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EQUIVALENCE OF THE NASH CONJECTURE FOR PRIMITIVE AND SANDWICHED SINGULARITIES

JESÚS FERNÁNDEZ-SÁNCHEZ

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ABSTRACT. We show that in order to prove the Nash Conjecture for sandwiched singularities it is enough to prove it for primitive singularities.

Let (X, Q) be a normal surface singularity and $f : \widehat{S} \longrightarrow (X, Q)$ its minimal resolution. An *arc* (an *arc of order* i) on (X, Q) is a local $\mathbb{C}$-morphism $\mathcal{O}_{X,Q} \longrightarrow \mathbb{C}[[t]]$ (a local $\mathbb{C}$-morphism $\mathcal{O}_{X,Q} \longrightarrow \mathbb{C}[[t]]/(t^{i+1})$). If $\{E_u\}_{u \in T_Q}$ are the irreducible exceptional components of $\widehat{S}$, let $\mathcal{F}_u^Q$ denote the set of arcs on (X, Q) such that the lifted arc $\widetilde{\varphi}$ on $\widehat{S}$ intersects E_u. Let $\mathcal{H}$ ($\mathcal{H}_i$ for $i \geq 0$) be the space of arcs (arcs of order i) on (X, Q). Define $Tr(i)$ to be the space of i-truncations of arcs on (X, Q), that is $Tr(i) = \rho_i(\mathcal{H})$, where $\rho_i : \mathcal{H} \longrightarrow \mathcal{H}_i$ is the morphism induced by the projection $\mathbb{C}[[t]] \longrightarrow \mathbb{C}[[t]]/(t^{i+1})$. Then, if $\mathcal{F}_u^Q(i) = \rho_i(\mathcal{F}_u^Q)$, the minimal resolution $f : \widehat{S} \longrightarrow (S, 0)$ induces a decomposition of the space of i-truncations $Tr(i) = \bigcup_{u \in T_Q} \mathcal{F}_u^Q(i)$ and, by taking the Zariski closure in $\mathcal{H}_i$,

$$\overline{Tr(i)} = \bigcup_{u \in T_Q} \overline{\mathcal{F}_u^Q(i)}.$$

The Nash Conjecture says that for $i \gg 0$, this is just the decomposition of $\overline{Tr(i)}$ into its irreducible components (see [5]).

A normal surface singularity (X, Q) is said to be *sandwiched* if it dominates birationally a non-singular surface. They arise when a complete $\mathbf{m}$-primary ideal in a local regular $\mathbb{C}$-algebra R of dimension two is blown up. A sandwiched singularity is said to be *primitive* if it can be obtained by blowing up a *simple* ideal, that is, a complete irreducible ideal of R ([7] II.3). It is known that any sandwiched singularity is the birational join of some primitive singularities ([7] Corollary II.1.5). In this note, we prove that the Nash Conjecture for sandwiched singularities and for primitive singularities are equivalent.

First, we need a lemma.

Lemma 1. *Let (X_1, Q_1) be a rational surface singularity, $g : X \longrightarrow (X_1, Q_1)$ a birational dominant morphism and E_p an exceptional component of Q such that E_p also appears in the minimal resolution of Q_1 modulo birational equivalence (see*

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Definition 2.1 *of* [4]*). Keeping the notation as above, assume that for some $i > 0$, $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$ and that the projection by g of any element of $\mathcal{F}_q^Q(i)$ is in $\mathcal{F}_u^{Q_1}(i)$, for some $u \in T_{Q_1}$. Then,*

$$\overline{\mathcal{F}_p^{Q_1}(i)} \subset \overline{\mathcal{F}_u^{Q_1}(i)}.$$

Proof. The morphism $g : X \longrightarrow (X_1, Q_1)$ is the blow-up of a complete ideal $J = (g_1, ..., g_m) \subset A = \mathcal{O}_{X_1,Q_1}$, and we may assume that $Q \in U_0$, where U_0 is an affine open set of X of the form $SpecA[g_1/g_0, ..., g_m/g_0] \subset \mathbb{A}_A^m$. Now, if $Spec(A) \subset \mathbb{A}_{\mathbb{C}}^n$, any arc γ on (X_1, Q_1) is written in the form $\gamma = (x_1, ..., x_n)$, $x_k \in \mathbb{C}[[t]]$. Thus, the lifting $\widetilde{\gamma}$ of γ on X is given by

$$(x_1(t), ..., x_n(t), \overline{g_1}(t)/\overline{g_0}(t), ..., \overline{g_m}(t)/\overline{g_0}(t))$$

where $\overline{g_k}(t) = g_k(x_1(t), ..., x_n(t))$, $k = 1, ..., m$.

If $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$, the i-truncation of any arc of $\mathcal{F}_p^Q$ can be approximated by the i-truncations of arcs of $\mathcal{F}_q^Q$. By taking the projections of these i-truncations on (X, Q), we see that $\mathcal{F}_p^{Q_1}(i) \subset \overline{g_*(\mathcal{F}_q^Q(i))} \subset \overline{\mathcal{F}_u^{Q_1}(i)}$ and hence, $\overline{\mathcal{F}_p^{Q_1}(i)} \subset \overline{\mathcal{F}_u^{Q_1}(i)}$. $\square$

Remark 1. A similar result has been proved independently by Camille Plénat [6].

