$ there exists a function $\varphi : \mathbb{R} \to \mathbb{R}$ of class $C^l$ such that $C^\infty(\tilde{\varphi}) \neq C^\infty(\varphi)$, where $\varphi \subset \mathbb{R} \times \mathbb{R}$ stands for the graph of the function $\varphi$.

**Proof of the Theorem.**

Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $(a_\nu)_\nu \subset \mathbb{R}$ be such that

(I) $a_1 > a_2 > \ldots > a_\nu > \ldots$, $a_\nu \to 0$ ($\nu \to \infty$);

(II) $\varphi : [a_\nu : \nu \in \mathbb{N}^*] \cup \{0\} : \mathbb{R} \to \mathbb{R}$ is $C^\infty$ ($\mathbb{N}^*$ (resp. $\mathbb{N}$) will denote the set of positive (resp. non-negative) integers);

(III) $\varphi(a_{\nu+1}, a_{\nu-1})$ is $C^\nu$ but not $C^{\nu+1}$ ($a_0 := +\infty$);

(IV) $\forall \nu \in \mathbb{N}$: $\lim_{x \to 0} \varphi^{(\nu)}(x)$ exists in $\mathbb{R}$ and $\lim_{x \to 0} \varphi(x) = \varphi(0)$.

**Lemma.** Fix $\nu$. If $f, g : U \to \mathbb{R}$ are $C^{\nu+1}$-functions in a neighbourhood $U$ of $(a_\nu, \varphi(a_\nu))$ in $\mathbb{R}^2$ such that $f = g$ in $U \cap \tilde{\varphi}$, then

$$\frac{\partial f}{\partial y}(a_\nu, \varphi(a_\nu)) = \frac{\partial g}{\partial y}(a_\nu, \varphi(a_\nu)).$$

**Proof of the Lemma.** Put $\omega(x) := f(x, \varphi(x)) = g(x, \varphi(x))$, for $x$ near $a_\nu$. Then

$$\omega^{(\nu)}(x) = P_k(\{(\frac{\partial^j + j f}{\partial x^j \partial y})(x, \varphi(x))\}_{i+j \leq \nu+1}, \varphi(x), \ldots, \varphi^{(\nu-1)}(x)) + \varphi^{(\nu)}(x)\frac{\partial f}{\partial y}(x, \varphi(x)),$$

for $x$ near $a_\nu, x \neq a_\nu$ and any $k \in \mathbb{N}$, where $P_k$ is a polynomial depending only on $k$.

In particular,

$$\omega^{(\nu+1)}(x) = P_{\nu+1}(\{(\frac{\partial^j + j f}{\partial x^j \partial y})(x, \varphi(x))\}_{i+j \leq \nu+1}, \varphi(x), \ldots, \varphi^{(\nu)}(x)) + \varphi^{(\nu+1)}(x)\frac{\partial f}{\partial y}(x, \varphi(x)).$$

$\forall k = 0, \ldots, \nu \exists \alpha_k \in \mathbb{R}$: $\lim_{x \to -a_\nu} \varphi^{(k)}(x) = \alpha_k$ and $\lim_{x \to a_\nu} \varphi^{(\nu+1)}(x)$ does not exist in $\mathbb{R}$.

Two cases:

(1) There are two sequences $(b_n)_n, (c_n)_n \subset \mathbb{R}$ converging to $a_\nu$ such that

$$\lim_{n \to \infty} \varphi^{(\nu+1)}(b_n) = \beta, \quad \lim_{n \to \infty} \varphi^{(\nu+1)}(c_n) = \gamma, \beta \neq \gamma.$$

(2) There is a sequence $(b_n)_n \subset \mathbb{R}$ converging to $a_\nu$ such that

$$\lim_{n \to \infty} \varphi^{(\nu+1)}(b_n) = \pm \infty.$$

In case (1),

$$\lim_{n \to \infty} \omega^{(\nu+1)}(b_n) = P_{\nu+1}(\{(\frac{\partial^j + j f}{\partial x^j \partial y})(a_\nu, \varphi(a_\nu))\}_{i+j \leq \nu+1}, a_1, \ldots, a_\nu) + \beta \frac{\partial f}{\partial y}(a_\nu, \varphi(a_\nu)),$$

