PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 3, Pages 751–758
S 0002-9939(04)07542-2
Article electronically published on October 7, 2004

EXACT NUMBER OF LIMIT CYCLES FOR A FAMILY OF RIGID SYSTEMS

A. GASULL AND J. TORREGROSA

(Communicated by Carmen C. Chicone)

ABSTRACT. For a given family of planar differential equations it is a very difficult problem to determine an upper bound for the number of its limit cycles. Even when this upper bound is one it is not always an easy problem to distinguish between the case of zero and one limit cycle. This note mainly deals with this second problem for a family of systems with a homogeneous nonlinear part. While the condition that allows us to separate the existence and the nonexistence of limit cycles can be described, it is very intricate.

1. INTRODUCTION AND MAIN RESULTS

Smooth planar systems whose angular speeds are constant are called *rigid* systems. When the origin is of center or focus type, all rigid systems are given by differential equations of the form

$$\begin{cases} \dot{x} &= -y + xF(x,y), \\ \dot{y} &= x + yF(x,y), \end{cases} \tag{1.1}$$

where $F(x,y)$ is a smooth real function. For rigid systems the center-focus problem (i.e. the distinction between a focus and a center) is equivalent to the isochronicity problem. This is one of the reasons for which they have already been studied by several authors; see for instance [1, 2, 3, 4, 5, 9].

In this paper we are interested in the study of the exact number of limit cycles that they can have when F has a special type. Concretely, we will study the system

$$\begin{cases} \dot{x} &= -y + x\left(a + f_n(x,y)\right), \\ \dot{y} &= x + y\left(a + f_n(x,y)\right), \end{cases} \tag{1.2}$$

where a is a real parameter and $f_n(x,y)$ is a homogeneous polynomial of degree n.

Our first result gives the uniqueness and hyperbolicity of the limit cycle for (1.2).

Theorem 1.1. *Consider system* (1.2)*, and define* $B := \int_0^{2\pi} f_n(\cos\theta, \sin\theta)d\theta$.

(i) *If* $B = 0$ *and* $a = 0$*, then there is a center at the origin and no limit cycles.*
(ii) *If* $a^2 + B^2 \neq 0$ *and* $aB \geq 0$*, then there are no periodic orbits.*

Received by the editors September 5, 2003 and, in revised form, October 6, 2003.
2000 *Mathematics Subject Classification.* Primary 34C07, 37G15; Secondary 34C25, 37C27.
Key words and phrases. Bifurcation, limit cycle, rotated vector field, rigid system.
This work was supported by DGES No. BFM2002-04236-C02-2 and CONACIT 2001SGR-00173.

(iii) *If $aB < 0$, then there is at most one periodic orbit, which is, whenever it exists, a hyperbolic limit cycle. Furthermore, if f_n is fixed, $B > 0$, and a is a parameter, then there exists a value $a^* = a^*(f_n) \in \mathbb{R}^- \cup \{-\infty\}$ such that the limit cycle exists if and only if $a^* < a < 0$.*

The rest of the paper is devoted to distinguishing between the existence and nonexistence of limit cycles in the case $aB < 0$, or in other words, to a study of the function $a^*(f_n)$. Without loss of generality, we will restrict our attention to the case where B is positive (see the proof of Theorem 1.1). Although we will see that equation (1.2) can be transformed into a linear equation, a determination of the existence and nonexistence of its limit cycles is not trivial.

To state our main result we introduce some notation. Fix a polynomial f_n of even degree, and let

$$f_n(\cos\theta, \sin\theta) = \sum_{k=-n/2}^{k=n/2} c_{2k} e^{2k\theta i}, \quad \text{where} \quad \overline{c_k} = c_{-k} \in \mathbb{C},$$

be its Fourier expansion. We define, for each real number a, the real-valued trigonometric polynomial

$$\rho_a^*(\theta) := -n \sum_{k=-n/2}^{k=n/2} \frac{c_{2k}}{na + 2ki} e^{2k\theta i}. \tag{1.3}$$

Consider also the function $\varphi(\theta) := f_n(\cos\theta, \sin\theta)$ and the discrete set

$$\Theta = \bigcup_{k\geq 0} \{\tau \in (0,\pi) \,:\, \varphi(\tau) = \varphi'(\tau) = \cdots = \varphi^{2k}(\tau) = 0 \text{ and } \varphi^{2k+1}(\tau) < 0\}. \tag{1.4}$$

Observe that Θ is the set of all odd multiplicity zeros of φ at which this function decreases.

