PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 4, Pages 1127–1134
S 0002-9939(04)07593-8
Article electronically published on September 16, 2004

PROBABILISTIC ASPECTS OF AL-SALAM–CHIHARA POLYNOMIALS

WŁODZIMIERZ BRYC, WOJCIECH MATYSIAK, AND PAWEŁ J. SZABŁOWSKI

(Communicated by Carmen C. Chicone)

ABSTRACT. We solve the connection coefficient problem between the Al-Salam–Chihara polynomials and the q-Hermite polynomials, and we use the resulting identity to answer a question from probability theory. We also derive the distribution of some Al-Salam–Chihara polynomials, and compute determinants of related Hankel matrices.

1. INTRODUCTION AND MAIN IDENTITY

The aim of the paper is to point out the connection of Al-Salam–Chihara polynomials with a regression problem in probability, and to use it to give a new simple derivation of their density. Our approach exploits identity (1.8) below, which connects the Al-Salam–Chihara polynomials to the continuous q-Hermite polynomials. This connection is more direct and elementary but less general than the technique of attachment exploited in [BI96, Section 2]. We also compute determinants of Hankel matrices with entries that are linear combinations of the q-Hermite polynomials.

The Al-Salam–Chihara polynomials were introduced in [ASC76], and their weight function was found in [AI84]. We are interested in the renormalized Al-Salam–Chihara polynomials $\{p_n(x|q,a,b)\}$, which are defined by the following three-term recurrence relation:

$$p_{n+1}(x) = (x - aq^n)\, p_n(x) - \left(1 - bq^{n-1}\right)[n]_q p_{n-1}(x) \quad (n \geq 0), \tag{1.1}$$

with the usual initial conditions $p_{-1} = 0$, $p_0 = 1$. Here, we use the standard notation

$$\begin{aligned} [n]_q &= 1 + q + \cdots + q^{n-1}, \\ [n]_q! &= [1]_q[2]_q \ldots [n]_q, \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[n-k]_q![k]_q!}, \end{aligned}$$

with the usual conventions $[0]_q = 0, [0]_q! = 1$.

Received by the editors April 22, 2003 and, in revised form, November 30, 2003.
2000 *Mathematics Subject Classification.* Primary 33D45; Secondary 05A30, 15A15, 42C05.
Key words and phrases. q-Hermite polynomials, matrix of moments, orthogonal polynomials, determinants, polynomial regression.
This research was partially supported by NSF grant #INT-0332062.

For $|q| < 1$, their generating function

$$f(t,x|q,a,b) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} p_n(x|q,a,b)$$

is given by

$$f(t,x|q,a,b) = \prod_{k=0}^{\infty} \frac{1-(1-q)atq^k + (1-q)bt^2q^{2k}}{1-(1-q)xtq^k + (1-q)t^2q^{2k}}; \tag{1.2}$$

compare [AI84, (3.6) and (3.10)].

The corresponding (renormalized) continuous q-Hermite polynomials $H_n(x|q) = p_n(x|q,0,0)$ satisfy the three-term recurrence relation

$$H_{n+1}(x) = xH_n(x) - [n]_q H_{n-1}(x). \tag{1.3}$$

For $|q| < 1$ their generating function $\phi(t,x|q) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} H_n(x|q)$ is

$$\phi(t,x|q) = \prod_{k=0}^{\infty} \left(1-(1-q)xtq^k + (1-q)t^2q^{2k}\right)^{-1}. \tag{1.4}$$

Of course, these are well-known special cases of (1.1) and (1.2); see [ISV87, (2.11) and (2.12)], which we state here for further reference.

We will also use polynomials $\{B_n(x|q)\}$ defined by the three-term recurrence relation

$$B_{n+1}(x) = -q^n x B_n(x) + q^{n-1}[n]_q B_{n-1}(x) \quad (n \geq 0) \tag{1.5}$$

with the usual initial conditions $B_{-1} = 0, B_0 = 1$. These polynomials are related to the q-Hermite polynomials by

$$B_n(x|q) = \begin{cases} i^n q^{n(n-2)/2} H_n(i\sqrt{q}\,x|\frac{1}{q}) & \text{if } q > 0, \\ (-1)^{n(n-1)/2}|q|^{n(n-2)/2} H_n(-\sqrt{|q|}\,x|\frac{1}{q}) & \text{if } q < 0 \end{cases} \tag{1.6}$$

and have been studied in [Ask89], [IM94]. Their generating function $\psi(t,x|q) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} B_n(x|q)$ is given by

$$\psi(t,x|q) = \prod_{k=0}^{\infty} \left(1-(1-q)xtq^k + (1-q)t^2q^{2k}\right). \tag{1.7}$$

We now point out the mutual relationship between the Al-Salam–Chihara polynomials $\{p_n(x|q,a,b)\}$ and the polynomials $\{H_n(x|q)\}_{n\geq 0}$ and $\{B_n(x|q)\}_{n\geq 0}$.

