PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 3, Pages 899–904
S 0002-9939(04)07393-9
Article electronically published on October 21, 2004

NEW PROPERTIES OF MROWKA'S SPACE $\nu\mu_0$

JOHN KULESZA

(Communicated by Alan Dow)

ABSTRACT. We extend the technique of Mrowka to show that his space $\nu\mu_0$ has the property that dim $\nu\mu_0^n = n$ while ind $\nu\mu_0^n = 0$, assuming his extra set-theoretic hypothesis. We also show that $\nu\mu_0$ is $N-$compact, so assuming the extra axiom, there is an $N-$compact metric space with no $N-$compact completion.

Recently, Mrowka has written two very interesting papers ([M2], [M3]) concerning an example of his, which he calls $\nu\mu_0$. This space is a zero-dimensional metric space, and he has shown, assuming a special set-theoretic axiom $S(\mathbf{c})$, that every completion of $\nu\mu_0$ contains an interval and that every completion of its square contains a square. Since dim $\nu\mu_0^2 \leq 2$, this implies that, at least consistently, there is a metric space for which ind $= 0$ while dim $= 2$, and also for which the ind of any completion is greater than 1.

In this note we show two further properties of $\nu\mu_0$. In particular we show, also assuming $S(\mathbf{c})$, that every completion of $\nu\mu_0^n$ contains an n-cube. It follows that dim $\nu\mu_0^n = n$ while ind $\nu\mu_0^n = 0$. This, at least consistently, answers the old question of what the possible values for dim $-$ ind are. This result should be viewed as merely the generalization and organization of ideas found in Mrowka's work on the two-dimensional case. We also show that $\nu\mu_0$ is $N-$compact; hence, there is an $N-$compact space for which (consistently) there is no $N-$compact completion. Previously, there were three known examples of complete, not strongly zero-dimensional $N-$compact metric spaces ([K1], [K3]), but no examples for which completion preserving $N-$compactness was known to be impossible.

The author would like to gratefully thank the referee, who offered many improvements to the presentation of these results.

1. Basics and background

1.1. **Notation.** All spaces are assumed metrizable unless otherwise noted. We use $\mathbf{c}$ to denote the cardinality of the continuum, and N to denote the positive integers. Let $\{r_i : i \in N\}$ enumerate $\mathcal{Q}$, the rationals in $[0,1]$, let $\mathcal{P}$ denote the irrationals in $[0,1]$ and let C denote the Cantor set (2^N with the product topology).

1.2. **The statement $S(\mathbf{c})$.** Let A be a set. $W \subseteq A^N$ is *countable on all lines parallel to the jth axis*, where $j \in N$, if whenever $\sigma \in W$, $\{\tau(j) : \tau \in W, \sigma(i) = \tau(i)$

Received by the editors May 2, 1998 and, in revised form, September 9, 2000.
2000 *Mathematics Subject Classification.* Primary 54F45.
Key words and phrases. Dimension, metric space, completion, $N-$compact, product space.

if $i \neq j\}$ is a countable set. The following axiom was formulated by Mrowka [M1], and was recently proved consistent by Dougherty ([D]).

$S(\mathbf{c})$: The space A^N, where A is a discrete space of cardinality $\mathbf{c}$, cannot be written as a countable union of closed sets each of which is countable on all lines parallel to some axis.

The statement $S(\mathbf{c})$ is a large cardinal assumption; in particular, it implies that in L there are weakly compact cardinals below $\mathbf{c}^V$.

1.3. **n-Bernstein sets.** A subset B of a complete metric space M is *Bernstein* provided that $B \cap K \neq \emptyset$ and $(M \backslash B) \cap K \neq \emptyset$ whenever K is a perfect subset of M. A subset S of $\prod_{i \in I} X_i$ is *oblique* if whenever σ and τ are in S, $\sigma(i) \neq \tau(i)$ for all $i \in I$. The Bernstein set $B \subset C$ is *n-Bernstein* provided B^n intersects each oblique perfect subset of C^n. It is not difficult to construct, by transfinite induction, a subset of C that is n-Bernstein for all n, but not all Bernstein sets are n-Bernstein.

