CLOSED SETS WHICH ARE NOT $C^\infty$-CRITICAL

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Abstract. In this paper we first observe that the complement of a countable closed subset of an $n$-dimensional manifold $M$ has large $(n-1)$-homology group. In the last section we use this information to prove that, under some topological conditions on the given manifold, certain families of fibers, in the total space of a fibration over $M$, are not critical sets for some special real or $S^1$-valued functions.

1. Introduction

Let $M, N$ be differentiable manifolds, $\mathcal{F} \subseteq C^\infty(M, N)$ be a family of smooth mappings and $f : M \to N$ be a differentiable mapping. Denote by $(f)$ its critical set and recall that $(f)$ is closed.

A closed subset of $M$ is called $\mathcal{F}$-critical if $C = (f)$ for some differentiable mapping $f \in \mathcal{F}$. A $C^\infty(M, N)$-critical set will be called $N$-critical and an $R$-critical set will be simply called critical. Given a closed subset $C$ of $M$, the question is: is it $\mathcal{F}$-critical? This is a fundamental problem which has been treated in [5] and [6] for $\mathcal{F} = C^\infty(M, R)$ and its subfamily of smooth proper functions. For instance the Antoine’s Necklace of $R^3$ is a properly critical set [5] while the circle $S^1 \subseteq R^2$ is not a critical set [6]. On the other hand the finite subsets of $M$ having cardinality strictly smaller than $\text{cat}(M)$-the Lusternick-Schnirelmann category of $M$, are not critical if $M$ is compact, because it is well known that any function $f : M \to R$ has at least $\text{cat}(M)$ critical points. Finally, when $M^m$ is immersible into $R^{m+1}$, the $S^m$-non-criticality of certain subsets of $M$ provides immediate information on the set of zeros of the Gauss-Kronecker curvature associated to an arbitrary immersion of $M$ into $R^{m+1}$, taking into account that the mentioned set of zeros is actually the critical set of the associated Gauss mapping of the given immersion. For instance, the product $S^k \times S^n, k + n \geq 3$ has, for any immersion $f : S^k \times S^n \to R^{k+n+1}$, infinitely many points of zero Gauss-Kronecker curvature, simply because the finite subsets of $S^k \times S^n$ are not $S^{k+n}$-critical [7].

Let us consider the family $CS^\infty(M, N) : = \{ f \in C^\infty(M, N) | B(f) \cap f(R(f)) = \emptyset \}$, where $R(f)$ is the set of regular points of $f$ while $B(f) = f(C(f))$ is the set of
its critical values. We say that a mapping from \(CS^\infty(M,N)\) separates the critical values by the regular ones.

**Remark 1.1.** Let \(p : \tilde{N} \to N\) be a covering mapping and \(\tilde{f} : M \to \tilde{N}\) be a differentiable mapping. If \(\tilde{f} \notin CS^\infty(M,\tilde{N})\), then \(p \circ \tilde{f} \notin CS^\infty(M,N)\). Therefore if \(g \in CS^\infty(M,N)\) and \(\tilde{g} \in C^\infty(M,\tilde{N})\) is a lifting of \(g\), then \(\tilde{g} \in CS^\infty(M,\tilde{N})\).

Indeed, since \(\tilde{f} \notin CS^\infty(M,\tilde{N})\), it follows that there exist \(x_0 \in C(f)\), \(x_1 \in R(f)\) such that \(\tilde{f}(x_0) = \tilde{f}(x_1)\). Because \(p\) is a local diffeomorphism, this implies that \(x_0 \in C(p \circ \tilde{f})\), \(x_1 \in R(p \circ \tilde{f})\) and obviously \((p \circ \tilde{f})(x_0) = (p \circ \tilde{f})(x_1) \in B(p \circ \tilde{f}) \cap (p \circ \tilde{f})(R(p \circ \tilde{f}))\), that is, \(p \circ \tilde{f} \notin CS^\infty(M,N)\).

**Proposition 1.2.** (i) \(f \in CS^\infty(M,N)\) iff \(C(f) = f^{-1}(B(f))\).

(ii) If \(M\) is a connected differentiable manifold and \(f \in CS^\infty(M,\mathbb{R})\) is such that \(R(f) = M \setminus C(f)\) is also connected, then \(f(R(f)) = (m_1, M_1)\), where \(m_1 = \inf f(x), M_1 = \sup f(x)\) and \(B(f) \subseteq \{m_1, M_1\} \cap \mathbb{R}\). Moreover, if \(M\) is compact, then \(m_1, M_1 \in \mathbb{R}\) and \(B(f) = \{m_1, M_1\}\).