Theorem 1.2. *Consider system* (1.2)*, and let $B = \int_0^{2\pi} f_n(\cos\theta, \sin\theta) d\theta > 0$.*

(i) *If n is odd, then there are no limit cycles.*

(ii) *If n is even, a limit cycle exists if and only if $a \in (a^*, 0)$ where*

$$a^* = \max_{\tau\in\Theta} \left[\max\{a \in \mathbb{R}^- \,:\, \rho_a^*(\tau) = 0\} \right],$$

with the function ρ_a^* and the set Θ defined in (1.3) *and* (1.4)*, respectively. If the sets involved in the definition of a^* are empty, then $a^* = -\infty$.*

We will prove the above-stated results in Section 2 and apply them in case $n = 2$ and $n = 4$ in Section 3. These two cases suffice to describe the shape of the bifurcation curve $a = a^*(f_n)$ for arbitrary even n. This is done in Section 4.

Our results are related to an old problem proposed by Coppel in [6]: Can the bifurcation diagram of a family of quadratic systems be described by algebraic inequalities? This question was answered in the negative by Dumortier and Fiddelaers in [7]. In fact, they proved that the bifurcation diagram is not even analytic for quadratic vector fields.

The system (1.2) studied in this paper is not quadratic in general, but as we will see in Sections 3 and 4, the bifurcation curve corresponding to the disappearance of limit cycles is piecewise algebraic. As far as we know there are no examples of this phenomenon in the literature. Certainly, the example that we present illustrates the complexity of the problem of determining the exact number of limit cycles for a polynomial family of planar differential equations.

In the easiest case for which system (1.1) cannot be transformed into a linear equation, i.e. when F is a complete polynomial of degree 2, there may be several limit cycles; hence, the analysis is even more complicated. This problem is studied in [8].

2. Proof of Theorem 1.1

A key point for our proof of Theorem 1.1 is that system (1.2) can be transformed into a linear differential equation. This is done in two steps: In polar coordinates system (1.2) is equivalent to the differential equation

$$r' = r\left(a + r^n f_n(\cos\theta, \sin\theta)\right), \tag{2.1}$$

where $r' := dr/d\theta$. The change of variables $\rho = r^{-n}$ converts (2.1) into the linear differential equation $\rho' + na\rho + nf_n(\cos\theta, \sin\theta) = 0$.

Consider the following Cauchy problem:

$$\rho' + na\rho + nf_n(\cos\theta, \sin\theta) = 0, \tag{2.2}$$

$$\rho(0) = \rho_0, \tag{2.3}$$

and denote its unique solution by $\rho(\theta, \rho_0)$. A useful characterization of the periodic orbits of system (1.2) is the following: *The orbit starting at* $(x_0, 0)$, *with* $x_0 > 0$, *is a periodic orbit for system* (1.2) *if and only if the solution of the Cauchy problem* (2.2)–(2.3) *with* $\rho_0 = x_0^{-n}$ *is a positive* 2π*-periodic function.* Hence the number of periodic solutions is bounded by the number of solutions of the boundary value problem $\rho(2\pi, \rho_0) = \rho_0$.