1.4. **Tree products.** If T and U are trees, then the *tree product* of T and U, denoted $T \otimes U$, is the collection $\{(t,u) : t \in T, u \in U, \text{ and } \mathrm{ht}_T(t) = \mathrm{ht}_U(u)\}$. Using the partial order $(t,u) \leq (t^*, u^*)$ iff $t \leq_T t^*$ and $u \leq_U u^*$, $T \otimes U$ is itself a tree. Let $T^{(n)} = T \otimes T \otimes \cdots \otimes T$ (n times). Tree products will be used to define a special base in section 3.

1.5. **The spaces $\nu\mu$ and $\nu\mu_0$.** Fix a subset B of C that is n-Bernstein for all $n \in N$. Let $A = C \backslash B$; then A and B both have cardinality $\mathbf{c}$. The space $\nu\mu$ is a subset of $C^N \times [0,1]$ (with a finer topology), and $\nu\mu_0$ is a subspace of $\nu\mu$. An element x of $\nu\mu$ will be denoted by $(\mathbf{x}, t)$ where $\mathbf{x} \in C^N$ and $t \in [0,1]$.

Our description of Mrowka's spaces differs slightly from [M2] and [M3], but it might help to relate them to the spaces in [R], [K2], and [K3].

The space $\nu\mu = P_1 \cup P_2$ where:

(i) $P_1 = \{(\mathbf{x}, t) : t \in \mathcal{P}, \text{ and } \mathbf{x} \in A^N\}$ and

(ii) $P_2 = \bigcup_{i \in N} P_{2,i}$ where

$P_{2,i} = \{(\mathbf{x}, r_i) : \mathbf{x}(j) \in A \text{ if } j \neq i \text{ and } \mathbf{x}(i) \in C\}$.

Also, $\nu\mu_0$ is the subset $P_1 \cup P_2^0$ of $\nu\mu$ with $P_2^0 = \bigcup_{i \in N} P_{2,i}^0$ where

$P_{2,i}^0 = \{(\mathbf{x}, r_i) : \mathbf{x}(j) \in A \text{ if } j \neq i \text{ and } \mathbf{x}(i) \in B\}$.

For $t \in [0,1]$ and $n \in N$, let $d_n(t) = \min(\{1/n\} \cup \{| t - r_i |: i \leq n, t \neq r_i\})$.

For $x = (\mathbf{x}, t) \in P_1$ and $n \in N$, let $U_n(x) = \{y = (\mathbf{y}, s) :| t - s |< d_n(t)$ and $\mathbf{y}(i) = \mathbf{x}(i)$, for $1 \leq i \leq n\}$.

For $x = (\mathbf{x}, r_i)$ in $P_{2,i}$, and $n \in N$, let $V_n(x) = \{y = (\mathbf{y}, s) : | r_i - s |< d_n(r_i), \mathbf{y}(j) = \mathbf{x}(j)$ if $j \neq i$ and $j \leq n$, and if $i \leq n$, then $\mathbf{y}(i)(k) = \mathbf{x}(i)(k)$ for $k \leq n\}$.

The $\{U_n(x) : n \in N, x \in P_1\} \cup \{V_n(x) : n \in N, x \in P_2\}$ form a base for a metrizable topology on $\nu\mu$ (for example, the metrization theorem of Frink (see [K2]) can be used to show this); the restriction of this topology to $\nu\mu_0$ is zero-dimensional.

Note that $\nu\mu$ can be viewed as a subset of $C^N \times [0,1]$, and P_1 can be viewed as $A^N \times \mathcal{P}$ where each factor of A^N is discrete. By rearranging coordinates, $P_{2,i}$ can be viewed as $A^N \times C \times \{r_i\}$, and $P_{2,i}^0$ can be viewed as $A^N \times B \times \{r_i\}$ (B has the subspace topology inherited from C). These facts will be used in section 3.

