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BORDISM GROUPS OF SPECIAL GENERIC MAPPINGS

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Abstract. The Pontrjagin-Thom construction expresses a relation between the oriented bordism groups of framed immersions $M^m \looparrowright \mathbb{R}^n, m < n$, and the stable homotopy groups of spheres. We apply the Pontrjagin-Thom construction to the oriented bordism groups $\mathcal{M}_{m,n}$ of mappings $M^m \to \mathbb{R}^n, m > n$, with mildest singularities. Recently, O. Saeki showed that for $m \geq 6$, the group $\mathcal{M}_{m,1}$ is isomorphic to the group of smooth structures on the sphere of dimension m. Generalizing, we prove that $\mathcal{M}_{m,n}$ is isomorphic to the n-th stable homotopy group $\pi_n^{st}(\mathrm{BSDiff}_r, \mathrm{BSO}_{r+1})$, $r = m - n$, where SDiff_r is the group of oriented auto-diffeomorphisms of the sphere S^r and SO_{r+1} is the group of rotations of S^r.

1. Introduction

A point x of a smooth mapping $f : M^m \to \mathbb{R}^n$, $m \geq n$, of a manifold is called *regular* if the rank of the differential $d_x f$ is maximal, i.e., equals n. Otherwise $x \in M^m$ is called *singular*. We say that a point x is *a definite fold singular point* if in some coordinate neighborhoods about x and $f(x)$ the mapping f has the form

$$f(t_1, \ldots, t_{n-1}, x_n, \ldots, x_m) = (t_1, \ldots, t_{n-1}, x_n^2 + \cdots + x_m^2).$$

In this paper we study *special generic mappings*, which are mappings with only regular and definite fold singular points.

Suppose $f_i : M_i^m \to \mathbb{R}^n$, $i = 0, 1$, are two special generic mappings of closed oriented manifolds and there exists a special generic mapping $F : M^{m+1} \to \mathbb{R}^n \times [0, 1]$ of an oriented manifold M^{m+1} with boundary $\partial M^{m+1} = M_0^m \cup (-M_1^m)$ such that

$$F|M_i = f_i : M_i^m \to \mathbb{R}^n \times \{i\}, \quad i = 0, 1.$$

Then we say that the special generic mappings f_0 and f_1 are *bordant*. The classes of bordant mappings constitute an abelian group $\mathcal{M}_{m,n}$ under the operation of taking a disjoint union of mappings. The identity element of this group is represented by the mapping of the empty manifold. The inverse $-[f]$ to the bordism class of a mapping $f : M^m \to \mathbb{R}^n$ is represented by the composition of f and a reflection of $\mathbb{R}^n$.

The bordism groups $\mathcal{M}_{m,n}$ were introduced by O. Saeki in [7] (see Problem 5.6). It turns out that for $m \geq 6$, the group $\mathcal{M}_{m,1}$ is isomorphic to the h-cobordism group of oriented homotopy m-spheres Θ_m [8]. The restriction on the dimension m

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appears as a consequence of the same restriction in the Cerf pseudo-isomorphism theorem [1]. In view of the isomorphism $\Theta_m \approx \pi_0(\mathrm{SDiff}_{m-1})$, $m \geq 6$, where SDiff_{m-1} is the group of orientation-preserving auto-diffeomorphisms of the sphere S^{m-1} [1], the Saeki theorem asserts that $\mathcal{M}_{m,1} \approx \pi_0(\mathrm{SDiff}_{m-1})$ provided that $m \geq 6$.

For greater values of n, we will show that the group $\mathcal{M}_{m,n}$ is closely related to the space $\mathrm{SDiff}_r/\mathrm{SO}_{r+1}$, $r = m-n$, where SO_{r+1} is the space of rotations of $\mathbb{R}^{m+1}$ and the symbol '/' stands for the quotient of spaces. We recall that the i-th stable homotopy group of a topological space X is defined as the direct limit of groups

$$\pi_i^{st}(X) := \lim_{N\to\infty} \pi_{i+N}(\Sigma^N X).$$

Here the space $\Sigma^N X$ is the N-th suspension of the space X. Let $\tilde{\pi}_k^{st}(X)$ denote the k-th reduced stable homotopy group of X, i.e., the kernel of the homomorphism $\pi_k^{st}(X) \to \pi_k^{st}(pt)$ induced by the projection of X onto a point pt. The i-th relative stable homotopy group $\pi_i^{st}(X,Y)$ of a pair of topological spaces (X,Y) is defined as $\tilde{\pi}_i^{st}(X/Y)$. For a group G, let BG be a space classifying principle G-bundles.