**Proof.** (i) Indeed if \(f \in CS^\infty(M,N)\) we only have to show that \(f^{-1}(B(f)) \subseteq C(f)\), because the other inclusion is always true. Hence if \(p \in f^{-1}(B(f))\), it follows that \(f(p) \in B(f) = f(C(f)) \subseteq N \setminus f(R(f))\), that is, \(f(p) \notin f(R(f))\), meaning that \(p \notin R(f)\).

Conversely, if \(C(f) = f^{-1}(B(f))\), assume that \(B(f) \cap f(R(f)) \neq \emptyset\) and consider \(q \in f(C(f)) \cap f(R(f))\). This means that there exists \(p_1 \in C(f), p_2 \in R(f)\) such that \(f(p_1) = f(p_2) = q\), that is \(p_2 \in f^{-1}(q) \subseteq f^{-1}(B(f)) = C(f)\), which is of course a contradiction with the obvious fact that \(R(f) \cap C(f) = \emptyset\).

(ii) If \(f\) is non-constant, then obviously \(f(R(f)) \subseteq \{m_1, M_1\}\). On the other hand, we can choose two regular values \(q_1, q_2 \in Im f \subseteq [m_1, M_1]\) arbitrarily close to \(m_1\) and \(M_1\), respectively. We can also take two regular points \(p_1 \in f^{-1}(q_1), p_2 \in f^{-1}(q_2)\) and connect \(p_1, p_2\) by a differential path in \(R(f)\). Because \(f\) is separating critical points by the regular ones, this path is applied by \(f\) on a segment of regular values in \(Im f\) connecting \(q_1\) with \(q_2\). Hence the interval \([q_1, q_2]\) is completely contained in \(Im f \setminus B(f)\). Consequently we have shown in this way that \(\{m_1, M_1\} \subseteq Im f \setminus B(f) = f(R(f))\) and that \(B(f) \subseteq \{m_1, M_1\} \cap \mathbb{R}\). \(\square\)

In this paper we are going to prove that the collection of fibers in the total space of certain fibrations \(p : E \to M\), over closed countable subsets of the base space, is neither \(CS^\infty(E,\mathbb{R})\)-critical nor \(CS^\infty(E,S^1)\)-critical under certain topological conditions on the total and base spaces and on the fiber of the considered fibrations.

Let us observe that any closed countable subset \(A\) of a manifold has countably many isolated points. Indeed otherwise the subset \(I \subseteq A\) of isolated points would be finite, possible empty, and \(A \setminus I\) would be a countable perfect subset of the given manifold. But it is folklore that perfect subsets of complete metric spaces are not countable. Therefore such a subset can be represented as \(A = I \cup A' = \{a_1, a_2, \ldots\} \cup A'\), where \(A'\) is the derived set of \(A\), that is, the set of accumulation points.

We close this section by recalling a previously proved theorem, which involves closed countable sets. It has been proved in [7] for some particular closed countable sets, the arguments for arbitrary ones being given in [8].
Theorem 1.3. Let $M$ be an $n$-dimensional differentiable manifold ($\partial M = \emptyset$) and $A$ be a closed countable subset of $M$. If $P$ is a compact differentiable $k$-dimensional manifold ($k < n$, $\partial P \neq \emptyset$) and $f : P \to M$ is a continuous map such that $f(\partial P) \subseteq M \setminus A$, then there exists a continuous map $g : P \to M$ such that $g(P) \subseteq M \setminus A$, $g|_{\partial P} = f|_{\partial P}$ and $f \simeq g|_{\partial P}$. If $M$ is connected, then one particularly gets, using the particular case $P = [0,1]$, that $M \setminus A$ is also connected.

2. Basic results

We start this section by proving that the complement of a closed countable subset of a given $n$-dimensional manifold has large $n-1$ homology group. The manifold $M$ will be with empty boundary all along the paper.

