PROBABILISTIC ASPECTS OF AL-SALAM–CHIHARA POLYNOMIALS

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Abstract. We solve the connection coefficient problem between the Al-Salam–Chihara polynomials and the $q$-Hermite polynomials, and we use the resulting identity to answer a question from probability theory. We also derive the distribution of some Al-Salam–Chihara polynomials, and compute determinants of related Hankel matrices.

1. Introduction and main identity

The aim of the paper is to point out the connection of Al-Salam–Chihara polynomials with a regression problem in probability, and to use it to give a new simple derivation of their density. Our approach exploits identity (1.8) below, which connects the Al-Salam–Chihara polynomials to the continuous $q$-Hermite polynomials. This connection is more direct and elementary but less general than the technique of attachment exploited in [B196, Section 2]. We also compute determinants of Hankel matrices with entries that are linear combinations of the $q$-Hermite polynomials.

The Al-Salam–Chihara polynomials were introduced in [ASC76], and their weight function was found in [AI84]. We are interested in the renormalized Al-Salam–Chihara polynomials $\{p_n(x|q,a,b)\}$, which are defined by the following three-term recurrence relation:

\begin{equation}
\begin{aligned}
  p_{n+1}(x) &= (x - aq^n)p_n(x) - (1 - bq^{n-1}) [n]_q p_{n-1}(x) \quad (n \geq 0),
\end{aligned}
\end{equation}

with the usual initial conditions $p_{-1} = 0$, $p_0 = 1$. Here, we use the standard notation

\[
\begin{align*}
[n]_q &= 1 + q + \cdots + q^{n-1}, \\
[n]_q! &= [1]_q [2]_q \cdots [n]_q, \\
\binom{n}{k}_q &= \frac{[n]_q!}{[n-k]_q! [k]_q!},
\end{align*}
\]

with the usual conventions $[0]_q = 0$, $[0]_q! = 1$.
For \(|q| < 1\), their generating function

\[ f(t, x|q, a, b) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} p_n(x|q, a, b) \]

is given by

\[ f(t, x|q, a, b) = \prod_{k=0}^{\infty} \frac{1 - (1 - q)atq^k + (1 - q)bt^2q^{2k}}{1 - (1 - q)xtq^k + (1 - q)t^2q^{2k}}; \]

compare [AI84 (3.6) and (3.10)].

The corresponding (renormalized) continuous q-Hermite polynomials \(H_n(x|q) = p_n(x|q, 0, 0)\) satisfy the three-term recurrence relation

\[ H_{n+1}(x) = xH_n(x) - [n]_q H_{n-1}(x). \]

For \(|q| < 1\) their generating function \(\phi(t, x|q) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} H_n(x|q)\) is

\[ \phi(t, x|q) = \prod_{k=0}^{\infty} \left( 1 - (1 - q)xtq^k + (1 - q)t^2q^{2k} \right)^{-1}. \]

Of course, these are well-known special cases of [1.1] and [1.2]; see [JSV87 (2.11) and (2.12)], which we state here for further reference.

We will also use polynomials \(\{B_n(x|q)\}\) defined by the three-term recurrence relation

\[ B_{n+1}(x) = -q^n xB_n(x) + q^{n-1}[n]_q B_{n-1}(x) \quad (n \geq 0) \]

with the usual initial conditions \( B_{-1} = 0, B_0 = 1 \). These polynomials are related to the q-Hermite polynomials by

\[ B_n(x|q) = \begin{cases} \frac{n^n q^{n(n-2)/2} H_n(i \sqrt{q} x^{1/2})}{\sqrt{q} H_n(-\sqrt{q} x^{1/2})} & \text{if } q > 0, \\ (-1)^{n(n-1)/2} [n]_q^{n(n-2)/2} H_n(-\sqrt{q} x^{1/2}) & \text{if } q < 0 \end{cases} \]

and have been studied in [Ask89], [IM94]. Their generating function \(\psi(t, x|q) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} B_n(x|q)\) is given by

\[ \psi(t, x|q) = \prod_{k=0}^{\infty} \left( 1 - (1 - q)xtq^k + (1 - q)t^2q^{2k} \right). \]

We now point out the mutual relationship between the Al-Salam–Chihara polynomials \(\{p_n(x|q, a, b)\}\) and the polynomials \(\{H_n(x|q)\}_{n \geq 0}\) and \(\{B_n(x|q)\}_{n \geq 0}\).