The solution of (2.2)–(2.3) is given by

$$\rho(\theta, \rho_0) = \left(\rho_0 - n\int_0^\theta e^{na\psi} f_n(\cos\psi, \sin\psi)\,d\psi\right) e^{-na\theta}. \tag{2.4}$$

If $a \neq 0$, then the equation $\rho(2\pi, \rho_0) = \rho_0$ has the unique solution

$$\rho_0^* = \frac{n\int_0^{2\pi} e^{na\psi} f_n(\cos\psi, \sin\psi)\,d\psi}{1 - e^{2\pi na}}. \tag{2.5}$$

If $a = 0$ and $B \neq 0$ it has no solution; and, if $a = 0$ and $B = 0$ it has a continuum of solutions. Furthermore, let $\rho^*(\theta) := \rho(\theta, \rho_0^*)$ denote the positive 2π-periodic solution of (2.2)–(2.3) with $\rho_0 = \rho_0^*$. Integrating the differential equation between 0 and 2π, we obtain the equation

$$na\int_0^{2\pi} \rho^*(\theta)\,d\theta + nB = 0; \tag{2.6}$$

hence, the existence of this positive periodic solution implies that $aB < 0$.

We have proved that at most one limit cycle exists in all cases, and we have proved parts (i) and (ii) of the theorem. It remains to prove part (iii).

Fix a function f_n such that $B \neq 0$. By making the change of variables $(x, y, t) \to (x, -y, -t)$ if necessary, there is no loss of generality if we consider only the case $B > 0$. It is easy to check that system (1.2) is a *rotated family of vector fields with parameter* a (see, for instance, [10]). Hence, it has the following properties.

(a) A limit cycle bifurcates from the origin when $a \lesssim 0$.

(b) The size of this limit cycle increases as $a < 0$ decreases; and, either the limit cycle exists for all negative values of a, or it exists only if $a \in (a^*, 0)$.

(c) For the value a^* of a given in item (b), the phase portrait of system (1.2) has an unbounded polycycle where the limit cycle disappears.

A proof of the nonexistence of periodic orbits for $a < a^*$ as well as a proof of the nonexistence of periodic orbits for $a \geq 0$, which is different from the proof that uses formula (2.6), can be constructed by using the nonintersection property of the periodic orbits for rotated families of vector fields; that is, if Γ_1 and Γ_2 are two periodic orbits corresponding to two different values of a, then $\Gamma_1 \cap \Gamma_2 = \emptyset$. Since by varying the parameter a between 0 and a^* the limit cycles cover a region from zero to infinity, no periodic orbits can exist for values of a not in the interval $(a^*, 0)$.

Notice that results (a) and (b) translated to equation (2.2) imply that its periodic orbit is born from infinity when $a \lesssim 0$, and its initial condition decreases as a decreases.

To finish the proof of the theorem, it suffices to show that if a limit cycle exists, then it is hyperbolic. This fact is a straightforward consequence of equation (2.4). Indeed, since $h(z) := \rho(2\pi, z) = e^{-2\pi na}z + K$, for some constant K, we have that $h'(z) = e^{-2\pi na} \neq 0$ at every point; in particular, $h'(z) \neq 0$ on the limit cycle.

Remark 2.1. By inserting ρ_0^*, the initial condition given in display (2.5) for the periodic solution of the differential equation (2.2), into the formula for the solution (2.4), we get the function $\rho_a^*(\theta)$ given in (1.3), which is a trigonometric polynomial of period π. Also recall that, after the change of variables $r = \rho^{-1/n}$, this periodic function corresponds to a limit cycle of system (1.2) only if it is strictly positive.

3. Proof of Theorem 1.2

The proof of (i) is a straightforward application of Theorem 1.1 (i)-(ii) because $B = 0$ when n is odd.

Consider the case where n is even, and fix a function f_n such that $B > 0$. In view of Theorem 1.1 (iii), we will finish the proof by characterizing the value a^* for which the limit cycle disappears. We already know that system (1.2) has a limit cycle whenever $a \lesssim 0$, and that our initial condition for the limit cycle decreases as a decreases. By Remark 2.1, the function $(\rho_a^*(\theta))^{-1/n}$ given in (1.3) is the candidate for a solution that corresponds to the limit cycle of system (1.2). The value a^* corresponds to the biggest negative value of a for which the function $\rho_a^*(\theta)$ takes the value 0. For this value of a, the function $\rho_{a^*}^*$ has to have a zero τ of even multiplicity (see Figure 1). Also, for $a = a^*$, the function ρ_a^* defined in display (1.3) is not transformed into a limit cycle of (1.2). On the contrary, it gives rise to the polycycle at which the limit cycle disappears.