Proposition 2.1. Let $M$ be an $n$-dimensional differentiable manifold, $n \geq 2$, and $A = I \cup A'$ be a closed countable subset of $M$, where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of $A$ and $A'$ is its derived set. If $H_{n-1}(M) \simeq 0$, then for each $k \geq 1$ there exists a surjective group homomorphism

$$\delta_k : H_{n-1}(M \setminus A) \to \mathbb{Z}^{k-1} \oplus H_{n-1}(M \setminus A_k),$$

where $A_k = \{a_{k+1}, a_{k+2}, \ldots\} \cup A'$. Moreover, if $M$ is either not compact or compact but not orientable, then $H_{n-1}(M \setminus A) \simeq \mathbb{Z}^k \oplus H_{n-1}(M \setminus A_k)$, for each $k \geq 1$, that is, $H_{n-1}(M \setminus A)$ has free abelian subgroups of arbitrarily large rank.

Proof. We first recall that $M \setminus A$ is connected, because of Theorem [13]. To prove the stated isomorphism, we will use the Mayer-Vietoris sequence for the following two spaces:

$$X_k = M \setminus \{a_1, a_2, \ldots, a_k\}, \quad Y_k = M \setminus A_k.$$

Taking into account the fact that $X \cap Y = M \setminus A$ and $X_k \cup Y_k = M$ for each $k \geq 1$, we get the following exact sequence:

$$0 \to H_n(M) \to H_{n-1}(M \setminus A) \xrightarrow{\Delta_k} H_{n-1}(X_k) \oplus H_{n-1}(Y_k) \to H_{n-1}(M) \to 0,$$

where $\Delta_k(\bar{z}) = (H_k(ik)(\bar{z}), H_k(jk)(\bar{z}))$ and $ik : M \setminus A \leftarrow X_k$, $jk : M \setminus A \leftarrow Y_k$ are the inclusions. Because of the exactness of the sequence (1) we can conclude that for each $k \geq 1$ the group homomorphism $\Delta_k$ is surjective because $H_{n-1}(M) \simeq 0$. Using the homology sequence of the pair $(M, X_k)$ we have

$$0 \to H_n(X_k) \to H_n(M) \to H_n(M, X_k) \to H_{n-1}(X_k) \to H_{n-1}(M) \to 0.$$

On the other hand, excising a suitable open subset of $X_k$, one can get an isomorphism $H_n(M, X_k) \simeq \mathbb{Z}^k$. Therefore, when $M$ is either not compact or compact and not orientable, we get that $H_{n-1}(X_k) \simeq \mathbb{Z}^k$, $H_n(M)$ being trivial in both cases [4] p. 166, 167].

If $M$ is compact orientable, the exact sequence (2) provide us, up to some group isomorphisms, the following short exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z}^k \to H_{n-1}(X_k) \to 0,$$

because $H_n(X_k) \simeq 0$, taking into account that $X_k$ is a connected non-compact $n$-dimensional manifold. It ensures us that, when $M$ is compact orientable, there exists $d \geq 1$ such that $H_{n-1}(X_k) \simeq \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}^{k-1}$. In any case there exists a surjective group homomorphism

$$\psi_k : H_{n-1}(X_k) \to \mathbb{Z}^{k-1},$$
which is defined by composing the isomorphism $H_{n-1}(X_k) \cong \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}^{k-1}$ or $H_{n-1}(X_k) \cong \mathbb{Z}^k$ with a suitable projection that simply forgets the first term of the direct sum. Consequently the group homomorphism

$$\delta_k : H_{n-1}(M\backslash A) \to \mathbb{Z}^{k-1} \oplus H_{n-1}(M\backslash A_k), \delta_k := (\psi_k \times \text{id}_{H_{n-1}(X_k)}) \circ \Delta_k$$

is obviously surjective, and the proof is now complete.

**Corollary 2.2.** Let $M$ be an $n$-dimensional differential manifold, $n \geq 2$, and $A = I \cup A'$ be a closed countable subset of $M$, where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of $A$ and $A'$ is its derived set. If $G$ is an abelian group and $\varphi : G \to H_{n-1}(M\backslash A)$ is a surjective group homomorphism, then $G$ is not finitely generated. In particular, $H_{n-1}(M\backslash A)$ is not finitely generated.

