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A q-SAMPLING THEOREM RELATED TO THE q-HANKEL TRANSFORM

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Abstract. A q-version of the sampling theorem is derived using the q-Hankel transform introduced by Koornwinder and Swarttouw. The sampling points are the zeros of the third Jackson q-Bessel function.

1. Introduction

The classical sampling theorem asserts that every function f in the Paley-Wiener space defined by

$$PW = \left\{ f \in L^2(\mathbf{R}) : f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ixt} u(t)\, dt, u \in L^2(-\pi, \pi) \right\}$$

can be represented by the interpolation series

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (x-n)}{\pi (x-n)}.$$

Hardy's proof of this fact [4] used properties from the kernel of the Fourier transform. Relying on properties of the Hankel transform kernel, Higgins [5] used the theory of reproducing kernels to obtain a sampling theorem where the sampling points are the zeros of the Bessel function. In this note, a q-Bessel analogue of the sampling theorem is derived by considering the kernel of the q-Hankel transform, H_q^ν, introduced by Koornwinder and Swarttouw [8],

$$\left(H_q^\nu f\right)(x) = \int_0^\infty (xt)^{\frac{1}{2}} J_\nu^{(3)}\left(xt; q^2\right) f(t)\, d_q t$$

where $J_\nu^{(3)}$ denotes the third Jackson q-Bessel function defined by the power series

$$J_\nu^{(3)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} x^\nu \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q^{\nu+1};q)_n (q;q)_n} x^{2n} \tag{1.1}$$

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with $0<q<1$, $(a;q)_n=(1-a)(1-aq)\ldots\left(1-aq^{n-1}\right)$ and $(a;q)_\infty=\lim_{n\to\infty}(a;q)_n$. We are using the definition of the q-integral. The q-integral in the interval $(0,1)$ is defined as

$$\int_0^1 f(t)\,d_qt=(1-q)\sum_{n=0}^{\infty}f(q^n)\,q^n \tag{1.2}$$

and in the interval $(0,\infty)$ as

$$\int_0^\infty f(t)\,d_qt=(1-q)\sum_{n=-\infty}^{\infty}f(q^n)\,q^n. \tag{1.3}$$

The sampling points will turn out to be $qj_{n\nu}\left(q^2\right)$, where $j_{n\nu}\left(q^2\right)$ is the n^{th} zero of $J_\nu^{(3)}\left(x;q^2\right)$. In [2] it was proved that $j_{n\nu}\left(q^2\right)=q^{-n+\epsilon_n}, 0<\epsilon_n<1$. This shows how big is the spacing between the sampling points.

2. Preliminaries on reproducing kernels

Let H be a class of complex-valued functions, defined in a set $X\subset\mathbf{C}$, such that X is a Hilbert space with the norm of $L^2(X,\mu)$. $g(s,x)$ is a *reproducing kernel* to H if

i) $g(t,x)\in H$ for every $x\in X$;

ii) $f(x)=\langle f(t),g(t,x)\rangle$ for every $f\in H$, $x\in X$.

The next result lists the properties of Hilbert spaces with reproducing kernel that will be used in the remainder. Properties (a), (c) and (d) are proved in [5]. Property (b) is a well-known property of the reproducing kernels, of primary importance, because it relates two different kinds of convergence. A proof of (b) can be found in [10], together with an introduction to the general theory.

Proposition 1. *In the Hilbert space $L^2[(a,b),\mu]$, an operator is defined by*

$$Ku=\langle K(x,t),u(t)\rangle_{L^2[(a,b),\mu]}.$$

The following properties hold:

(a) If K^{-1} is bounded, the range of K, denoted by N, is a Hilbert space with reproducing kernel.

(b) If the sequence $\{f_n\}$ converges strongly to f in the norm of H, with reproducing kernel g, then $\{f_n\}$ converges pointwise in X to f. The convergence is uniform in every set of X where $g(x,x)$ is bounded.

(c) If K is an isometry, then $g(s,x)=\langle K(s,t),K(x,t)\rangle_{L^2[(a,b),\mu]}$.

(d) Let $\{f_n\}$ be a complete orthogonal sequence in H and (x_n) such that $f_n(x_m)=\delta_{nm}$. Then

$$f_n(t)=\frac{g(t,x_n)}{g(x_n,x_n)}.$$

We will suppose $N\subset L^2(X,\mu)$, and this implies K bounded. K^{-1} is a transformation of N over $L^2[(a,b),\mu]$, also bounded.