Theorem 1.1. *There is a natural homomorphism $\mathcal{M}_{m,n} \to \pi_n^{st}(\mathrm{BSDiff}_r, \mathrm{BSO}_{r+1})$, $r \geq 0$, which is an isomorphism for $r > 0$.*

Corollary 1.2. *The group $\mathcal{M}_{m,n}$ is isomorphic to $\tilde{\pi}_{n-1}^{st}(\mathrm{SDiff}_r/\mathrm{SO}_{r+1})$, $r \geq 0$.*

Proof. The long exact sequences of stable homotopy groups for pairs $(\mathrm{SDiff}_r, \mathrm{SO}_{r+1})$ and $(\mathrm{BSDiff}_r, \mathrm{BSO}_{r+1})$ fit into the commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \to & \pi_n^{st}(\mathrm{BSDiff}_r) & \to & \pi_n^{st}(\mathrm{BSDiff}_r, \mathrm{BSO}_{r+1}) & \to & \pi_{n-1}^{st}(\mathrm{BSO}_{r+1}) & \to & \cdots \\
 & & \downarrow\approx & & \downarrow & & \downarrow\approx & & \\
\cdots & \to & \pi_{n-1}^{st}(\mathrm{SDiff}_r) & \to & \pi_{n-1}^{st}(\mathrm{SDiff}_r, \mathrm{SO}_{r+1}) & \to & \pi_{n-2}^{st}(\mathrm{SO}_{r+1}) & \to & \cdots
\end{array}$$

with vertical isomorphisms given by transgressions of the universal bundles over BSO_{r+1} and BSDiff_r. By the Five Lemma, the remaining vertical homomorphisms

$$\pi_n^{st}(\mathrm{BSDiff}_r, \mathrm{BSO}_{r+1}) \longrightarrow \pi_{n-1}^{st}(\mathrm{SDiff}_r, \mathrm{SO}_{r+1})$$

of the diagram are isomorphisms. Hence $\mathcal{M}_{m,n} \approx \pi_{n-1}^{st}(\mathrm{SDiff}_r, \mathrm{SO}_{r+1})$. □

Corollary 1.3. *The isomorphism $\mathcal{M}_{m,1} \approx \pi_0(\mathrm{SDiff}_{m-1})$ holds for all $m > 0$.*

Proof. For $n = 1$, Theorem 1.1 asserts that $\mathcal{M}_{m,n}$ is isomorphic to the group

$$\pi_1^{st}(\mathrm{BSDiff}_{m-1}, \mathrm{BSO}_m) \approx \tilde{\pi}_1^{st}(\mathrm{BSDiff}_{m-1}/\mathrm{BSO}_m).$$

Let us prove that

$$\tilde{\pi}_1^{st}(\mathrm{BSDiff}_{m-1}/\mathrm{BSO}_m) \approx \pi_0(\mathrm{SDiff}_{m-1}).$$

The reduced Atiyah-Hirzebruch spectral sequence with coefficients in $\pi_*^{st}(pt)$ for the space $X = \mathrm{BSDiff}_{m-1}/\mathrm{BSO}_m$ has the second term $\tilde{E}^2_{s,t} = \tilde{H}_s(X; \pi_t^{st}(pt))$ and converges to $\tilde{\pi}_{s+t}(X)$. From $\tilde{E}^2_{0,1} = 0$ and $\tilde{E}^2_{1,0} = \tilde{H}_1(X; \pi_0^{st}(pt))$ we deduce that $\tilde{\pi}_1^{st}(X) = \tilde{H}_1(X; \pi_0^{st}(pt)) = H_1(X; \mathbb{Z})$. There is a commutative diagram

$$\begin{array}{ccc}
H_1(\mathrm{BSDiff}_{m-1}) & \approx & H_1(\mathrm{BSDiff}_{m-1}, \mathrm{BSO}_m) \\
\uparrow & & \uparrow \\
\pi_1(\mathrm{BSDiff}_{m-1}) & \approx & \pi_1(\mathrm{BSDiff}_{m-1}, \mathrm{BSO}_m),
\end{array}$$