1.6. **Properties of $\nu\mu_0$.** Mrowka discovered $\nu\mu$ and $\nu\mu_0$ and proved the following two theorems, the first in [M2] and the second in [M3].

Theorem 1.1. *$S(\mathbf{c})$ implies that every completion of $\nu\mu_0$ contains an interval.*

Theorem 1.2. *$S(\mathbf{c})$ implies that every completion of $\nu\mu_0^2$ contains a square.*

In section 3 we prove the general analogue of these:

Theorem 1.3. *$S(\mathbf{c})$ implies that, for all $n \geq 1$, every completion of $\nu\mu_0^n$ contains an n-cube.*

In section 2 we prove:

Theorem 1.4. *The space $\nu\mu_0$ is $N-$compact.*

This theorem assumes no extra set-theoretic conditions, but the conclusion that $\nu\mu_0$ has no $N-$compact completion requires Theorem 1.1.

2. Proof of Theorem 1.4

We make use of the following well-known characterization of $N-$compactness.

Theorem 2.1. *A zero-dimensional space X is $N-$compact if and only if every clopen ultrafilter on X with the countable intersection property is fixed.*

Suppose that $\mathcal{U}$ is a clopen ultrafilter on $\nu\mu_0$ with the countable intersection property. For each $k, n \in N$ and $b \in 2^n$, let $H(k,b)$ denote $\{x \in \nu\mu_0 : b \subset \mathbf{x}(k)\}$. Then $H(k,b)$ is a clopen set in $\nu\mu_o$. Fixing k and n, $\{H(k,b) : b \in 2^n\}$ is a finite clopen partition of $\nu\mu_0$, and thus there is $b_n \in 2^n$ with $H(k,b_n) \in \mathcal{U}$. Clearly $b_m \subset b_n$ when $m > n$; let $s_k = \bigcup_n b_n$.

Letting $W = \bigcap_{k\in N} \bigcap_{n\in N} H(k,b_n)$, W is nonempty by the countable intersection property of $\mathcal{U}$, and if $(\mathbf{x}, t) \in W$, then $\mathbf{x}(k) = b_k$ for all $k \in N$. Now either $\mathbf{x}(i) \in A$ for all i or, for exactly one i, $\mathbf{x}(i) \in B$. In the latter case, $t = r_i$ and W is a singleton; hence $\mathcal{U}$ is fixed. In the other case, $W \subseteq \{\mathbf{x}\} \times \mathcal{P}$, so W is a zero-dimensional separable and metrizable space and is therefore N-compact. It follows that the trace of $\mathcal{U}$ on W has a nonempty intersection, and $\mathcal{U}$ is fixed.

3. Proof of Theorem 1.3

From the description of $\nu\mu$ in section 1, by a rearrangement of coordinates, $\nu\mu^n$ can be viewed as a subset of $(C^N)^n \times [0,1]^n$, with $(A^N)^n \times [0,1]^n \subset \nu\mu^n$.

Suppose $F = \bigcup_{i\in N} F_i$ is an F_σ set in $\nu\mu^n \backslash \nu\mu_0^n$. The goal is to find $x \in (A^N)^n$ such that the set $\{x\} \times [0,1]^n$ is disjoint from F. From this it follows that the G_δ set $\nu\mu^n \backslash F$ of $\nu\mu^n$ contains an n-cube. An application of Lavrentieff's Theorem (see [E]) then yields Theorem 1.3.

Partition $[0,1]^n$ and along with it $\nu\mu^n$ into countably many sets as follows. For each $u \subseteq \{1, 2, ..., n\}$ and $f : u \to \mathcal{Q}$, let $T(u,f) = \{y \in [0,1]^n :$ if $i \in u$, then $y(i) = f(i)$, and $y(i) \in \mathcal{P}$ otherwise$\}$. Correspondingly, let $S(u,f) = \{(x_1, x_2, ..., x_n) \in \nu\mu^n : (t_1, t_2, \ldots, t_n) \in T(u,f)\}$. The $x \in (A^N)^n$ that we want must satisfy, for all u and f, $\{x\} \times T(u,f) \cap F = \emptyset$.