We will localize the zeros of ρ_a^* that have even multiplicity. Since this function is a solution of (2.2), if $\theta = \tau$ is one of its zeros with multiplicity at least two, then $\partial(\rho_a^*(\theta))/\partial\theta|_{\theta=\tau} = 0$ and we see that τ is also a zero of the function $\theta \mapsto f_n(\cos\theta, \sin\theta)$. By taking derivatives of (2.2), it follows that if τ is a zero of ρ_a^* of multiplicity $2k$ and if the $2k$th derivative of ρ_a^* at τ is positive, then τ is also a zero of $f_n(\cos\theta, \sin\theta)$ of multiplicity $2k - 1$ and its derivative of order $(2k - 1)$ at τ is negative. In other words, all zeros of even multiplicity of ρ_a^* are contained in the set Θ given in the statement of the theorem, as required. Notice that it suffices to consider the zeros in $(0, \pi)$, because the function ρ_a^* is π-periodic and $\rho_a^*(0) \neq 0$. This completes the proof of the theorem.

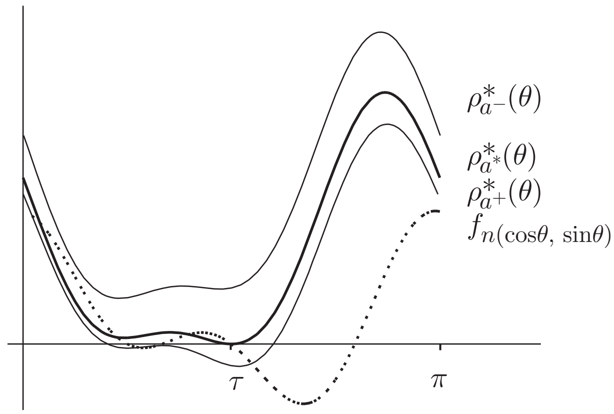

FIGURE 1. Periodic solutions of system (2.2) for $a^+ < a^* < a^-$.

Remark 3.1. As a corollary of the above result, if n is even and the function $\theta \mapsto f_n(\cos\theta, \sin\theta)$ does not change sign, then system (1.2) has a limit cycle if and only if $a \cdot \operatorname{sgn}(f_n(\cos\theta, \sin\theta)) < 0$. This result follows because, under these hypotheses, the set Θ is empty; hence, system (1.2) cannot have polycycles. Therefore, the limit cycle born for a sufficiently small can never disappear.

4. EXAMPLES

In this section we apply our results to the cases $n = 2$ and $n = 4$.

The case $n = 2$ is easy. Suppose that the function f_2 does not change sign. By Remark 3.1, we can determine if system (1.2) has limit cycles. Suppose that f_2 changes sign. Since f_2 is homogeneous, there it vanishes in some direction. By a rotation and a scaling of the variables and of the time, there is no loss of generality if we assume that

$$f_2(\cos\theta, \sin\theta) = d_0 \cos^2\theta + d_1 \cos\theta \sin\theta, \quad \text{with } d_1 \neq 0 \text{ and } d_0 > 0. \tag{4.1}$$

Proposition 4.1. *System* (1.2) *with* $n = 2$ *and* f_2 *given in expression* (4.1) *has a limit cycle if and only if* $a \in (-|d_0/d_1|, 0)$.

Proof. To determine the value a^* for which the limit cycle disappears we will apply Theorem 1.2.