Theorem 1. *For all $a, c, q \in \mathbb{C}$, $c \neq 0$, and $n \geq 1$ we have*

$$p_n(x|q,a,b) = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q c^{n-k} B_{n-k}(\frac{a}{c}|q) \left(H_k(x|q) - c^k H_k(\frac{a}{c}|q)\right), \tag{1.8}$$

where $b = c^2$.

Proof. From the recurrence relations (1.1), (1.3), and (1.5), it is clear that $p_n(x|q,a,b)$, $H_n(x|q)$, and $B_n(x|q)$ are given by polynomial expressions in the variable q. The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is also a polynomial in q. Therefore, we see that identity (1.8) is equivalent to a polynomial identity in variable $q \in \mathbb{C}$.

Hence it is enough to prove that (1.8) holds true for all $|q| < 1$. When $|q| < 1$, inspecting (1.2), (1.4), and (1.7) we notice that for $b = c^2$ we have

$$f(t, x|q, a, b) = \psi(tc, a/c|q)\phi(t, x|q) \tag{1.9}$$

and

$$\psi(t, x|q)\phi(t, x|q) = 1. \tag{1.10}$$

Therefore,

$$f(t, x|q, a, b) = 1 + \psi(ct, a/c|q)\left(\phi(t, x|q) - \phi(ct, a/c|q)\right),$$

which is valid for all small enough $|t|$. Comparing the coefficients at t^n for $n \geq 1$ and taking into account that $H_k(x|q) - c^k H_k(\frac{a}{c}|q) = 0$ for $k = 0$, we get (1.8). □

Remark 1. One could split (1.8) into the following two separate identities, which are implied by (1.9) and (1.10) respectively:

$$\forall n \geq 0: \ p_n(x|q, a, c^2) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q c^{n-k} B_{n-k}(\frac{a}{c}|q) H_k(x|q), \tag{1.11}$$

$$\forall n \geq 1: \ \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q B_{n-k}(x|q) H_k\,(x|q) = 0. \tag{1.12}$$

Formula (1.11) is a renormalized inverse of formula [IRS99, (4.7)], which expresses the q-Hermite polynomials as linear combinations of Al-Salam–Chihara polynomials. Formula (1.12) resembles [Car56, (2.28)], which considers q-Hermite polynomials of the form $h_n(x|q) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k$, paired with $b_n(x|q) = h_n(x|1/q)$.

2. Probabilistic aspects

Quadratic regression questions in the paper [Bry01] lead to the problem of determining all probability distributions μ which are defined indirectly by the relationships

$$\int H_n(x|q)\mu(dx) = \rho^n H_n(y|q), \ n = 1, 2, \ldots, \tag{2.1}$$

where $y, \rho, q \in \mathbb{R}$ are fixed parameters, and $\{H_n\}_{n\geq 0}$ is the family of the q-Hermite polynomials.

Our next result shows that this problem can be solved using the Al-Salam–Chihara polynomials.

Theorem 2. *If $\mu = \mu(dx|\rho, y)$ satisfies* (2.1)*, then its orthogonal polynomials are Al-Salam–Chihara polynomials* $\{p_n(x|q, a, b)\}$ *with $a = \rho y$, $b = \rho^2$.*

Proof. Recall that $H_n(x|q) = p_n(x|q, 0, 0)$. Thus if $\rho = 0$, then (2.1) implies that $\int p_n(x|q, a, b)\mu(dx) = 0$ for all $n = 1, 2, \ldots$. Suppose now that $\rho \neq 0$. Combining (1.8) with (2.1) we get

$$\int p_n(x|q, a, b)\mu(dx) = \sum_{k=1}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q \rho^{n-k} B_{n-k}(y|q) \int \left(H_k(x|q) - \rho^k H_k(y|q)\right)\mu(dx) = 0$$

for all $n = 1, 2, \ldots$. Since $\{p_n\}$ satisfy a three-step recurrence, this implies $\int p_k(x)p_n(x)\mu(dx) = 0$ for all $0 \leq k < n$. □

Next we answer an unresolved case from [Bry01].