**Proof.** Assume that $G$ is a finitely generated group and $\varphi : G \to H_{n-1}(M\backslash A)$ is a surjective group homomorphism. It follows that

$$\delta_k \circ \varphi : G \to \mathbb{Z}^{k-1} \oplus H_{n-1}(M\backslash A_k)$$

is also surjective for each $k \geq 1$. But since $\delta_k \circ \varphi$ maps the torsion part $t(G)$ of $G$ into the torsion part $t(\mathbb{Z}^{k-1} \oplus H_{n-1}(M\backslash A_k)) = \mathbb{Z}^{k-1} \oplus t(H_{n-1}(M\backslash A_k))$ of $\mathbb{Z}^k \oplus H_{n-1}(M\backslash A_k)$, it follows that

$$\frac{G}{t(G)} \to \mathbb{Z}^{k-1} \oplus \frac{H_{n-1}(M\backslash A_k)}{t(\mathbb{Z}^{k-1} \oplus H_{n-1}(M\backslash A_k))}, g \mapsto (\delta_k \circ \varphi)(g) + t(G)$$

is a well-defined group homomorphism denoted by $\gamma_k$. But since $G$ is finitely generated, there exists a unique natural number $q$ such that $G \cong \mathbb{Z}^q \oplus t(G)$ and obviously $\frac{G}{t(G)} \cong \mathbb{Z}^q$. Therefore, up to some group isomorphisms, $\gamma_{q+2}$ acts surjectively from $\mathbb{Z}^q$ to $\mathbb{Z}^{q+1} \oplus \frac{H_{n-1}(M\backslash A_{q+2})}{t(H_{n-1}(M\backslash A))}$. Because the projection

$$pr_1 : \mathbb{Z}^{q+1} \oplus \frac{H_{n-1}(M\backslash A_{q+2})}{t(H_{n-1}(M\backslash A))} \to \mathbb{Z}^{q+1}$$

is surjective, it follows that $pr_1 \circ \gamma_{q+2} : \mathbb{Z}^q \to \mathbb{Z}^{q+1}$ is also surjective, which is impossible because such a group homomorphism does not exist.

Let $p : E \to M$ be a fibration whose base space $M$ is an $n$-dimensional manifold, and let $A \subseteq M$ be a closed countable subset. In order to prove that the natural group homomorphism $h_{n-1} : \pi_{n-1}(E\backslash p^{-1}(A)) \to H_{n-1}(E\backslash p^{-1}(A))$ is an isomorphism, we will show that each homotopy class

$$[F] \in [(P, \partial P), (E, E\backslash p^{-1}(A))]$$

contains a mapping whose image avoids the subset $p^{-1}(A)$, where $P$ is a compact connected differentiable manifold such that $\dim P < \dim N$ and $\partial P \neq \emptyset$.

**Theorem 2.3.** Let $p : E \to M$ be a fibration whose base space $M$ is an $n$-dimensional differentiable manifold, and let $A$ be a closed countable subset of $M$. If $P$ is a compact differentiable $k$-dimensional manifold ($k < n$, $\partial P \neq \emptyset$) and $F : P \to E$ is a continuous map such that $F(\partial P) \subseteq E\backslash p^{-1}(A)$, then there exists a continuous map $G : P \to E$ such that $G(P) \subseteq E\backslash p^{-1}(A)$, $G|_{\partial P} = F|_{\partial P}$ and $F \cong G|_{\text{rel } \partial P}$. If $E$ is connected, then one particularly gets, using the particular case $P = [0, 1]$, that $E\backslash p^{-1}(A)$ is also connected.
Proof. Applying Theorem 1.3 there exists a homotopy \( H : P \times [0, 1] \to M \) of \( p \circ F \) such that \( 1mH(\cdot, 1) \subseteq M \setminus A \). Because \( p : E \to M \) is a fibration, there exists a homotopy \( H' \) of \( F \) that covers the homotopy \( H \), namely \( H = p \circ H' \). It is easy to check that \( 1mH'(\cdot, 1) \subseteq E \setminus p^{-1}(A) \).