Notice that $S(u,f) = R_1 \times R_2 \times \ldots \times R_n$ where, if $i \in u$, and $f(i) = r_{j(i)}$, then R_i is $P_{2,j(i)}$, and if i is not in u, then $R_i = P_1$. It follows that $S(\emptyset,\emptyset) = (A^N)^n \times \mathcal{P}^n$, which is a subset of $\nu\mu_0^n$ and so does not intersect F; hence we focus on nonempty u.

Fix a nonempty u and an f; if $x \in (A^N)^n$ is such that $\{x\} \times T(u,f)$ intersects F, then it intersects some F_i. This means that x is in the projection along $[0,1]^n$

of $F_i \cap S(u,f)$. By rearranging coordinates and letting $k = \mid u \mid$ we can view $S(u,f)$ as $A^N \times C^k \times T(u,f)$. In this view, $F_i \cap S(u,f)$ is disjoint from $A^N \times B^k \times T(u,f)$, which is a subset of $\nu\mu_0$.

We now need:

Theorem 3.1. *Suppose $0 \leq k \leq n, u \subseteq \{1,2,\ldots,n\}$ with $\mid u \mid = k$, $f : u \to \mathcal{Q}$ and F is a closed subset of $A^N \times C^k \times T(u,f)$ that does not intersect $A^N \times B^k \times T(u,f)$. Then there are closed sets $\{K_i : i \in N\}$ of $A^N \times C^k$ each of which is countable on all lines parallel to some axis, and such that the projection of F to $A^N \times C^k$ is contained in $\bigcup\{K_i\}$.*

Observe that the intersection of a K_i from Theorem 3.1 with $A^N \times A^k$ is closed in $A^N \times A^k$ if each factor is assumed discrete.

To finish the argument, for each u, f and i, let $\mathcal{K}(u,f,i)$ denote the countable collection guaranteed by Theorem 3.1 when F_i is substituted for F in the statement of the theorem, and let $\mathcal{K}^*(u,f,i)$ denote the collection of intersections of elements of $\mathcal{K}(u,f,i)$ with $A^N \times A^k$. By appropriately rearranging coordinates, this means that if $x \in (A^N)^n$ is not in $\mathcal{K}^* = \bigcup_{u,f,i} \mathcal{K}^*(u,f,i)$, then $\{x\} \times T(u,f) \cap F_i = \emptyset$. Hence $\mathcal{K}^*$ is a countable collection of closed sets in $(A^N)^n$, each of which is countable on all lines parallel to some axis, and if $x \in (A^N)^n, x \notin \bigcup \mathcal{K}^*$, then $\{x\} \times [0,1]^n \cap F = \emptyset$. But $S(\mathbf{c})$ can be applied, giving that such an x exists.

To prove Theorem 3.1, we need some lemmas.

Lemma 3.2. *Let X be a space, and suppose $M \subseteq X^n$, $M \neq \emptyset$. Then one of the following holds:*

(1) There is an open $U \subseteq X^n$ with $U \cap M \neq \emptyset$ and k with $1 \leq k \leq n$ such that if $\sigma, \tau \in U \cap M$, then $\sigma(k) = \tau(k)$.

(2) There are $\sigma, \tau \in M$ with $\sigma(i) \neq \tau(i)$ for $1 \leq i \leq n$.

Proof. Suppose (2) does not hold and fix $\sigma \in M$. For $\delta \in M$ let $s(\delta) = \{i : \delta(i) = \sigma(i)\}$; choose τ so that $\mid s(\tau) \mid$ is minimal; $\mid s(\tau) \mid \geq 1$. Then for each $j \in \{1,2,...,n\} \backslash s(\tau)$, $F_j = \{\delta \in X^n : \delta(j) = \sigma(j)\}$ is a closed set not containing τ, so there is a neighborhood U of τ missing $\bigcup_{j \notin s(\tau)} F_j$. It follows that if $\delta \in U \cap M$, then $\delta(i) = \tau(i) = \sigma(i)$ if $i \in s(\tau)$. So (1) holds.