Corollary 1. *Fix* $q > 1, y \in \mathbb{R}$. *Let* $\mathcal{R}_q = \{1, 1/q, 1/q^2, \dots, 1/q^n, \dots, 0\}$.

(i) *If* $\rho^2 \notin \mathcal{R}_q$, *then* (2.1) *has no probabilistic solution* μ.
(ii) *If* $\rho^2 \in \mathcal{R}_q$ *is non-zero, then the probabilistic solution of* (2.1) *exists, and is a discrete measure supported on* $1 + \log_q 1/\rho^2$ *points.*

Proof. Suppose that μ is positive and solves (2.1). Therefore its monic orthogonal polynomials satisfy the three-term recurrence relation

$$p_{n+1}(x) = (x - \rho y q^n) p_n(x) - (1 - \rho^2 q^{n-1}) p_{n-1}(x). \tag{2.2}$$

For a positive non-degenerate measure $\mu_y(dx)$, and $n \geq 1$ we have

$$\int p_n^2(x)\mu_y(dx) = (1 - \rho^2 q^{n-1}) \int p_{n-1}^2(x)\mu_y(dx). \tag{2.3}$$

If $\rho^2 \notin \mathcal{R}_q$, then $(1 - \rho^2 q^{n-1}) \neq 0$ for all n. Since $\int p_0^2(x)\mu_y(dx) > 0$, this shows that $\int p_n^2(x)\mu_y(dx) > 0$ for all $n \geq 0$. But then the coefficients $1 - \rho^2 q^{n-1}$ must be non-negative for all n, which is false. This proves (i).

To conclude the proof it remains to notice that if $\rho^2 = 1/q^m$, then from (2.2) and (the proof of) Favard's theorem, see [Fre71, Theorem II.1.5], it follows that the solution of (2.1) is given by a measure supported on the roots of the polynomial p_{m+1}. Indeed, (2.2) implies that the polynomial p_{m+2} is divisible by p_{m+1}. Therefore, p_{m+1} is the common factor of all polynomials $\{p_k : k \geq m+1\}$. It is also known, see [Fre71, Theorem I.2.2], that p_{m+1} has exactly $m+1$ distinct real roots $x_1, \dots, x_{m+1}$. Thus, any measure $\mu(dx) = \sum \lambda_j \delta_{x_j}$ supported on the roots of the polynomial p_{m+1} satisfies $\int p_{m+1+k}\mu(dx) = 0$. Solving the remaining $m+1$ equations $\int p_0 \mu(dx) = 1$, and $\int p_k(x)\mu(dx) = 0, k = 1, 2, \dots, m$ for λ_j, we get a measure that solves (2.1). This measure is non-negative since the coefficients at the third term in the recurrence (2.2) are non-negative for $n = 1, \dots, m$; see [Fre71, page 58]. ☐

From Theorem 2 it follows that if the solution of (2.1) exists, then it is given by the distribution of the Al-Salam–Chihara polynomials. The distribution of the Al-Salam–Chihara polynomials is derived in [AI84, Chapter 3]. However, in [Bry01, Proposition 8.1] we found the solution of (2.1) that relies solely on the facts about the q-Hermite polynomials. We repeat the latter argument here, and then use it to re-derive the distribution of the corresponding Al-Salam–Chihara polynomials.

Corollary 2. *If* $\rho, q, y \in \mathbb{R}$ *are such that* $|\rho| < 1$, $|q| < 1$, *and* $y^2(1-q) < 4$, *then the probabilistic solution of* (2.1) *is given by the absolutely continuous measure* μ *with the density on* $x^2 < 4/(1-q)$ *given by*

$$\frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}} \prod_{k=0}^{\infty} \frac{(1-\rho^2 q^k)\left(1-q^{k+1}\right)\left((1+q^k)^2 - (1-q)x^2 q^k\right)}{(1-\rho^2 q^{2k})^2 - (1-q)\rho q^k (1+\rho^2 q^{2k}) xy + (1-q)\rho^2 (x^2+y^2) q^{2k}}.$$