**Theorem 1.** For all \(a, c, q \in \mathbb{C}, c \neq 0, a \neq 0\), and \(n \geq 1\) we have

\[ p_n(x|q, a, b) = \sum_{k=1}^{n} \binom{n}{k}_q e^{n-k} B_{n-k}(\frac{a}{c}|q) \left( H_k(x|q) - e^k H_k(\frac{a}{c}|q) \right), \]

where \(b = c^2\).

**Proof.** From the recurrence relations [1.1], [1.3], and [1.5], it is clear that \(p_n(x|q, a, b), H_n(x|q),\) and \(B_n(x|q)\) are given by polynomial expressions in the variable \(q\). The q-binomial coefficient \(\binom{n}{k}_q\) is also a polynomial in \(q\). Therefore, we see that identity (1.5) is equivalent to a polynomial identity in variable \(q \in \mathbb{C}\).
Hence it is enough to prove that (1.8) holds true for all $|q| < 1$. When $|q| < 1$, inspecting (1.2), (1.4), and (1.7) we notice that for $b = c^2$ we have

\begin{equation}
\psi(1.10) f(t, x|q, a, b) = \psi(tc, a/c|q)\phi(t, x|q)
\end{equation}

and

\begin{equation}
\psi(t, x|q)\phi(t, x|q) = 1.
\end{equation}

Therefore,

\begin{equation}
f(t, x|q, a, b) = 1 + \psi(tc, a/c|q)\left(\phi(t, x|q) - \phi(tc, a/c|q)\right),
\end{equation}

which is valid for all small enough $|t|$. Comparing the coefficients at $t^n$ for $n \geq 1$ and taking into account that $H_k(x|q) = c^kH_k\left(\frac{a}{c}|q\right) = 0$ for $k = 0$, we get (1.8). \hfill \Box

Remark 1. One could split (1.8) into the following two separate identities, which are implied by (1.9) and (1.10) respectively:

\begin{equation}
\forall n \geq 0 : p_n(x|q, a, c^2) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] c^{n-k}B_{n-k}\left(\frac{a}{c}|q\right)H_k(x|q),
\end{equation}

\begin{equation}
\forall n \geq 1 : \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] B_{n-k}(x|q)H_k(x|q) = 0.
\end{equation}

Formula (1.11) is a renormalized inverse of formula \cite{IRS99} (4.7), which expresses the $q$-Hermite polynomials as linear combinations of Al-Salam–Chihara polynomials. Formula (1.12) resembles \cite{Car56} (2.28), which considers $q$-Hermite polynomials of the form $b_n(x|q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \frac{1}{q} x^k$, paired with $b_n(x|q) = h_n(x|1/q)$.

2. Probabilistic aspects

Quadratic regression questions in the paper \cite{Bry01} lead to the problem of determining all probability distributions $\mu$ which are defined indirectly by the relationships

\begin{equation}
\int H_n(x|q)\mu(dx) = \rho^n H_n(y|q), \quad n = 1, 2, \ldots,
\end{equation}

where $y, \rho, q \in \mathbb{R}$ are fixed parameters, and $\{H_n\}_{n \geq 0}$ is the family of the $q$-Hermite polynomials.

Our next result shows that this problem can be solved using the Al-Salam–Chihara polynomials.

Theorem 2. If $\mu = \mu(dx|\rho, y)$ satisfies (2.1), then its orthogonal polynomials are Al-Salam–Chihara polynomials $\{p_n(x|q, a, b)\}$ with $a = \rho y, b = \rho^2$.

Proof. Recall that $H_n(x|q) = p_n(x|q, 0, 0)$. Thus if $\rho = 0$, then (2.1) implies that $\int p_n(x|q, a, b)\mu(dx) = 0$ for all $n = 1, 2, \ldots$. Suppose now that $\rho \neq 0$. Combining (1.8) with (2.1) we get

\begin{equation}
\int p_n(x|q, a, b)\mu(dx) = \sum_{k=1}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \rho^{n-k}B_{n-k}(y|q) \int \left( H_k(x|q) - \rho^k H_k(y|q) \right) \mu(dx) = 0
\end{equation}

for all $n = 1, 2, \ldots$. Since $\{p_n\}$ satisfy a three-step recurrence, this implies $\int p_k(x)p_n(x)\mu(dx) = 0$ for all $0 \leq k < n$. \hfill \Box
Next we answer an unresolved case from [Bry01].