where the horizontal isomorphisms are parts of the homology and homotopy long exact sequences for the pair $(\mathrm{BSDiff}_{m-1}, \mathrm{BSO}_m)$ and the vertical maps are Hurewicz homomorphisms. The group

$$\pi_1(\mathrm{BSDiff}_{m-1}) \approx \pi_0(\mathrm{SDiff}_{m-1})$$

is abelian. Therefore all the groups of the diagram are isomorphic and $\tilde{\pi}_1^{st}(X) \approx \pi_0(\mathrm{SDiff}_{m-1})$. $\square$

Remark 1.4. For a definition of un-oriented bordism groups and generalizations of the group $\mathcal{M}_{m,n}$ we refer the reader to [7]. In [8] it is proved that for $m \geq 6$, the un-oriented bordism group of special generic functions is isomorphic to $\Theta_n \otimes \mathbb{Z}_2$. The reasoning in the proof of Theorem 1.1 can be modified to show that in general the un-oriented bordism group of mappings of m-manifolds into $\mathbb{R}^n$ is isomorphic to $\pi_n^{st}(\mathrm{BDiff}_r, \mathrm{BO}_{r+1})$, $r = m - n > 0$, where Diff_r is the group of auto-diffeomorphisms of S^{r-1} and O_{r+1} is the group of orthogonal transformations of $\mathbb{R}^{r+1}$.

Remark 1.5. For $r = 0$, the homomorphism of Theorem 1.1 may not be an isomorphism. The group $\pi_n^{st}(\mathrm{BSDiff}_0, \mathrm{BSO}_1)$ is trivial for all n. On the other hand, every homotopy sphere admits a special generic mapping and if the homotopy sphere does not bound a compact parallelizable manifold, then the special generic mapping represents a non-trivial element in $\mathcal{M}_{n,n}$ [7].

Remark 1.6. By [7, (5.5.7)], the groups $\mathcal{M}_{m,n}$ are trivial for $m - n = 1, 2, 3$. We can deduce this fact from Theorem 1.1. Indeed, by [2] and [10], the natural inclusion $\mathrm{SO}_{r+1} \to \mathrm{SDiff}_r$ is a homotopy equivalence for $r = 2, 3$. Therefore $\pi_n^{st}(\mathrm{BSDiff}_r, \mathrm{BSO}_{r+1}) = 0$ for $r = 1, 2, 3$.

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2. Proof of Theorem 1.1

Let $f : M^m \to N^n$ be a smooth mapping of manifolds. We say that two points x_1 and x_2 of the domain are *Stein equivalent* if for some point $y \in N^n$ the points x_1 and x_2 belong to the same path component of the inverse image $f^{-1}(y)$. Identification of the Stein equivalent points leads to a space $W = W_f$ called *the Stein factorization complex* associated with the mapping f. If the space W is endowed with the maximal topology with respect to which the factorization mapping $q_f : M^m \to W$ is continuous, then the mapping f factors into the composition of q_f and some continuous mapping $i_f : W \to N^n$.

In the case where f is a special generic mapping, the Stein factorization complex has some especially nice properties, which are discussed in detail in [6]. Here we recall some of the properties without proofs.

- The topological space W is a manifold with boundary. The dimension of W is n.
- The manifold W has a differentiable structure with respect to which the mappings q_f and i_f are smooth.
- The boundary of W is diffeomorphic to the submanifold $S(f) \subset M$ of singular points of f, and the diffeomorphism is given by the restriction $q_f|S(f)$.

- The mapping $i_f : W \to \mathbb{R}^n$ is an immersion.
- The manifold W is parallelizable.

Let $\overset{\circ}{W}$ denote the complement in W to an open collar neighborhood of the boundary ∂W. Then $\overset{\circ}{W}$ is a smooth manifold (with boundary) diffeomorphic to W. The restriction of the mapping q_f to the inverse image $q_f^{-1}(\overset{\circ}{W})$ is a bundle with fiber diffeomorphic to the standard sphere [6]. We will denote the resulting sphere bundle by $\xi = \xi_f$.