From now on, assume that (X, Q) is a sandwiched singularity and that $X = Bl_I(R)$ is the surface obtained by blowing up a complete $\mathbf{m}$-primary ideal I in a regular local two-dimensional $\mathbb{C}$-algebra R. Assume also that I satisfies the conditions of Corollary 1.14 of [7]. In particular, if

$$I = \prod_{j=1}^{N} I_j$$

is the factorization of I into simple complete ideals, we have that $I_j \neq I_k$ for $j \neq k$, and for $j = 1, \ldots, N$ the surface $X_j = Bl_{I_j}(R)$ has only one singularity, that will be denoted by Q_j.

Following the notation introduced in [3], we denote by $\mathcal{K} = BP(I)$ and $\mathcal{K}_j = BP(I_j)$ the clusters of base points of I and the simple ideals I_j, for $j = 1, \ldots, N$. We also denote by S_K and S_j the surfaces obtained by blowing up all the points of the clusters $\mathcal{K}$ and $\mathcal{K}_j$. The reader is referred to [2] for the connection between sandwiched singularities and clusters, and to chapters 3 and 4 of [1] for conventions and definitions about clusters.

It is known that the morphisms $f : S_K \longrightarrow X$ and $f_j : S_j \longrightarrow X_j$ induced by the universal property of the blowing up are the minimal resolutions of X and $X_j, j = 1, \ldots, N$ respectively. Moreover, X and S_K are the birational join of the surfaces X_j and S_j for $j = 1, \ldots, N$ ([7] Proposition 3.6). If we write $\alpha_j : X \longrightarrow X_j$ for the blow up of $I\mathcal{O}_{X_j}$ in X_j and $\sigma_j : S_K \longrightarrow S_j$ for the induced morphism, we have commutative diagrams of birational morphisms:

$$\begin{array}{ccc} S_K & \xrightarrow{f} & X \\ \downarrow \sigma_j & & \downarrow \alpha_j \\ S_j & \xrightarrow{f_j} & X_j \end{array}$$

By abuse of notation, we will write $\{E_u\}_{u \in T_{Q_j}}$ for the exceptional components of $f_j : S_j \longrightarrow X_j$.

Theorem 2. *Let $p, q \in T_Q$, $p \neq q$, be such that $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$. Then there exists some $j \in \{1, \ldots, N\}$ with $p \in T_{Q_j}$ and some $u \in T_{Q_j}, u \neq p$, such that $\overline{\mathcal{F}_p^{Q_j}(i)} \subset \overline{\mathcal{F}_u^{Q_j}(i)}$.*

Proof. Assume that $p, q \in T_Q$ are points such that $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$ for $i \gg 0$. By Theorem 3.4 of [3], we know that q is not infinitely near to p. Let $\mathcal{K}_j$ be any irreducible subcluster of $\mathcal{K}$ containing p.

If p is infinitely near to q, then q is also in $\mathcal{K}_j$ and hence, the exceptional components E_p and E_q also appear in the minimal resolution S_j of Q_j modulo birational equivalence. By Lemma 1 applied to $\alpha_j : X \longrightarrow X_j$, we deduce that $\overline{\mathcal{F}_p^{Q_j}(i)} \subset \overline{\mathcal{F}_q^{Q_j}(i)}$.

If p is not infinitely near to q, let u_0 be the maximal point of $\mathcal{K}_j$ to which q is infinitely near. Then the projection by α_j of any element in $\mathcal{F}_q^Q$ gives an arc γ on (X_j, Q_j) whose lifting to S_j intersects (transversally or not) the exceptional component E_{u_0} and therefore, $\gamma \in \mathcal{F}_{u_0}^{Q_j}$. Thus, the projection of any element of $\mathcal{F}_q^Q(i)$ is in $\mathcal{F}_{u_0}^{Q_1}(i)$ and by Lemma 1 again, we deduce that $\overline{\mathcal{F}_p^{Q_j}(i)} \subset \overline{\mathcal{F}_{u_0}^{Q_j}(i)}$.

In any case, we see that an inclusion of spaces of arcs on the sandwiched singularity (X, Q) implies a non-trivial inclusion of some spaces of arcs on the primitive singularity (X_j, Q_j). The claim follows. □

As a direct consequence, we have the following corollary.

Corollary 3. *If the Nash Conjecture is true for primitive singularities, then it is also true for sandwiched singularities.*

Remark 3. Notice that if (X, Q) is a primitive singularity and we have $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$ for $p, q \in T_Q$ with $p \neq q$, then by 3.4 of [3], p must be infinitely near to q and moreover, $e_O(I_p) > e_O(I_q)$ where $e_O(J)$ is the minimum of the multiplicities of the curves defined by elements of the complete ideal J.

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Departament d'Àlgebra i Geometria, Universitat de Barcelona, Gran Via, 585, E–08007, Barcelona, Spain

E-mail address: `jsan@cerber.mat.ub.es`