**Corollary 1.** Fix $q > 1, y \in \mathbb{R}$. Let $\mathcal{R}_q = \{1, 1/q, 1/q^2, \ldots, 1/q^n, \ldots, 0\}$.

(i) If $\rho^2 \notin \mathcal{R}_q$, then (2.1) has no probabilistic solution $\mu$.

(ii) If $\rho^2 \in \mathcal{R}_q$ is non-zero, then the probabilistic solution of (2.1) exists, and is a discrete measure supported on $1 + \log_q 1/\rho^2$ points.

**Proof.** Suppose that $\mu$ is positive and solves (2.1). Therefore its monic orthogonal polynomials satisfy the three-term recurrence relation

\begin{equation}
(p_{n+1}(x) = (x - \rho y q^n)p_n(x) - (1 - \rho^2 q^{n-1})p_{n-1}(x).
\end{equation}

For a positive non-degenerate measure $\mu_y(dx)$, and $n \geq 1$ we have

\begin{equation}
\int p_n^2(x)\mu_y(dx) = (1 - \rho^2 q^{n-1}) \int p_{n-1}^2(x)\mu_y(dx).
\end{equation}

If $\rho^2 \notin \mathcal{R}_q$, then $(1 - \rho^2 q^{n-1}) \neq 0$ for all $n$. Since $\int p_0^2(x)\mu_y(dx) > 0$, this shows that $\int p_n^2(x)\mu_y(dx) > 0$ for all $n \geq 0$. But then the coefficients $1 - \rho^2 q^{n-1}$ must be non-negative for all $n$, which is false. This proves (i).

To conclude the proof it remains to notice that if $\rho^2 = 1/q^m$, then from (2.2) and (the proof of) Favard’s theorem, see [Fre71, Theorem II.1.5], it follows that the solution of (2.1) is given by a measure supported on the roots of the polynomial $p_{m+1}$. Indeed, (2.2) implies that the polynomial $p_{m+2}$ is divisible by $p_{m+1}$. Therefore, $p_{m+1}$ is the common factor of all polynomials $\{p_k : k \geq m + 1\}$. It is also known, see [Fre71, Theorem I.2.2], that $p_{m+1}$ has exactly $m + 1$ distinct real roots $x_1, \ldots, x_{m+1}$. Thus, any measure $\mu(dx) = \sum \lambda_j \delta_{x_j}$, supported on the roots of the polynomial $p_{m+1}$ satisfies $\int p_{m+1+k}\mu(dx) = 0$. Solving the remaining $m + 1$ equations $\int p_k\mu(dx) = 1$, and $\int p_k(x)\mu(dx) = 0$, $k = 1, 2, \ldots, m$ for $\lambda_j$, we get a measure that solves (2.1). This measure is non-negative since the coefficients at the third term in the recurrence (2.2) are non-negative for $n = 1, \ldots, m$; see [Fre71, page 58].

From Theorem 2 it follows that if the solution of (2.1) exists, then it is given by the distribution of the Al-Salam–Chihara polynomials. The distribution of the Al-Salam–Chihara polynomials is derived in [A183, Chapter 3]. However, in [Bry01, Proposition 8.1] we found the solution of (2.1) that relies solely on the facts about the $q$-Hermite polynomials. We repeat the latter argument here, and then use it to re-derive the distribution of the corresponding Al-Salam–Chihara polynomials.

**Corollary 2.** If $\rho, q, y \in \mathbb{R}$ are such that $|\rho| < 1$, $|q| < 1$, and $y^2(1 - q) < 4$, then the probabilistic solution of (2.1) is given by the absolutely continuous measure $\mu$ with the density on $x^2 < 4/(1 - q)$ given by

\[\frac{\sqrt{1-q}}{2\pi\sqrt{4 - (1-q)x^2}} \prod_{k=0}^{\infty} \frac{(1 - \rho^2 q^{2k})((1 + q^{2k})^2 - (1-q)x^2)^2}{(1 - \rho^2 q^{2k})^2 - (1-q)\rho q(1 + \rho^2 q^{2k})xy + (1-q)\rho^2(x^2 + y^2)q^{2k}}.\]

**Proof.** The distribution of the $q$-Hermite polynomials $H_n(x|q)$ is supported on $x^2 < 4/(1 - q)$ with the density

\[f_H(x) = \frac{\sqrt{1-q}}{2\pi\sqrt{4 - (1-q)x^2}} \prod_{k=0}^{\infty} ((1 + q^k)^2 - (1-q)x^2 q^k) \prod_{k=0}^{\infty} (1-q^{k+1});\]
see [ISV87] (2.15). Moreover, since \(|H_n(x)| \leq C_q(n + 1)(1 - q)^{-n/2}\) when \(x^2, y^2 \leq 4/(1 - q)\), the series

\[
g_H(x, y, \rho) = \sum_{n=0}^{\infty} \frac{\rho^n}{|n|_q} H_n(x)H_n(y)
\]