$$\lim_{n \to \infty} \omega^{(\nu+1)}(c_n) = P_{\nu+1}(\{(\frac{\partial^j + j f}{\partial x^j \partial y})(a_\nu, \varphi(a_\nu))\}_{i+j \leq \nu+1}, a_1, \ldots, a_\nu) + \gamma \frac{\partial f}{\partial y}(a_\nu, \varphi(a_\nu)).$$
Consequently,
\[ \lim_{n \to \infty} \frac{\omega^{(\nu + 1)}(b_n) - \omega^{(\nu + 1)}(c_n)}{\beta - \gamma} = \frac{\partial f}{\partial y}(a_\nu, \varphi(a_\nu)) = \frac{\partial g}{\partial y}(a_\nu, \varphi(a_\nu)). \]

In case (2),
\[ \omega^{(\nu + 1)}(b_n) = \text{sequence with a finite limit} + \phi^{(\nu + 1)}(b_n) \frac{\partial f}{\partial y}(b_n, \varphi(b_n)). \]

Since \( \phi^{(\nu + 1)}(b_n) \to \pm \infty \) we have
\[ \lim_{n \to \infty} \frac{\omega^{(\nu + 1)}(b_n)}{\phi^{(\nu + 1)}(b_n)} = \frac{\partial f}{\partial y}(a_\nu, \varphi(a_\nu)) = \frac{\partial g}{\partial y}(a_\nu, \varphi(a_\nu)). \]

To finish the proof of the theorem first take a \( C^1 \)-function \( \lambda : \mathbb{R} \to \mathbb{R} \) such that \( \lambda^{(k)}(0) = \lim_{x \to 0} \phi^{(k)}(0) \) for each \( k \in \mathbb{N} \) (by Borel’s theorem), and then define
\[ f(x, y) := \frac{y - \lambda(x)}{x}, \quad \text{for } (x, y) \in \varphi \setminus \{(0, \varphi(0))\}, \quad \text{and } f(0, \varphi(0)) := 0. \]

Fix any \( k \in \mathbb{N} \). For \( (x, y) \neq (0, \varphi(0)) \), \( f(x, y) = \psi(x) \), where
\[ \psi(x) := \frac{\varphi(x) - \lambda(x)}{x}, \quad \text{for } x \in \mathbb{R} \setminus \{0\}, \quad \text{and } \psi(0) := 0. \]

\( \psi \) is \( C^k \) on the set \( (-\infty, a_{k-1}) \setminus \{0\} \), due to the properties (II)-(IV). On the other hand, by l’Hôpital’s rule,
\[ \forall p, q \in \mathbb{N} : \lim_{x \to 0} \frac{\varphi^{(p)}(x) - \lambda^{(p)}(x)}{x^q} = 0. \]

This implies in an easy way that \( \lim_{x \to 0} \psi^{(p)}(x) = 0 \), for all \( p \in \mathbb{N} \).

Consequently, \( \psi \) is a \( C^k \)-function on \( (-\infty, a_{k-1}) \), which can be treated as a \( C^k \)-function on \( (-\infty, a_{k-1}) \times \mathbb{R} \) not depending on \( y \). On the other hand, \( \frac{y - \lambda(x)}{x} \) is a \( C^\infty \)-function on \( (a_k, +\infty) \times \mathbb{R} \), so it suffices now to glue smoothly these two functions along the strip \( (a_k, a_{k-1}) \times \mathbb{R} \).

To check that \( f \) cannot be extended to a \( C^\infty \)-function \( F : \mathbb{R}^2 \to \mathbb{R} \), suppose that such an extension \( F \) exists. Then from the Lemma
\[ \frac{\partial F}{\partial y}(a_\nu, \varphi(a_\nu)) = \frac{\partial (\varphi - \lambda(x))}{\partial y}(a_\nu, \varphi(a_\nu)) = \frac{1}{a_\nu} \to +\infty, \]
but, of course,
\[ \frac{\partial F}{\partial y}(a_\nu, \varphi(a_\nu)) \to \frac{\partial F}{\partial y}(0, \varphi(0)), \]
a contradiction.

Remark. It follows from [G] (the author is indebted to Rémi Soufflet for this reference) that the function \( \varphi \) in our theorem can be chosen in such a way that the germ of \( \varphi \) at 0 belongs to a Hardy field of germs of real functions at 0.
References


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