Consider the homotopies \( \psi : P \times [0, 1] \to P \) and \( \varphi : P \times [0, 1] \to E \):

\[
\psi(x, t) = \begin{cases} 
x & \text{if } x \in P \setminus Q(\partial P \times [0, 2]) \\
Q((\pi_1 \circ Q^{-1})(x), \frac{2}{t} \pi_2 \circ Q^{-1})(x) + \frac{2t}{t-2} & \text{if } x \in Q(\partial P \times [t, 2]) \\
(\pi_1 \circ Q^{-1})(x) & \text{if } x \in Q(\partial P \times [0, t]),
\end{cases}
\]

\[
\varphi(x, t) = \begin{cases} 
H'(\psi(x, t)) & \text{if } x \in P \setminus Q(\partial P \times [0, t]) \\
H'(Q^{-1}(x)) & \text{if } x \in Q(\partial P \times [0, t]),
\end{cases}
\]

where \( Q : \partial P \times [0, \infty) \to U \subseteq P \) is a collar neighbourhood of \( \partial P \) and \( \pi_1 : \partial P \times [0, \infty) \to \partial P \), \( \pi_2 : \partial P \times [0, \infty) \to [0, \infty) \) are obviously the projections.

Denoting \( \varphi(\cdot, 1) \) by \( G \) and observing that \( \varphi(\cdot, 0) = F \), one can easily see that \( F \approx \varphi G(\text{rel } \partial P) \) and also that \( G(P) \subseteq E \setminus p^{-1}(A) \), the theorem being now completely proved.

Corollary 2.4. If \( p : E \to M \) is a fibration whose base space \( M \) is an \( n \)-dimensional differentiable manifold and \( A \) is a closed countable subset of \( M \), then the pair \( (E, E \setminus p^{-1}(A)) \) is \((n-1)\)-connected, that is, \( \pi_q(E, E \setminus p^{-1}(A)) \approx 0 \) for all \( q \in \{1, \ldots, n-1\} \). In particular we get that \( H_q(E, E \setminus p^{-1}(A)) \approx 0 \) for all \( q \in \{1, \ldots, n-1\} \) and the natural group homomorphism

\[ \chi_n : \pi_n(E, E \setminus p^{-1}(A)) \to H_n(E, E \setminus p^{-1}(A)) \]

is surjective. On the other hand the inclusion \( i_{E \setminus p^{-1}(A)} : E \setminus p^{-1}(A) \hookrightarrow E \) is \((n-1)\)-connected, that is, the induced group homomorphism

\[ \pi_q(i_{E \setminus p^{-1}(A)}) : \pi_q(E \setminus p^{-1}(A)) \to \pi_q(E) \]

is an isomorphism for \( q \leq n-2 \) and it is an epimorphism for \( q = n-1 \). Hence the morphism \( \chi_n \) is an isomorphism if \( E \) is simply connected and \( n \geq 3 \).

Proof. The fact that \( \pi_q(E, E \setminus p^{-1}(A)) \approx 0 \) for all \( q \in \{1, 2, \ldots, n-1\} \) is an immediate consequence of Theorem 2.3 and of the fact that \( [a] \in \pi_q(E, E \setminus p^{-1}(A)) \) is zero if and only if there exists \( \beta \in [a] \) such that \( \beta(D^q) \subseteq E \setminus p^{-1}(A) \). From the Hurewicz theorem \( \pi_q(E, E \setminus p^{-1}(A)) \approx 0 \) for all \( q \in \{1, \ldots, n-1\} \) as well as that \( \chi_n : \pi_n(E, E \setminus p^{-1}(A)) \to H_n(E, E \setminus p^{-1}(A)) \) is surjective. Further on, using the exact homotopy sequence

\[ \cdots \to \pi_{r+2}(E, E \setminus p^{-1}(A)) \to \pi_r(E, E \setminus p^{-1}(A)) \to \pi_r(E) \to \pi_r(E, E \setminus p^{-1}(A)) \to \cdots, \]

and the triviality of \( \pi_q(E, E \setminus p^{-1}(A)) \) for \( q \in \{1, 2, \ldots, n-1\} \), it follows that the inclusion \( i_{E \setminus p^{-1}(A)} : E \setminus p^{-1}(A) \hookrightarrow E \) is \((n-1)\)-connected. Finally, since \( E \) is simply connected, it follows that \( E \setminus p^{-1}(A) \) is also simply connected such that \( \pi_1(E \setminus p^{-1}(A)) \) acts trivially on \( \pi_n(E, E \setminus p^{-1}(A)) \), which means that \( \chi_n \) is an isomorphism [3 p. 166].

