ON THE EVALUATION OF SALIÉ SUMS

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Abstract. The Salié sum $S(m, n; c)$ can be evaluated as the product of a Gauss sum and an exponential sum involving square roots of $mn \mod c$. We give a new proof of this fact that can simultaneously handle a twisted version of these sums that arise in the theory of half-integral weight modular forms.

The exponential sum
$$K(m, n; c) = \sum_{a \equiv 1 \pmod{c}} \varepsilon_a \left( \frac{c}{a} \right) e((ma + n\overline{a})/c)$$
arises in the theory of modular forms of half-integral weight. $K(m, n; c)$ is only defined when $4|c$, and then $\varepsilon_a = 1$ or $i$ according to whether $a \equiv 1$ or $3 \pmod{4}$, $(-)$ is the extension of the Legendre-Jacobi symbol as in [5], and $e(z) = \exp(2\pi i z)$. In Iwaniec’s celebrated estimates for the Fourier coefficients of said forms (see [2]), an essential role is played by the identity

$$K(D,1; c) = \frac{G(1,0; c)}{2} \sum_{x^2 \equiv D \pmod{c}} e(2x/c)$$

valid when $8|c$ and the analogous formula

$$S(D,1; c) = \sum_{a \equiv 1 \pmod{c}} \left( \frac{a}{c} \right) e((Da + \overline{a})/c) = G(1,0; c) \sum_{x^2 \equiv D \pmod{c}} e(2x/c),$$

which is valid whenever $c$ is odd. In the formula above $G(a,b; c)$ is the Gauss sum $G(a,b;c) = \sum_{x(c)} e((ax^2 + bx)/c)$.

When the modulus is prime this identity was first proved by Salié [3]; see also [6]. Iwaniec derived a formula for the general modulus by pasting the local results together using quadratic reciprocity, while Sarnak gave a “global” proof of (2) that works when $(2D,c) = 1$; see [4].

The purpose of this note is to give a simple argument that leads to (1) when $c$ is even and to (2) when $c$ is odd, without any assumption on $D$. It is based on a very simple idea: we start by the sum

$$A = \sum_{x^2 \equiv D \pmod{c}} e(2x/c)$$
and sieve out the support of the sum by

\[ A = \frac{1}{c} \sum_{x(c)} e(2x/c) \sum_{a(c)} e(a(x^2 - D)/c). \]

Interchanging the two sums, we are led to

\[ A = \frac{1}{c} \sum_{a(c)} G(a, 2; c)e(-aD/c) = \sum_{d|c} A_d \]

where

\[ A_d = \frac{1}{c} \sum_{a(c), (a,c)=d} G(a, 2; c)e(-aD/c). \]

It is well known that \( G(a, b; c) = dG(a/d, b/d; c/d) \) if \( d = (a, c)b \) and is zero otherwise. This shows that \( A_d = 0 \) for all \( d > 1 \), when \( c \) is odd or \( 8|c \), because we even have \( G(a', 1; c') = 0 \) if \( 4|c' \).

Now, when \( (a, c) = 1 \),

\[ G(a, 2; c) = e(-\pi/c)G(a, 0; c), \]

and so

\[ A = A_1 = \frac{G(1, 0; c)}{c} \sum_{a(c), (a,c)=1} G(a, 0; c) e(-\pi/c)e(-aD/c). \]

This proves (1) and (2), since

\[ G(a, 0; c)/G(1, 0; c) = \begin{cases} \left( \frac{a}{c} \right) & \text{if } c \text{ is odd}, \\ \left( \frac{c}{a} \right) e_a^{-1} & \text{if } 4|c. \end{cases} \]

Since the explicit form of the \( \theta \)-multiplier arises from \( G(a, 0; c)/G(1, 0; c) \), see e.g. [3], identity (3) can be eliminated from the argument. In this respect, note that in [1], using the Davenport-Hasse relation, Duke evaluated generalized Salié sums that bear the same relation to metaplectic forms on \( GL_n \) as \( K(m, n; c) \) to half-integral weight forms. Although the method presented here does not seem to be applicable in a direct fashion, this fact still suggests that Duke’s theorem can be generalized to non-prime arguments, without any recourse to the explicit evaluation of generalized Gauss sums.

References


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