3. A q-sampling theorem

We introduce a q-Bessel version of the Paley-Wiener space, and call it PW_q^ν:

$$PW_q^\nu=\left\{f\in L_q^2(0,\infty):f(x)=\int_0^1(tx)^{\frac12}J_\nu^{(3)}\left(xt;q^2\right)u(t)\,d_qt,u\in L_q^2(0,1)\right\}. \tag{3.1}$$

The notation $L_q^2(0,1)$ stands for the Hilbert space associated to the measure of the q-integral in $(0,1)$. In [8] the following inversion formula was proved:

$$f(t)=\int_0^\infty (xt)^{\frac{1}{2}}\left(H_q^\nu f\right)(x)\,J_\nu^{(3)}\left(xt;q^2\right)d_qx=\left(H_q^\nu\left(H_q^\nu f\right)\right)(t)\,. \tag{3.2}$$

Let $f\in L_q^2(0,\infty)$ such that $\left(H_q^\nu f\right)(q^{-n})=0$, $n=1,2,....$ Then $f\in PW_q^\nu$. To see this use the formula (3.2) and compare (1.2) and (1.3) to write f as an element of PW_q^ν.

Now, in the language of the preceding section, consider $X=(0,\infty)$, $(a,b)=(0,1)$ and the kernel $K(x,t)=(xt)^{\frac{1}{2}}J_\nu^{(3)}\left(xt;q^2\right)$. The corresponding operator K is

$$(Ku)(x)=\langle K(x,t),u(t)\rangle_{L_q^2(0,1)}=\int_0^1 (xt)^{\frac{1}{2}}J_\nu^{(3)}\left(xt;q^2\right)u(t)\,d_qt.$$

By (3.2), H_q^ν is a self-inverse operator and consequently, an isometry. Thus, K is also an isometry. The range of K, N, is the set of functions $f\in L_q^2(0,\infty)$ such that $f=Ku$ for some $u\in L_q^2(0,1)$. By (3.1), $N=PW_q^\nu$. In the next lemma, the reproducing kernel of the space PW_q^ν is evaluated.

Lemma 1. *The set PW_q^ν is a Hilbert space with reproducing kernel given by*

$$g(s,x)=(1-q)\,q^v\,\frac{(xs)^{\frac{1}{2}}\left[xJ_{\nu+1}^{(3)}\left(x;q^2\right)J_\nu^{(3)}\left(sq^{-1};q^2\right)-sJ_{\nu+1}^{(3)}\left(s;q^2\right)J_\nu^{(3)}\left(xq^{-1};q^2\right)\right]}{x^2-s^2}. \tag{3.3}$$

Proof. By Proposition 1 (a), PW_q^ν is a space with reproducing kernel $g(s,x)$. From Proposition 1 (c), since K is an isometry,

$$g(s,x)=\langle K(s,t),K(x,t)\rangle_{L_q^2(0,1)}=\int_0^1 t\,(xs)^{\frac{1}{2}}J_\nu^{(3)}\left(xt;q^2\right)J_\nu^{(3)}\left(st;q^2\right)d_qt.$$

In [7], the following formula was proved:

$$\left(a^2-b^2\right)\int_0^z tJ_\nu^{(3)}\left(aqt;q^2\right)J_\nu^{(3)}\left(bqt;q^2\right)d_qt \tag{3.4}$$

$$=(1-q)\,q^{\nu-1}z\left[aJ_{\nu+1}^{(3)}\left(aqz;q^2\right)J_\nu^{(3)}\left(bz;q^2\right)-bJ_{\nu+1}^{(3)}\left(bqz;q^2\right)J_\nu^{(3)}\left(az;q^2\right)\right].$$

Setting $z=1$, $a=xq^{-1}$ and $b=sq^{-1}$ in (3.4), (3.3) follows. □

The q-sampling theorem can now be stated and proved.

Theorem 1. *If $f\in PW_q^\nu$, then f has the unique representation*

$$f(x)=\sum_{n=1}^\infty f\left(qj_{n\nu}\left(q^2\right)\right)\frac{2\left(xqj_{n\nu}\left(q^2\right)\right)^{\frac{1}{2}}J_\nu^{(3)}\left(x;q^2\right)}{\frac{d}{dx}\left[J_\nu^{(3)}\left(x;q^2\right)\right]_{x=qj_{n\nu}(q^2)}\left(x^2-q^2j_{n\nu}^2\left(q^2\right)\right)} \tag{3.5}$$

where $\left(j_{n\nu}\left(q^2\right)\right)$ denotes the sequence of positive zeros of $J_\nu^{(3)}\left(x;q^2\right)$. The series converges uniformly in compact subsets of $(0,\infty)$.