Corollary 2.5. Let \( p : E \to M \) be a fibration whose base space \( M \) is an \( n \)-dimensional differentiable manifold, and let \( A \) be a closed countable subset of \( M \).
If the total space $E$ is simply connected and the natural group homomorphisms $h^E_q : \pi_q(E) \to H_q(E)$, $q \in \{n-1, n\}$ are isomorphisms, then the natural group homomorphism

$$h_{n-1} : \pi_n\left(E \setminus p^{-1}(A)\right) \to H_{n-1}\left(E \setminus p^{-1}(A)\right)$$

is also an isomorphism.

**Proof.** Consider the following ladder with exact rows and commutative rectangles:

$$\begin{array}{ccc}
\pi_n(E) & \to & \pi_n(E, E \setminus p^{-1}(A)) \\
\pi_n(E) & \to & \pi_n(E, E \setminus p^{-1}(A)) \\
\pi_n(E) & \to & \pi_n(E) \\
\pi_n(E) & \to & \pi_n(E)
\end{array}
\quad
\begin{array}{ccc}
h^E_q & \to & h_{n-1} \chi_n \\
h^E_q & \to & h_{n-1} \\
h^E_q & \to & h_{n-1} \\
h^E_q & \to & h_{n-1}
\end{array}
\quad
\begin{array}{ccc}
H_n(E) & \to & H_n(E, E \setminus p^{-1}(A)) \\
H_n(E) & \to & H_n(E, E \setminus p^{-1}(A)) \\
H_n(E) & \to & H_n(E) \\
H_n(E) & \to & H_n(E)
\end{array}
\quad
\begin{array}{ccc}
\chi_n^{-1} & \to & \chi_n^{-1} \\
\chi_n^{-1} & \to & \chi_n^{-1} \\
\chi_n^{-1} & \to & \chi_n^{-1} \\
\chi_n^{-1} & \to & \chi_n^{-1}
\end{array}
$$

and conclude, using the hypothesis, Corollary 2.4 and the five lemma, that $h_{n-1}$ is indeed an isomorphism. \qed

**Remark 2.6.** (i) If $M$ is an $n$-dimensional differentiable manifold and $A$ is a closed countable subset of $M$, then the pair $(M, M \setminus A)$ is $(n-1)$-connected, that is, $\pi_q(M, M \setminus A) \simeq 0$ for all $q \in \{1, \ldots, n-1\}$.

In particular we get that $H_q(M, M \setminus A) \simeq 0$ for all $q \in \{1, \ldots, n-1\}$ and the natural group homomorphism $\chi_n : \pi_n(M, M \setminus A) \to H_n(M, M \setminus A)$ is surjective. On the other hand the inclusion $i_{\setminus M \setminus A} : M \setminus A \hookrightarrow M$ is $(n-1)$-connected, that is, the induced group homomorphism $\pi_q(i_{\setminus M \setminus A}) : \pi_q(M \setminus A) \to \pi_q(M)$ is an isomorphism for $q \leq n-2$ and it is an epimorphism for $q = n-1$. Hence the morphism $\chi_n$ is an isomorphism if $M$ is simply connected and $n \geq 3$.

(ii) Let $M$ be an $n$-dimensional differentiable manifold, and let $A$ be a closed countable subset of $M$. If the natural group homomorphism $h_{n-1}^M : \pi_n(M) \to H_{n-1}(M)$ is surjective, then the natural group homomorphism

$$h_{n-1}^{M \setminus A} : \pi_n(M \setminus A) \to H_{n-1}(M \setminus A)$$

is also surjective. Moreover, if $h_{n-1}^M, h_n^M$ are isomorphisms, then $h_{n-1}^{M \setminus A}$ is an isomorphism too.

(iii) If $M$ is an $n$-dimensional differentiable manifold and $A$ is a closed countable subset of $M$, then $\pi_{n-1}(M \setminus A)$ is not finitely generated.

(iv) Let $M$ be an $n$-dimensional differential contractible manifold, $n \geq 2$, and let $A = I \cup A'$ be a closed countable subset of $M$, where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of $A$ and $A'$ is its derived set. Then $M \setminus A$ is $(n-2)$-connected and $\pi_{n-1}(M \setminus A) \cong \mathbb{Z}^I \oplus \pi_{n-1}(M \setminus A_i)$, where $A_i = \{a_{i+1}, a_{i+2}, \ldots\} \cup A'$. In particular $\pi_{n-1}(M \setminus A)$ has free abelian subgroups of arbitrarily large rank.