We have that $B = \int_0^{2\pi} f_2(\cos\theta, \sin\theta) = d_0\pi > 0$. Also, the formula for the function ρ_a^* given in (1.3), expressed in real variables, is

$$\rho_a^*(\theta) = \frac{\big(-(2a^2+1)d_0 + ad_1\big)\cos^2\theta - \big(2ad_0 + 2a^2 d_1\big)\sin\theta\cos\theta - (d_0 + ad_1)\sin^2\theta}{2a(a^2+1)}.$$

Let β and $\pi/2$ denote the two roots of f_2 in $(0, \pi)$. Notice that $f_2'(\pi/2) = -d_1$, $f_2'(\beta) = d_1$, and $\tan\beta = -d_0/d_1$. Hence, the set Θ given in (1.4) is exactly $\{\pi/2\}$ (resp. $\{\beta\}$) when $d_1 > 0$ (resp. $d_1 < 0$). In the first case, $\rho_a^*(\pi/2) = -(ad_1 + d_0) / \big(2a(a^2+1)\big)$, and we have that $a^* = -d_0/d_1$. In the second case, $\rho_a^*(\beta) = \big(d_1^2 + d_0^2\big)\cos^2\beta\,(ad_1 - d_0) / \big(2a(a^2+1)d_1^2\big)$ and $a^* = d_0/d_1$. $\square$

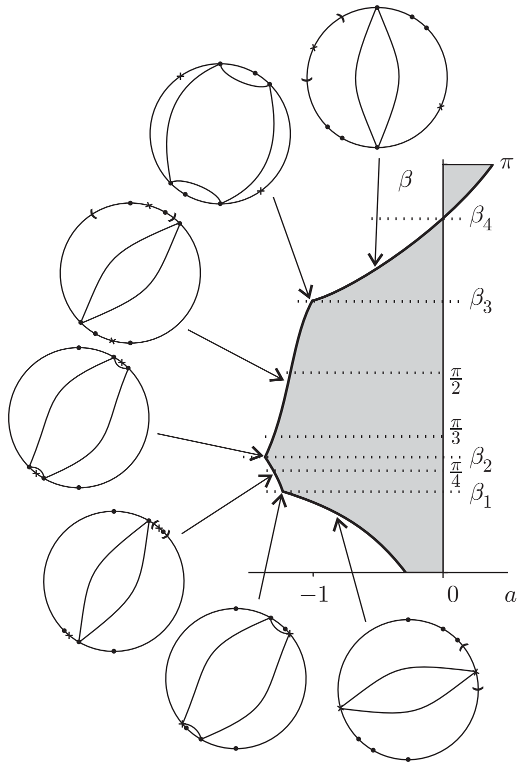

FIGURE 2. Bifurcation diagram of system (1.2) with f_4 given in (4.2) in the (a, β)-plane, where $\beta := \arctan(b^{-1}) \in (0, \pi)$.

The case $n = 4$ is more complicated. Instead of presenting an exhaustive study, we will analyze the typical two-parameter family

$$f_4(\cos\theta, \sin\theta) = \cos\theta\,(\cos\theta - b\sin\theta)\,(\cos\theta - \sin\theta)\left(\cos\theta - \frac{\sqrt{3}}{3}\sin\theta\right) \tag{4.2}$$

with parameters a and b, such that a is the parameter in system (1.2) and b is chosen so that the roots of f_4 are $\pi/2, \pi/3, \pi/4$ and $\beta := \arctan\left(b^{-1}\right)$, with $\beta \in (0, \pi)$.

Theorem 4.2. *System* (1.2) *with* $n = 4$ *and* f_4 *given in expression* (4.2) *has a limit cycle if and only if the parameters* a *and* β *are in the grey region depicted in Figure 2 whose boundary is a piecewise algebraic curve. The algebraic curves can be given explicitly in the variables* a *and* b *(see the proof of the theorem). Also, the polycycles in the Poincaré sphere where the limit cycle disappears are as depicted in the figure.*

Proof. System (1.2) with f_4 given in (4.2) has parameters a and b. For simplicity of exposition, some of the statements in the proof will be presented for the corresponding parameters a and β, where $\beta := \arctan\left(b^{-1}\right)$ for $b \neq 0$ and $\beta = \pi/2$ for $b = 0$. We choose the branch of arctan so that $\beta \in (0, \pi)$. The proof is similar to the proof of Proposition 4.1.