Proof. The distribution of the q-Hermite polynomials $H_n(x|q)$ is supported on $x^2 < 4/(1-q)$ with the density

$$f_H(x) = \frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}} \prod_{k=0}^{\infty} \left((1+q^k)^2 - (1-q)x^2 q^k\right) \prod_{k=0}^{\infty} (1-q^{k+1});$$

see [ISV87, (2.15)]. Moreover, since $|H_n(x)| \le C_q(n+1)(1-q)^{-n/2}$ when $x^2, y^2 \le 4/(1-q)$, the series

$$g_H(x,y,\rho) = \sum_{n=0}^{\infty} \frac{\rho^n}{[n]_q!} H_n(x)H_n(y) \tag{2.4}$$

converges uniformly and defines the Poisson-Mehler kernel, which is given by

$$g_H(x,y,\rho) = \prod_{k=0}^{\infty} \frac{(1-\rho^2 q^k)}{(1-\rho^2 q^{2k})^2 - (1-q)\rho q^k (1+\rho^2 q^{2k})xy + (1-q)\rho^2(x^2+y^2)q^{2k}}; \tag{2.5}$$

this is the renormalized version of the well-known result; see e.g. [IS88, (2.2)], which considers the q-Hermite polynomials given by $\{(1-q)^{n/2} H_n(2x/\sqrt{1-q}|q)\}$ instead of our $\{H_n(x|q)\}$.

Since (1.3) implies that $\int H_n^2(x|q) f_H(x)dx = [n]_q!$, it follows from (2.4) that

$$\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} H_n(x|q) g_H(x,y,\rho) f_H(x)\, dx = \rho^n H_n(y).$$

□

Corollary 3. *If $q,a,b \in \mathbb{R}$ are such that $|q|<1$, $0<b<1$, and $a^2(1-q) < 4b$, then the distribution of the Al-Salam–Chihara polynomials $\{p_n(x|q,a,b)\}$ is absolutely continuous with the density on $x^2 < 4/(1-q)$ given by*

$$\frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}} \prod_{k=0}^{\infty} \frac{(1-bq^k)\left(1-q^{k+1}\right)\left((1+q^k)^2 - (1-q)x^2 q^k\right)}{(1-bq^{2k})^2 - (1-q)aq^k(1+bq^{2k})x + (1-q)(bx^2+a^2)q^{2k}}.$$

Proof. By Theorem 2, the distribution of polynomials p_n solves (2.1) with $\rho = \sqrt{b}, y = a/\rho$. Thus the formula follows from Corollary 2. □

Remark 2. Iterating (2.1) we see that the measure corresponding to the parameter $\rho_1\rho_2$ instead of ρ is given by

$$\mu(\cdot|\rho_1\rho_2, x) = \int \mu(\cdot|\rho_1, y)\mu(dy|\rho_2, x). \tag{2.6}$$

For $|q|<1, |\rho|<1$ the density of μ is given in Corollary 2; hence, after simplifying common factors and substituting $x = 2\xi/\sqrt{1-q}$, $y = 2\eta/\sqrt{1-q}$, $z = 2\zeta/\sqrt{1-q}$, the relationship (2.6) takes the following form:

$$\begin{aligned}
&\int_{-1}^{1} \prod_{k=0}^{\infty} \frac{(1-\rho_1^2 q^k)\left(1-q^{k+1}\right)\left((1+q^k)^2 - 4\eta^2 q^k\right)}{(1-\rho_1^2 q^{2k})^2 - 4\rho_1 q^k(1+\rho_1^2 q^{2k})\eta\zeta + 4\rho_1^2(\eta^2+\zeta^2)q^{2k}} \\
&\quad \times \prod_{k=0}^{\infty} \frac{(1-\rho_2^2 q^k)}{(1-\rho_2^2 q^{2k})^2 - 4\rho_2 q^k(1+\rho_2^2 q^{2k})\xi\eta + 4\rho_2^2(\eta^2+\xi^2)q^{2k}} \frac{d\eta}{2\pi\sqrt{1-\eta^2}} \\
&= \prod_{k=0}^{\infty} \frac{(1-\rho_1^2\rho_2^2 q^k)}{(1-\rho_1^2\rho_2^2 q^{2k})^2 - 4\rho_1\rho_2 q^k(1+\rho_1^2\rho_2^2 q^{2k})\xi\zeta + 4\rho_1^2\rho_2^2(\zeta^2+\xi^2)q^{2k}}.
\end{aligned}$$

3. Determinants of Hankel matrices

In this section we are interested in calculating the determinants of the Hankel matrices

$$M_n = [m_{i+j}]_{i,j=0,\dots n-1},$$

where $m_i = \int x^i \mu(dx)$ are the moments of a certain (perhaps signed) measure μ. It is well known that for positive measures we must have $\det M_n \geq 0$, and that these determinants can be read out from the three-term recurrence for the corresponding monic orthogonal polynomials.