For $\sigma \in X \times Y^n$, write $\sigma = (\sigma_x, \sigma(1), \sigma(2), ..., \sigma(n))$ where $\sigma_x \in X$ and $(\sigma(1), \sigma(2), ..., \sigma(n)) \in Y^n$. □

The next lemma follows easily from Lemma 3.2.

Lemma 3.3. *Let X, Y be spaces, and suppose $M \subseteq X \times Y^n$, $M \neq \emptyset$. Then one of the following holds:*

(1) There is an open $U \subseteq X \times Y^n$ with $U \cap M \neq \emptyset$ and i with $1 \leq i \leq n$ such that if $\sigma, \tau \in U \cap M$, then $\sigma(i) = \tau(i)$.

(2) There are $\sigma, \tau \in M$ with $\sigma(i) \neq \tau(i)$ for $1 \leq i \leq n$.

Lemma 3.4. *If each of X and Y is a complete metric space, $n \in N$, and F is a closed subset of $X \times Y^n$, then one of the following holds:*

(1) The projection of F to Y^n contains an oblique Cantor set, or

(2) There is an open set U with $U \cap F \neq \emptyset$ and i, $1 \leq i \leq n$, such that if $\sigma, \tau \in U \cap F$, then $\sigma(i) = \tau(i)$.

Proof. We suppose (2) does not hold and show (1) does. For $\sigma \in \{0,1\}^k$, $k \geq 1$, let $\sigma^- = (\sigma(1), \sigma(2), ..., \sigma(k-1)) \in \{0,1\}^{k-1}$. Let $O_\emptyset = X \times Y^n$. By induction on k, for each $\sigma \in \{0,1\}^k$ we find a point $x_\sigma \in F$ and an open set O_σ such that:

(i) $x_\sigma \in O_\sigma \subseteq cl(O_\sigma) \subseteq O_{\sigma^-}$;

(ii) the diameter of O_σ is less than $1/k$;

(iii) if $\sigma, \tau \in \{0,1\}^k, \sigma \neq \tau$ and $x \in O_\sigma, y \in O_\tau$, then for the coordinates $i \in \{1, 2, ..., n\}$ of the projection to Y^n, $x(i) \neq y(i)$.

Having achieved this, letting $\mathcal{O}_k = \bigcup_{\sigma \in \{0,1\}^k} O_\sigma$, we have $\bigcap_{k \in N} \mathcal{O}_k$ is a Cantor set in $X \times Y^n$ whose projection to Y^n is an oblique Cantor set, and (1) holds.

Suppose the induction is complete through stage $k-1$, and $\sigma \in \{0,1\}^k$. We simultaneously get $x_\sigma, x_\tau, O_\sigma, O_\tau$ where τ is the sequence differing from σ in only the last position. Because (2) is not satisfied, letting $M = O_{\sigma^-} \cap F$, there is no open U with $U \cap M$ satisfying (1) of Lemma 3.3, so (2) of 3.3 holds, and there are $x_\sigma, x_\tau \in O_{\sigma^-} \cap F$ with all different Y coordinates. Then O_σ, O_τ can be chosen to be small enough neighborhoods of σ and τ to satisfy (i), (ii), and (iii). □

Proof of Theorem 3.1. Because the projection of F into C^k is disjoint from B^k, it does not contain any oblique Cantor sets, nor does any closed subset of F. It follows from Lemma 3.4 that if G is a closed subset of F, there is a nonempty open set O with $O \cap G \neq \emptyset$ and $O \cap G$ has constant projection to one of the factors of C^k.