Indeed the first points (i) and (ii) are immediate consequences of Corollaries 2.3 and 2.5 respectively by considering the particular fibration $id_M : M \to M$, while the third point (iii) follows easily by combining Corollary 2.2 and Remark 2.6 (ii). At the fourth point (iv) the $(n-2)$-connectedness follows easily from Remark 2.6 (i) while the isomorphism $\pi_{n-1}(M \setminus A) \cong \mathbb{Z}^I \oplus \pi_{n-1}(M \setminus A_i)$ follows from Proposition 2.1 by using the Hurewitz theorem and Remark 2.6 (ii).

3. **Application**

In this section we will give the topological conditions on the total and base spaces and on the fiber of a fibration in order that the collection of fibers over a closed countable subset of the base space not be $CS^\infty$-critical.
Theorem 3.1. Let $F \hookrightarrow E \xrightarrow{p} M^n$ be a differential fibration with compact total space, and let $A$ be a closed countable subset of $M$.

(i) If $n \geq 2$, $M$ is a homotopy sphere and $H_1(F) \cong 0$, then $p^{-1}(A)$ is not $CS^\infty(E, R)$-critical. Moreover, if $E$ is simply connected, then $p^{-1}(A)$ is not $CS^\infty(E, S^1)$-critical too.

(ii) If $n \geq 3$, $E$ is simply connected, $H_{n-1}(M) \cong 0$, $\pi_{n-2}(F)$ is finitely generated and commutative when $n = 3$ and the natural group homomorphisms $h_{n-1}^E, h_n^E$ are isomorphisms, then $p^{-1}(A)$ is neither $CS^\infty(E, R)$-critical nor $CS^\infty(E, S^1)$-critical.

Proof. Assume that there exists a mapping $f \in CS^\infty(E, R)$ such that $C(f) = p^{-1}(A)$. This means that $B(f) = \{m_j, M_j\}$ and that its restriction

$$E\backslash C(f) \to \text{Im} f \backslash B(f) = (m_j, M_j), \ p \mapsto f(p)$$

is a proper submersion, that is, via Ehresmann’s theorem, a locally trivial fibration whose compact fiber we are denoting by $\mathcal{F}$. Its base space $(m_j, M_j)$ being contractible, it follows that the inclusion $i_f : \mathcal{F} \hookrightarrow E\backslash C(f)$ is a weak homotopy equivalence, namely the induced group homomorphisms

$$\pi_q(i_f) : \pi_q(\mathcal{F}) \to \pi_q(E\backslash C(f))$$

are all isomorphisms. Consequently, using the Whitehead theorem [3, p. 167] or [9, p. 399], it follows that the induced group homomorphisms

$$H_q(i_f) : H_q(\mathcal{F}) \to H_q(E\backslash C(f)) = H_q(E\backslash p^{-1}(A))$$

are also isomorphisms.

(i) The Serre homotopy sequence of the fibration $F \hookrightarrow E\backslash p^{-1}(A) \xrightarrow{p} M\backslash A$ is:

$$\cdots \to H_{n-1}(F) \to H_{n-1}(E\backslash p^{-1}(A)) \xrightarrow{H_0} H_{n-1}(M\backslash A) \to H_{n-2}(F) \to \cdots.$$ 

Because $H_{n-1}(\mathcal{F}) \cong H_{n-1}(E\backslash p^{-1}(A))$ and $H_{n-1}(\mathcal{F}), H_{n-2}(F)$, are finitely generated, it follows that $H_{n-1}(M\backslash A)$ must be finitely generated, which is a contradiction with Corollary 2.2. In order to prove that $p^{-1}(A)$ is not $CS^\infty(E, S^1)$-critical, under the additional hypothesis of simply connectedness of $E$, we assume that there exists $g \in CS^\infty(E, S^1)$ such that $C(g) = p^{-1}(A)$ and consider a lifting $\tilde{g} : E \to R$ that does not belong to $CS^\infty(E, R)$ since $C(\tilde{g}) = C(g) = p^{-1}(A)$. Using Remark 1.1, one can deduce that $g = \exp \circ \tilde{g} \notin CS^\infty(E, S^1)$.