Remark 2.1. There exist two mappings f and g of non-diffeomorphic manifolds into $\mathbb{R}^n$ that lead to equivalent fiber bundles $\xi_f = \xi_g$. Indeed, let Σ_f and Σ_g be two different 7-dimensional homotopy spheres and $f : \Sigma_f \to \mathbb{R}$, $g : \Sigma_g \to \mathbb{R}$ be two Morse functions with two critical points each. Then f and g are special generic, the Stein factorization complexes W_f, W_g are segments and the bundles ξ_f, ξ_g are trivial sphere bundles over subsegments $\overset{\circ}{W}_f$ and $\overset{\circ}{W}_g$ respectively. In particular, ξ_f and ξ_g are equivalent. We also indicate that the result of Saeki [8] implies that two special generic functions on different homotopy spheres are not bordant.

Let B denote the closure of a collar neighborhood of the boundary ∂W in the manifold W. The inverse image $q_f^{-1}(B)$ can be identified with the total space of the normal bundle of $S(f)$ in M. Since the n-manifold W admits an immersion into $\mathbb{R}^n$, we have that W is orientable and the singular manifold $S(f)$, being diffeomorphic to the boundary ∂W, is orientable as well.

The boundary of $q_f^{-1}(B)$ coincides with the set of points $q_f^{-1}(\partial B) \setminus S(f)$. We say that a bundle is *linear* if its structure group is a group of orthogonal transformations. Since the structure group of the normal bundle of $S(f)$ reduces to the group of rotations SO_{m-n+1}, we obtain

Lemma 2.2 (O. Saeki, [6, Proposition 2.1])**.** *The quotient mapping q_f restricted to $q_f^{-1}(\partial B) \setminus S(f)$ is an orientable linear bundle.*

In general the sphere bundle ξ over $\overset{\circ}{W}$ has a structure group different from the orthogonal group. Let BSDiff_r be the space classifying smooth orientable bundles with fiber r-dimensional sphere (see construction in [5]) and BSO_{r+1} be the classifying space of $(r+1)$-dimensional orientable vector bundles. Then BSO_{r+1} is contained in the space BSDiff_r. Since the bundle ξ restricted to the boundary $\partial\overset{\circ}{W}$ has orthogonal structure group, the mapping classifying the bundle ξ over $\overset{\circ}{W}$ takes the pair $(\overset{\circ}{W}, \partial\overset{\circ}{W})$ into the pair $(\mathrm{BSDiff}_r, \mathrm{BSO}_{r+1})$. To motivate the subsequent definition of a relative group $\mathcal{P}_n(X, Y)$, we observe that the manifold $\overset{\circ}{W}$ carries a framing τ of the stable tangent bundle defined by the immersion $i_f : \overset{\circ}{W} \to \mathbb{R}^n$.

Let $Y \subset X$ be a subset of an arcwise connected topological space X. By definition, an element of $\mathcal{P}_n(X, Y)$ is represented by a mapping $f : (N, \partial N; \tau) \to (X, Y)$ of an n-dimensional manifold N with a framing τ of the stable tangent bundle. Two mappings $f_i : (N_i, \partial N_i; \tau_i) \to (X, Y)$, $i = 0, 1$, represent the same element in $\mathcal{P}_n(X, Y)$ if and only if there is a mapping $f : (N, \partial N \setminus (N_0 \cup N_1); \tau) \to (X, Y)$ such that

(1) $N_0 \cup N_1$ is a submanifold of ∂N,

(2) $f(\partial N \setminus (N_0 \cup N_1)) \subset Y$,

(3) $f|N_i = f_i$ for $i = 0, 1$,

(4) the framing τ_0 (respectively τ_1) followed by the inward (respectively outward) vector normal to TN_0 (respectively TN_1) in TN coincides with the framing $\tau|N_0$ (respectively $\tau|N_1$).

In terms of representatives the operation in $\mathcal{P}_n(X, Y)$ is given by taking the disjoint union of mappings.

We recall that X is assumed arcwise connected.