Since $C = 2^N, \mathcal{P} = \omega^N$ and $T(u, f)$ is identifiable with $\mathcal{P}^{n-k}$, a clopen base for the space $A^N \times C^k \times T(u, f)$ is naturally identified with the tree product $\mathbf{T} = A^{<\omega} \otimes (2^{<\omega})^{(k)} \otimes (\omega^{<\omega})^{(n-k)}$. Let $[t]$ denote the open set associated with $t \in \mathbf{T}$, and identify the separate groups of coordinates of t by writing $t = (t_A, t_{C^k}, t_{\mathcal{P}^{(n-k)}})$. If $t, v \in \mathbf{T}$, $t \neq v$, and $\mathrm{ht}(t) = \mathrm{ht}(v)$, then $[t] \cap [v] = \emptyset$.

By transfinite induction define subsets U_α of $\mathbf{T}$ and F_α of F as follows. Let $F_0 = F$ and given F_α, let U_α be the set of $t \in \mathbf{T}$ such that $[t] \cap F_\alpha$ is nonempty, has a constant projection into one of the factors of C^k and t is minimal with respect to this property. Let $F_{\alpha+1} = F_\alpha \setminus \bigcup_{t \in U_\alpha}$, and $F_\alpha = \bigcap_{\beta < \alpha} F_\beta$ if α is a limit. By the first paragraph of this proof, if $\beta < \alpha$ and $F_\beta \neq \emptyset$, then F_α is a proper subset of F_β, so the induction stops when $F_\alpha = \emptyset$.

Observe that each U_α is an antichain and that $r \neq t$ if $r \in U_\alpha$, $t \in U_\beta$ and $\beta < \alpha$, because $[r]$ meets F_α and $[t]$ does not. Now split $U = \bigcup_\alpha U_\alpha$ into countably many sets $\{W_{m,i,s} : m \in N, i \leq k, s \in (\omega^{<\omega})^{(n-k)}\}$ as follows: $t \in W_{m,i,s}$ iff $\mathrm{ht}(t) = m$, if $t \in U_\alpha$, then $F_t = [t] \cap F_\alpha$ has constant projection to the ith factor of C^k and $s = t_{\mathcal{P}^{(n-k)}}$.

Finally let $K_{m,i,s}$ be the closure of the projection of $\bigcup\{F_t : t \in W_{m,i,s}\}$ into $A^N \times C^k$. Then $K_{m,i,s}$ is the union of the individual closures of the projections of the F_t, because $\{(t_A, t_{C^{(k)}}) : t \in W_{m,i,s}\}$ represents a discrete family of clopen sets in $A^N \times C^k$.

It follows that $K_{m,i,s}$ meets every line parallel to the ith coordinate of C^k in exactly one point. The family $\{K_{m,i,s} : m \in N, i \leq k, s \in (\omega^{<\omega})^{(n-k)}\}$ is as desired. □

References

[D] R. Dougherty, Narrow coverings of ω-ary product spaces, *Annals of Pure and Applied Logic*, **88** (1997), 47-91. MR1478522 (99j:03035)

[E] R. Engelking, General Topology, Berlin, (1989). MR1039321 (91c:54001)

[K1] J.S. Kulesza, An example in the Dimension Theory of Metrizable spaces, *Top. Appl.*, **35** (1990), 109-120. MR1058791 (91g:54045)

[K2] J.S. Kulesza, Metrizable Spaces Where the Inductive Dimensions Disagree, *Trans. AMS*, **318** (1990), 763-781. MR0954600 (90g:54030)

[K3] J.S. Kulesza, Dissertation, SUNY at Binghamton, (1987).

[M1] S. Mrowka, $N-$compactness, Metrizability, and Covering Dimension, *Rings of Continuous Functions*, Marcel Dekker Inc., (1995), 247-275. MR0789276 (86i:54034)

[M2] S. Mrowka, Small Inductive Dimension of Completions of Metric Spaces, *Proceedings AMS*, **125** (1997), 1545-1554. MR1423324 (97i:54043)

[M3] S. Mrowka, Small Inductive Dimension of Completions of Metric Spaces II, Preprint.

[R] P. Roy, Nonequality of Dimensions for Metric Spaces, *Transactions AMS*, **134** (1968), 117-132. MR0227960 (37:3544)

Department of Mathematics, George Mason University, Fairfax, Virginia 22030-4444
E-mail address: jkulesza@gmu.edu