Note that

$$B := B(b) = \frac{(3+\sqrt{3})b + 9 + \sqrt{3}}{12}\pi.$$

Hence, the sign of the parameter a for which the limit cycle exists is negative (resp. positive) for β such that $\beta < \beta_4$ (resp. $\beta > \beta_4$), where $\beta_4 := \arctan\left(b_4^{-1}\right)$ and $b_4 = \sqrt{3} - 4$. In case $\beta = \beta_4$, Theorem 1.1 implies that system (1.2) has no limit cycles and that it has a center when $a = 0$. By a (lengthy) computation, the values $C_i(a,b) := \rho_a^*(\tau_i)$, for $\tau_1 = \beta = \arctan\left(b^{-1}\right)$, $\tau_2 = \pi/3$, $\tau_3 = \pi/4$ and $\tau_4 = \pi/2$ are given by

$$\begin{aligned}
C_1(a,b) &= \big(8b - 8b^2 - 24b^3 + 24b^4\big)a^3 + \big(4 + (-8-4\sqrt{3})b + (-32+4\sqrt{3})b^2 \\
&\quad +(-4\sqrt{3}+40)b^3 + (4\sqrt{3}+12)b^4\big)a^2 + \big(-3 - 3\sqrt{3} + (6\sqrt{3}-10)b \\
&\quad +28b^2 + (6\sqrt{3}+6)b^3 + (3\sqrt{3}+15)b^4\big)a + 3\sqrt{3} + 1 + (2\sqrt{3}+10)b \\
&\quad +(4\sqrt{3}+4)b^2 + (2\sqrt{3}+10)b^3 + (3+\sqrt{3})b^4, \\
C_2(a,b) &= \big(-2\sqrt{3}b + 2\big)a^3 + \big(-2 + \sqrt{3} + (1+2\sqrt{3})b\big)a^2 \\
&\quad +\big(-3\sqrt{3} - 1 + (3+\sqrt{3})b\big)a - 5 - 2\sqrt{3} - (\sqrt{3}+2)b, \\
C_3(a,b) &= \big(-4b + 4\big)a^3 + \big(8 + 2\sqrt{3} - (4+2\sqrt{3})b\big)a^2 \\
&\quad +\big(10 + 3\sqrt{3} - b\big)a + 5 + 2\sqrt{3} + (2+\sqrt{3})b, \\
C_4(a,b) &= 8ba^3 + \big(-4 - (4+4\sqrt{3})b\big)a^2 + \big(3 + 3\sqrt{3} + (3\sqrt{3}+5)b\big)a \\
&\quad -3\sqrt{3} - 1 - (\sqrt{3}+1)b,
\end{aligned}$$

and the intersection points of the curves $C_i=0$ and $C_j=0$, $C_{ij} := \{C_i=0\}\cap\{C_j=0\}$, which appear in Figure 2, are

$$\begin{aligned}
C_{12} &= \left(-\frac{1-\sqrt{5+4\sqrt{3}}}{2}, \left(-1-\sqrt{3}+\sqrt{5+4\sqrt{3}}\right)^{-1}\right) := (a_1,b_1), \\
C_{23} &= \left(\frac{-1-\sqrt{3}}{2}, 6-3\sqrt{3}\right) := (a_2,b_2), \\
C_{34} &= \left(-1, -\frac{\sqrt{3}}{3}\right) := (a_3,b_3),
\end{aligned}$$

where $\beta_i = \arctan\left(b_i^{-1}\right) \in (0,\pi)$ for $i = 1,2,3$.

By Theorem 1.2, a^* is obtained by solving for a in the equations $C_i(a,b) = 0$ corresponding to $\tau_i \in \Theta$. By comparing all the solutions obtained, the value for which the limit cycle disappears is

$$a^*(b) = \begin{cases}
\max\{a : C_1(a,b) = 0, a < 0\} & \text{if } 0 < b \le b_1\, (0 < \beta \le \beta_1), \\
\max\{a : C_2(a,b) = 0, a < 0\} & \text{if } b_1 \le b \le b_2\, (\beta_1 \le \beta \le \beta_2), \\
\max\{a : C_3(a,b) = 0, a < 0\} & \text{if } b_2 \le b,\ b \le b_3 \text{ or } b = \infty\, (\beta_2 \le \beta \le \beta_3), \\
\max\{a : C_4(a,b) = 0, a < 0\} & \text{if } b_3 \le b < b_4\, (\beta_3 \le \beta < \beta_4), \\
\min\{a : C_4(a,b) = 0, a > 0\} & \text{if } b_4 < b < 0\, (\beta_4 < \beta < \pi).
\end{cases}$$