Consider first the moments $m_k(y) = \int x^k \mu(dx)$ of the (perhaps signed) measure $\mu = \mu_{y,\rho}$, which solves (2.1). Then $m_k(y)$ are polynomials of degree k in the variable y and can be written as follows. Let $a_{n,2i}$, $i \leq \lfloor n/2 \rfloor$ be the coefficients in the expansion of the monomial x^n into the q-Hermite polynomials,

$$x^n = \sum_{i=0}^{\lfloor n/2 \rfloor} a_{n,2i} H_{n-2i}\left(x|q\right),\ n \geq 0.$$

Then

$$m_n(y) = \int x^n d\mu\left(x\right) = \sum_{k=0}^{\lfloor n/2 \rfloor} \rho^{n-2k} a_{n,2k} H_{n-2k}\left(y|q\right).$$

Let S_n be the Hankel matrix of moments $m_k(y)$,

$$S_n(y|q,\rho) = \begin{bmatrix} m_0(y) & m_1(y) & \dots & m_{n-1}(y) \\ m_1(y) & m_2(y) & & \\ \vdots & & \ddots & \vdots \\ m_{n-1}(y) & & \dots & m_{2n-2}(y) \end{bmatrix}.$$

It is well known that $\det S_n$ is the product of the coefficients at the third term of (2.2), which implies the following.

Corollary 4. $\det S_{n+1} / \det S_n = [n]_q! \prod_{i=1}^{n} \left(1 - \rho^2 q^{i-1}\right).$

Our second Hankel matrix has an even simpler form. As indicated in [IS97], [IS02] the q-Hermite polynomials can be viewed as moments of a signed measure, $H_n(x|q) = \int u^n \mu(du|x,q)$. It turns out that if $q \neq 0$, the measure $\mu(du|x,q)$ cannot be positive even for a single value of x. To see this, consider the following $n \times n$ matrices:

$$M_n(x|q) = \begin{bmatrix} H_0(x|q) & H_1(x|q) & H_2(x|q) & \dots & H_{n-1}(x|q) \\ H_1(x|q) & H_2(x|q) & & & H_n(x|q) \\ H_2(x|q) & & \ddots & & H_{n+1}(x|q) \\ \vdots & & & \ddots & \vdots \\ H_{n-1}(x|q) & H_n(x|q) & & \dots & H_{2n-2}(x|q) \end{bmatrix}.$$

The following q-generalization of [Kra99, (3.55)] shows that the determinants $\det M_n(x|q)$ are free of the variable x and take negative values.

Theorem 3.

$$\frac{\det M_{n+1}}{\det M_n} = (-1)^n q^{n(n-1)/2} [n]_q!.$$

Proof. Using (1.3), we row-reduce the first column of the matrix. Namely, from the second row of M_{n+1}, we subtract the first one multiplied by x. Similarly, for $i \geq 3$, we subtract x times row $i-1$ and add the $(i-2)$-th row multiplied by $[i-1]_q$. Taking (1.3) into account, $\det M_{n+1}(x|q)$ becomes

$$\det \begin{bmatrix} H_0 & H_1 & H_2 & \dots & H_n \\ 0 & ([0]-[1])H_0 & ([0]-[2])H_1 & & ([0]-[n])H_{n-1} \\ 0 & ([1]-[2])H_1 & ([1]-[3])H_2 & & ([1]-[n+1])H_n \\ \vdots & & & \ddots & \vdots \\ 0 & ([n-1]-[n])H_{n-1} & ([n-1]-[n+1])H_n & \dots & ([n-1]-[2n-1])H_{2n-2} \end{bmatrix}.$$

Now, we use the fact that for $m \leq n$ we have $[n]_q - [m]_q = q^m [n-m]_q$. Thus $\det M_{n+1}(x|q)$ becomes

$$\det \begin{bmatrix} H_0 & H_1 & H_2 & \dots & H_{n-1} \\ 0 & -[1]H_0 & -[2]H_1 & & -[n]H_{n-1} \\ 0 & -q[1]H_1 & -q[2]H_2 & & -q[n]H_n \\ \vdots & & & \ddots & \vdots \\ 0 & -q^{n-1}[1]H_{n-1} & -q^{n-1}[2]H_n & \dots & -q^{n-1}[n]H_{2n-2} \end{bmatrix}.$$

Expanding $\det M_{n+1}$ with respect to the first column, and factoring out the common factors $-q^{i-1}$ from the i-th row and $[j]_q$ from the j-th column of the resulting matrix, we get

$$\det M_{n+1} = (-1)^n \, q^{\sum_{i=1}^{n-1} i} \prod_{j=1}^{n} [j] \det M_n = (-1)^n \, q^{n(n-1)/2} [n]_q! \det M_n. \qquad \square$$

The formula stated in Corollary 4 was originally discovered through symbolic computations and motivated this paper. We were unable to find a direct algebraic proof along the lines of the proof of Theorem 3, and our search for the explanation of why $\det S_n(y)$ does not depend on y led us to Al-Salam–Chihara polynomials and identity (1.8).