Lemma 2.3. *For each element of $\mathcal{P}_n(X, Y)$ there is a representative given by a mapping of a connected manifold with non-empty boundary.*

Every special generic mapping $f : M^m \to \mathbb{R}^n$ gives rise to a manifold $\mathring{W}$ with stable tangent framing τ and a mapping

$$\theta(f) : (\mathring{W}, \partial\mathring{W}; \tau) \to (\mathrm{BSDiff}_{m-n}, \mathrm{BSO}_{m-n+1})$$

representing some element $[\theta(f)]$ in $\mathcal{P}_n(\mathrm{BSDiff}_{m-n}, \mathrm{BSO}_{m-n+1})$. Special generic mappings that represent the same element in $\mathcal{M}_{m,n}$ lead to the same class in $\mathcal{P}_n(\mathrm{BSDiff}_{m-n}, \mathrm{BSO}_{m-n+1})$. Indeed, suppose F is a bordism between special generic mappings f_0 and f_1. Then, being special generic, F determines a classifying mapping $\theta(F)$ that provides a framed bordism between $\theta(f_0)$ and $\theta(f_1)$. Hence $[\theta(f_0)] = [\theta(f_1)]$. Since the group operation in $\mathcal{M}_{m,n}$ agrees with that in $\mathcal{P}_n(\mathrm{BSDiff}_{m-n}, \mathrm{BSO}_{m-n+1})$, we obtain a homomorphism

$$h : \mathcal{M}_{m,n} \to \mathcal{P}_n(\mathrm{BSDiff}_{m-n}, \mathrm{BSO}_{m-n+1}).$$

Let us find the kernel of the homomorphism h. If a mapping f represents an element of the kernel, then, changing f by bordism, we may assume that $\theta(f)$ sends $\mathring{W}_f$ into $\mathrm{BSO}_{m-n+1} \subset \mathrm{BSDiff}_{m-n}$. In [7] (see 5.5.6) O. Saeki showed that if $m > n$ and the bundle over $\mathring{W}_f$ determined by a mapping f is linear, then $[f] = 0 \in \mathcal{M}_{m,n}$. Thus, for $m > n$, the kernel of h is trivial.

Lemma 2.4. *The homomorphism $h : \mathcal{M}_{m,n} \to \mathcal{P}_n(\mathrm{BSDiff}_{m-n}, \mathrm{BSO}_{m-n+1})$ is an isomorphism.*

Proof. The homomorphism h is injective. To show that h is surjective, pick an arbitrary element in $\mathcal{P}_n(\mathrm{BSDiff}_{m-n}, \mathrm{BSO}_{m-n+1})$. By Lemma 2.3, this element can be represented by some mapping

$$\theta : (\mathring{W}, \partial\mathring{W}; \tau) \to (\mathrm{BSDiff}_{m-n}, \mathrm{BSO}_{m-n+1})$$

of a connected manifold $\mathring{W}$ with non-empty boundary. The mapping θ induces a sphere bundle E over $\mathring{W}$ such that ∂E, which is the part of E over $\partial\mathring{W}$, is the boundary of the total space of some linear disc bundle D over $\partial\mathring{W}$, $\partial D = \partial E$. After smoothing the corners of $D \cup E$ we obtain a manifold M.

Since $\mathring{W}$ has a non-empty boundary, the framing τ of the stable tangent bundle $T\mathring{W} \oplus \varepsilon$ is the sum of a framing of $T\mathring{W}$ and the framing of the trivial 1-dimensional bundle ε. By the Smale-Hirsch theorem [3], the tangent framing of $T\mathring{W}$ determines a regular homotopy class of an immersion $i : \mathring{W} \to \mathbb{R}^n$.

On the part E of $M = E \cup D$ we define a regular mapping $f : E \to \mathbb{R}^n$ as the composition of the projection $E \to \overset{\circ}{W}$ and the immersion $i : \overset{\circ}{W} \to \mathbb{R}^n$. The mapping f extends to a special generic mapping $f(\theta) : M \to \mathbb{R}^n$ (see [6, Proposition 2.1]). The class of the mapping $f(\theta)$ in $\mathcal{M}_{m,n}$ maps under h into θ. Therefore h is surjective. $\square$

By the Pontrjagin-Thom construction for manifolds with stable tangent framings, the group $\mathcal{P}_n(\mathrm{BSDiff}_{m-n}, \mathrm{BSO}_{m-n+1})$ is isomorphic to the stable homotopy group $\pi_n^{st}(\mathrm{BSDiff}_{m-n}, \mathrm{BSO}_{m-n+1})$ (e.g., see [11]). This completes the proof of Theorem 1.1.

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