The phase portraits of system (1.2) in the Poincaré sphere are obtained by plotting the function $\theta \mapsto \rho^*_{a^*}(\theta)^{-1/n}$. For instance, when the parameter values are a corner of the curve in parameter space that corresponds to the disappearance of the limit cycle, the polycycle goes to infinity in two different directions corresponding to the two zeros of $\rho^*_{a^*}$ in $(0, \pi)$. □

5. Conclusions

Although the system (1.2) studied in this paper appears to be simple, its analysis illustrates just how difficult it can be to determine the exact number of limit cycles of a planar system. In general, suppose that n is even, and let $d_0, d_1, \ldots, d_n$ denote the real coefficients of the trigonometric polynomial f_n. The value a^* given in Theorem 1.2 for which the limit cycle disappears is a function of these parameters $a^* = a^*(d_0, d_1, \ldots, d_n)$. By using arguments similar to those used to study the case $n = 4$, it follows that this function is piecewise "algebraic" in the sense that each piece is given implicitly by a polynomial in a, $\cos\beta_i$ and $\sin\beta_i$, where the degree of a is at most $n-1$, the degrees of the trigonometric variables are at most n, and the values β_i, $i = 1, \ldots, n$, are the roots in $(0, \pi)$ of the homogeneous trigonometric equation $f_n(\cos\theta, \sin\theta) = 0$. In the case $n > 4$, the above equation is in general not solvable by radicals; hence it is very difficult to even determine implicit expressions for each piece of the function a^* in the variables $d_0, d_1, \ldots, d_n$.

References

[1] A. Algaba and M. Reyes, *Centers with degenerate infinity and their commutators*, J. Math. Anal. Appl. **278** (2003), no. 1, 109–124. MR2004b:34090

[2] A. Algaba and M. Reyes, *Computing center conditions for vector fields with constant angular speed*, J. Comput. Appl. Math. **154** (2003), no. 1, 143–159. MR2004c:34078

[3] M. A. M. Alwash, *On the center conditions of certain cubic systems*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3335–3336. MR99a:34073

[4] C. B. Collins, *Algebraic conditions for a centre or a focus in some simple systems of arbitrary degree*, J. Math. Anal. Appl. **195** (1995), no. 3, 719–735. MR96j:34050

[5] R. Conti, *Uniformly isochronous centers of polynomial systems in* $\mathbf{R}^2$, Differential equations, dynamical systems, and control science, Lecture Notes in Pure and Appl. Math., vol. 152, Dekker, New York, 1994, 21–31. MR94i:34061

[6] W. A. Coppel, *A survey of quadratic systems*, J. Differential Equations **2** (1966), 293–304. MR33:4374

[7] F. Dumortier and P. Fiddelaers, *Quadratic models for generic local* 3*-parameter bifurcations on the plane*, Trans. Amer. Math. Soc. **326** (1991), no. 1, 101–126. MR91j:58118

[8] A. Gasull, R. Prohens, and J. Torregrosa, *Limit cycles for cubic rigid systems*, to appear in J. Math. Anal. Appl.

[9] L. Mazzi and M. Sabatini, *Commutators and linearizations of isochronous centers*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **11** (2000), no. 2, 81–98. MR2001h:34046

[10] L. Perko, *Differential equations and dynamical systems*, third ed., Texts in Applied Mathematics, vol. 7, Springer-Verlag, New York, 2001. MR2001k:34001

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C 08193, Bellaterra, Barcelona, Spain

E-mail address: gasull@mat.uab.es

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C 08193, Bellaterra, Barcelona, Spain

E-mail address: torre@mat.uab.es