(ii) Because $H_{n-1}(M) \cong 0$, this implies that $h_{n-1}^E$ is obviously surjective. According to Corollary 2.5 and Remark 2.6 (ii), the natural group homomorphism

$$h_{n-1} : \pi_n-1(E\backslash p^{-1}(A)) \to H_{n-1}(E\backslash p^{-1}(A))$$

is an isomorphism, while $h_{n-1}^M : \pi_{n-1}(M\backslash A) \to H_{n-1}(M\backslash A)$ is surjective. Therefore, $h_{n-1}^E \circ H_{n-1}(\mathcal{F}) : H_{n-1}(\mathcal{F}) \to \pi_{n-1}(E\backslash p^{-1}(A))$ is an isomorphism.

The homotopy sequence of the fibration $F \hookrightarrow E\backslash p^{-1}(A) \xrightarrow{p} M\backslash A$ is

$$\cdots \to \pi_{n-1}(F) \to \pi_{n-1}(E\backslash p^{-1}(A)) \xrightarrow{H_0} \pi_{n-1}(M\backslash A) \to \pi_{n-2}(F) \to \cdots,$$

and it forces $\pi_{n-1}(M\backslash A)$ to be finitely generated since $H_{n-1}(\mathcal{F}) \cong \pi_{n-1}(E\backslash p^{-1}(A))$ and $\pi_{n-2}(F), H_{n-1}(\mathcal{F})$ are finitely generated. But this is impossible because Remark 2.6 (iii) ensures us that $\pi_{n-1}(M\backslash A)$ is not finitely generated. The $CS^\infty(E, S^1)$-non-criticality of $p^{-1}(A)$ can be proved similarly. 

□
Corollary 3.2. If $F \hookrightarrow E \overset{p}{\rightarrow} M$ is one of the following fibrations:

(i) $\text{Spin}(n) \hookrightarrow \text{Spin}(n+1) \rightarrow S^n$, $n \geq 3$,
(ii) $SU(n) \hookrightarrow SU(n+1) \rightarrow S^{2n+1}$, $n \geq 2$,
(iii) $Sp(n) \hookrightarrow Sp(n+1) \rightarrow S^{4n+3}$, $n \geq 1$,
(iv) $S^3 \hookrightarrow S^{4n+3} \rightarrow PH^n$, $n \geq 1$,
(v) $S^1 \hookrightarrow S^{2n+1} \rightarrow PC^n$, $n \geq 2$,

and if $A$ is a closed countable subset of the base space, then $p^{-1}(A)$ is neither $C^\infty(E,\mathbb{R})$-critical nor $C^\infty(E,S^1)$-critical.

Proof. Indeed, (i), (ii) and (iii) follow immediately from Theorem 3.1 (i) taking into account that $S^k$ is $(k-1)$-connected, $\text{Spin}(n)$, $U(n)$, $Sp(n)$ are compact and $H_i(\text{Spin}(n))$, $H_i(SU(n))$, $H_i(\text{Sp}(n))$, $i \in \{1, 2\}$ are trivial because $\pi_i(\text{Spin}(n)), \pi_i(SU(n)), \pi_i(\text{Sp}(n))$, $i \in \{1, 2\}$ are trivial [1] p. 368, [2] p. 224.

(iv) and (v) follow from Theorem 3.1 (i) since $H_{4n-1}(PH^n) \simeq 0 \cong H_{2n-1}(PC^n)$ while $\pi_{2n-2}(S^1)$, $\pi_{4n-2}(S^3)$ are finitely generated, $\pi_{2n-2}(S^1)$ being actually trivial for $n \geq 2$, while $\pi_{4n-2}(S^3)$ is even finite [3] p. 318. Finally the natural group homomorphisms $h_{2n-1}^{4n+3}$, $h_{2n}^{4n+3}, h_{2n-1}^{2n+1}, h_{2n}^{2n+1}$ are obviously isomorphisms, because $\pi_q(S^{4n+3}), H_q(S^{4n+3}), q \in \{4n-1, 4n\}$ and $\pi_r(S^{2n+1}), H_r(S^{2n+1}), r \in \{2n-1, 2n\}$ are trivial.□

Remark 3.3. The quaternionic Hopf fibration $S^3 \hookrightarrow S^7 \rightarrow S^4$ can also be treated by means of Theorem 3.1 (i).

References


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