The fact that Hankel determinants formed of certain linear combinations of the q-Hermite polynomials do not depend on the argument of these polynomials as exposed in Theorem 3 and Corollary 4 is striking and unexpected to us. A natural question arises whether other linear combinations have this property.

Acknowledgement

Part of the research of the first author was conducted while visiting the Warsaw University of Technology. The authors thank Jacek Wesołowski for several helpful discussions.

References

[AI84] Richard Askey and Mourad Ismail. Recurrence relations, continued fractions, and orthogonal polynomials. *Mem. Amer. Math. Soc.*, 49(300):iv+108, 1984. MR85g:33008

[ASC76] W. A. Al-Salam and T. S. Chihara. Convolutions of orthonormal polynomials. *SIAM J. Math. Anal.*, 7(1):16–28, 1976. MR53:3381

[Ask89] Richard Askey. Continuous q-Hermite polynomials when $q > 1$. In *q-series and partitions (Minneapolis, MN, 1988)*, volume 18 of *IMA Vol. Math. Appl.*, pages 151–158. Springer, New York, 1989. MR90h:33019

[BI96] Christian Berg and Mourad E. H. Ismail. q-Hermite polynomials and classical orthogonal polynomials. *Canad. J. Math.*, 48(1):43–63, 1996. MR97b:33020

[Bry01] Włodzimierz Bryc. Stationary random fields with linear regressions. *Annals of Probability*, 29:504–519, 2001. MR2002d:60014

[Car56] L. Carlitz. Some polynomials related to theta functions. *Ann. Mat. Pura Appl. (4)*, 41:359–373, 1956. MR17:1205e

[Fre71] Géza Freud. *Orthogonal Polynomials*. Pergamon Press, Oxford, 1971.

[IM94] M. E. H. Ismail and D. R. Masson. q-Hermite polynomials, biorthogonal rational functions, and q-beta integrals. *Trans. Amer. Math. Soc.*, 346(1):63–116, 1994. MR96a:33022

[IRS99] Mourad E. H. Ismail, Mizan Rahman, and Dennis Stanton. Quadratic q-exponentials and connection coefficient problems. *Proc. Amer. Math. Soc.*, 127(10):2931–2941, 1999. MR2000a:33027

[IS88] Mourad E. H. Ismail and Dennis Stanton. On the Askey-Wilson and Rogers polynomials. *Canad. J. Math.*, 40(5):1025–1045, 1988. MR89m:33003

[IS97] Mourad E. H. Ismail and Dennis Stanton. Classical orthogonal polynomials as moments. *Canad. J. Math.*, 49(3):520–542, 1997. MR98f:33033

[IS02] Mourad E. H. Ismail and Dennis Stanton. q-integral and moment representations for q-orthogonal polynomials. *Canad. J. Math.*, 54(4):709–735, 2002. MR2003k:33024

[ISV87] M. E. H. Ismail, D. Stanton, and G. Viennot. The combinatorics of q-Hermite polynomials and the Askey-Wilson integral. *Europ. J. Combinatorics*, 8:379–392, 1987. MR89h:33015

[Kra99] Christian Krattenthaler. Advanced determinant calculus. *Sém. Lothar. Combin.*, 42:Art. B42q, 67 pp. (electronic), 1999. MR2002i:05013

Department of Mathematics, University of Cincinnati, P.O. Box 210025, Cincinnati, Ohio 45221–0025

E-mail address: `Wlodzimierz.Bryc@UC.edu`

Faculty of Mathematics and Information Science, Warsaw University of Technology, pl. Politechniki 1, 00-661 Warszawa, Poland

E-mail address: `wmatysiak@elka.pw.edu.pl`

Faculty of Mathematics and Information Science, Warsaw University of Technology, pl. Politechniki 1, 00-661 Warszawa, Poland

E-mail address: `pszablowski@elka.pw.edu.pl`