Proof. Consider the sequence $\{f_n(x)\}$ defined by

$$f_n(x)=\left(xqj_{n\nu}\left(q^2\right)\right)^{\frac{1}{2}}J_\nu^{(3)}\left(qxj_{n\nu}\left(q^2\right);q^2\right).$$

It was proved in [1] that $\{f_n(x)\}$ is a complete orthogonal sequence in $L_q^2(0,1)$. Taking into account that K is an isometry, the sequence $(Kf_n)(x)$ is also orthogonal and complete in PW_q^ν. Now set

$$F_n(x) = \frac{(Kf_n)(x)}{(Kf_n)(qj_{n\nu}(q^2))}.$$

The orthogonality of $\{f_n(x)\}$ implies

$$F_n\left(qj_{m\nu}\left(q^2\right)\right) = \delta_{nm}. \tag{3.6}$$

Proposition 1 (d) allows us to write

$$F_n(x) = \frac{g\left(x, qj_{n\nu}\left(q^2\right)\right)}{g\left(qj_{n\nu}(q^2), qj_{n\nu}(q^2)\right)}.$$

Substituting in (3.3) yields

$$F_n(x) = \frac{2\left(xqj_{n\nu}(q^2)\right)^{\frac{1}{2}} J_\nu^{(3)}\left(x;q^2\right)}{\frac{d}{dx}\left[J_\nu^{(3)}(x;q^2)\right]_{x=qj_{n\nu}(q^2)}\left(x^2 - q^2 j_{n\nu}^2(q^2)\right)}.$$

$F_n(x)$ is an orthonormal complete sequence in N. Thus, every $f \in PW_q^\nu$ has a unique series expansion in the form

$$f(x) = \sum_{n=1}^{\infty} a_n F_n(x) \tag{3.7}$$

where a_n are the Fourier coefficients of f in $\{F_n(x)\}$. The series in (3.7) is convergent in the norm of $L_q^2(0,1)$ and also in the norm of PW_q^ν. The real-valued function $g(x,x)$ is continuous, thus bounded in every compact subset of $(0,\infty)$. It follows from Proposition 1 (b) that (3.7) converges uniformly in compact subsets of $(0,\infty)$. Finally, setting $x = qj_{n\nu}\left(q^2\right)$ in (3.7), (3.6) implies $f\left(qj_{m\nu}\left(q^2\right)\right) = a_m$ and thus, (3.7) can be written in the form (3.5). □

4. Application

The following formula is a consequence of the product representation for the classical Bessel function:

$$\frac{\frac{d}{dx}J_\nu(x)}{J_\nu(x)} = 2x\sum_{n=1}^{\infty}\frac{1}{j_{n\nu}^2 - x^2} + \frac{\nu}{x}. \tag{4.1}$$

Using the recurrence $x\frac{d}{dx}J_\nu(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x)$, (4.1) becomes

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} = -2x\sum_{n=1}^{\infty}\frac{1}{j_{n\nu}^2 - x^2} \tag{4.2}$$

where $j_{n\nu}$ stands for the zeros of $J_\nu(x)$. In the case of the q-analogues of the Bessel function, this analysis cannot be done, for there are no formulas to establish a simple relation between a q-Bessel function and its derivative. While the q-analogue of (4.1) is very simple to derive from the Hadamard factorization theorem or using residues, the q-analogue of (4.2) is harder to obtain. In [6], Ismail studied the second Jackson q-Bessel function, $J_\nu^{(2)}(x;q)$, and found such a q-analogue using the orthogonality measure of the modified q-Lommel polynomials associated to $J_\nu^{(2)}(x;q)$. Kvitsinsky [9] found a recurrence relation for the coefficients h_n in the identity

$$\frac{J_{\nu+1}^{(3)}(x;q)}{J_\nu^{(3)}(x;q)} = \sum_{n=1}^{\infty} h_n x^{2n-1}. \tag{4.3}$$

In this section an explicit formula for the coefficients h_n will be obtained as a special case of the expansion of a particular function as a sampling series. Preliminary to this expansion, a q-integral formula connecting two q-Bessel functions of different orders is established.

Lemma 2. *For $y > 0$, $\nu > -\frac{1}{2}$ and $x \in \mathbf{R}$, the following relation holds:*

$$\frac{(q;q)_\infty}{(q^y;q)_\infty} x^{-y} J_{\nu+y}^{(3)}(x;q) = \int_0^1 t^{\frac{\nu}{2}} \frac{(tq;q)_\infty}{(tq^y;q)_\infty} J_\nu^{(3)}(xt^{\frac{1}{2}};q) d_q t. \tag{4.4}$$

Proof. The q-analogues of the gamma and beta functions will be critical in the proof. According to [3, 1.10], the q-gamma function, $\Gamma_q(x)$, is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \tag{4.5}$$

and the q-beta function, $\beta_q(x,y)$, by

$$\beta_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{4.6}$$

The q-beta function has the q-integral representation

$$\beta_q(x,y) = \int_0^1 t^{x-1} \frac{(tq;q)_\infty}{(tq^y;q)_\infty} d_q t,\ Re(x) > 0,\ y \neq 0, -1, -2, ... \tag{4.7}$$

Using the series representation (1.1) and the q-integral representation (4.7), it is easy to see that, if $\nu > -\frac{1}{2}$ and $y > 0$,

$$\begin{aligned} &\int_0^1 t^{\frac{\nu}{2}} \frac{(tq;q)_\infty}{(tq^y;q)_\infty} J_\nu^{(3)}(xt^{\frac{1}{2}};q) d_q t \\ &\quad = x^\nu \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\frac{k(k+1)}{2}}}{(q;q)_k (q^{\nu+1};q)_k} x^{2k} \beta_q(k+\nu+1, y). \end{aligned} \tag{4.8}$$

Now use (4.5) and (4.6) to express $\beta_q(k+\nu+1,y)$ as a quotient of infinite products. Then, some algebraic manipulations using the formula $(a;q)_\infty = (a;q)_n (aq^n;q)_\infty$ allow us to see that the right-hand member of (4.8) is equal to the left-hand member of identity (4.4). $\square$

Before moving to the next theorem, it is convenient to point out that, from the definition (1.2), one can verify the relation:

$$\int_0^1 f(t^{\frac{1}{2}}) d_{q^2} t = (1+q) \int_0^1 t f(t)\, d_q t. \tag{4.9}$$

Theorem 2. *If $u > \nu > -\frac{1}{2}$, the following identity holds:*

$$x^{\nu-u} \frac{J_u^{(3)}(x;q^2)}{J_\nu^{(3)}(x;q^2)} = -2 \sum_{n=1}^{\infty} \frac{\left(q j_{n\nu}(q^2)\right)^{\nu-u+1} J_u^{(3)}\left(q j_{n\nu}(q^2); q^2\right)}{\frac{d}{dx}\left[J_\nu^{(3)}(x;q^2)\right]_{x=qj_{n\nu}(q^2)} \left(q^2 j_{n\nu}^2(q^2) - x^2\right)}. \tag{4.10}$$

Proof. Setting $y = u - \nu$ in (4.4) and replacing q by q^2, the result is, if $u > \nu$,

$$\frac{(q^2;q^2)_\infty}{(q^{2u-2\nu};q^2)_\infty} x^{\nu-u} J_u^{(3)}(x;q^2) = \int_0^1 t^{\frac{\nu}{2}} \frac{(tq^2;q^2)_\infty}{(tq^{2u-2\nu};q^2)_\infty} J_\nu^{(3)}(xt^{\frac{1}{2}};q^2) d_{q^2}t.$$

Taking (4.9) into account, this can be rewritten as

(4.11)

$$\frac{(q^2;q^2)_\infty}{(q^{2u-2\nu};q^2)_\infty} x^{\nu-u} J_u^{(3)}(x;q^2) = (1+q) \int_0^1 t^{\nu+1} \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2u-2\nu};q^2)_\infty} J_\nu^{(3)}(xt;q^2) d_q t.$$

Considering

$$u(t) = t^{\nu+\frac{1}{2}} \frac{(1+q)\left(q^{2u-2\nu};q^2\right)_\infty \left(t^2q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty \left(t^2q^{2u-2\nu};q^2\right)_\infty},$$

relation (4.11) yields

$$x^{\nu-u+\frac{1}{2}} J_u^{(3)}\left(x;q^2\right) = \int_0^1 (tx)^{\frac{1}{2}} J_\nu^{(3)}(xt;q^2) u(t)\, d_q t.$$

Thus,

$$f(x) = x^{\nu-u+\frac{1}{2}} J_u^{(3)}\left(x;q^2\right) \in PW_q^\nu.$$

Now it is possible to apply Theorem 1 to f. The result of this application is (4.10). □

Taking $u = \nu + 1$ in (4.10) and replacing q^2 by q, the result is the analogue of (4.2) previously mentioned:

$$\frac{J_{\nu+1}^{(3)}(x;q)}{J_\nu^{(3)}(x;q)} = -2x \sum_{n=1}^{\infty} \frac{J_{\nu+1}^{(3)}\left(q^{\frac{1}{2}} j_{n\nu}(q);q\right)}{\frac{d}{dx}\left[J_\nu^{(3)}(x;q)\right]_{x=q^{\frac{1}{2}} j_{n\nu}(q)}} \frac{1}{q j_{n\nu}^2(q) - x^2}. \tag{4.12}$$

Expanding $1/\left(j_{n\nu}^2(q) - x^2\right)$ in power series of x and substituting in (4.12), the coefficients h_n in (4.3) can be seen to be

$$h_n = \sum_{k=1}^{\infty} \frac{J_{\nu+1}^{(3)}\left(q^{\frac{1}{2}} j_{k\nu}(q);q\right)}{\frac{d}{dx}\left[J_\nu^{(3)}(x;q)\right]_{x=q^{\frac{1}{2}} j_{n\nu}(q)}} \left(\frac{1}{q j_{k\nu}^2(q)}\right)^{2n}.$$

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