(2.4)

converges uniformly and defines the Poisson-Mehler kernel, which is given by

\[
g_H(x, y, \rho) = \prod_{k=0}^{\infty} \frac{(1 - \rho^2q^{2k})^2 - (1 - q)\rho^k(1 + \rho^2q^{2k})xy + (1 - q)\rho^2(x^2 + y^2)q^{2k};}
\]

this is the renormalized version of the well-known result; see e.g. [IS88] (2.2)], which considers the \(q\)-Hermite polynomials given by \((1 - q)^{n/2} H_n(2x/\sqrt{1 - q})\) instead of our \(H_n(x|q)\).

Since \((1.3)\) implies that \(\int H_n^2(x|q)f_H(x)dx = |n|_q!\), it follows from (2.4) that

\[
\int_{-2/\sqrt{1 - q}}^{2/\sqrt{1 - q}} H_n(x|q)g_H(x, y, \rho)f_H(x)dx = \rho^n H_n(y).
\]

\[\square\]

**Corollary 3.** If \(q, a, b \in \mathbb{R}\) are such that \(|q| < 1, 0 < b < 1,\) and \(a^2(1 - q) < 4b\), then the distribution of the \(Al-Salam–Chihara polynomials\) \(\{p_n(x|q, a, b)\}\) is absolutely continuous with the density on \(x^2 < 4/(1 - q)\) given by

\[
\frac{\sqrt{1 - q}}{2\pi \sqrt{4 - (1 - q)x^2}} \prod_{k=0}^{\infty} \frac{(1 - bq^k)(1 - q^{k+1})(1 + q^k)^2 - (1 - q)x^2q^k}{(1 - bq^{2k})^2 - (1 - q)aq^k(1 + bq^{2k})x + (1 - q)(bx^2 + a^2)q^{2k}}.
\]

**Proof.** By Theorem 2, the distribution of polynomials \(p_n\) solves (2.1) with \(\rho = \sqrt{b}, y = a/\rho\). Thus the formula follows from Corollary [2]. \[\square\]

**Remark 2.** Iterating (2.1) we see that the measure corresponding to the parameter \(\rho_1\rho_2\) instead of \(\rho\) is given by

\[
\mu(\cdot|\rho_1\rho_2, x) = \int \mu(\cdot|\rho_1, y)\mu(dy|\rho_2, x).
\]

(2.6)

For \(|q| < 1, |\rho| < 1\) the density of \(\mu\) is given in Corollary 2 hence, after simplifying common factors and substituting \(x = 2\zeta/\sqrt{1 - q}, y = 2\eta/\sqrt{1 - q}, z = 2\zeta/\sqrt{1 - q},\) the relationship (2.6) takes the following form:

\[
\int_{-1}^{1} \prod_{k=0}^{\infty} \frac{(1 - \rho_1^2q^k)(1 - q^{k+1})(1 + q^k)^2 - 4q^k\rho_1^k(1 + \rho_1^2q^{2k})\eta\xi + 4\rho_1^2(\eta^2 + \xi^2)q^{2k}}{(1 - \rho_1^2q^{2k})^2 - 4\rho_1q^k(1 + \rho_1^2q^{2k})\eta\xi + 4\rho_1^2(\eta^2 + \xi^2)q^{2k}}\frac{d\eta}{2\pi \sqrt{1 - \eta^2}}
\]

\[
\times \prod_{k=0}^{\infty} \frac{(1 - \rho_2^2q^{2k})^2 - 4\rho_2q^k(1 + \rho_2^2q^{2k})\xi\zeta + 4\rho_2^2(\xi^2 + \zeta^2)q^{2k}}{(1 - \rho_2^2q^{2k})^2 - 4\rho_2q^k(1 + \rho_2^2q^{2k})\xi\zeta + 4\rho_2^2(\xi^2 + \zeta^2)q^{2k}}
\]

\[
= \prod_{k=0}^{\infty} \frac{(1 - \rho_1^2\rho_2^2q^{2k})^2 - 4\rho_1\rho_2q^k(1 + \rho_1^2\rho_2^2q^{2k})\xi\zeta + 4\rho_1^2\rho_2^2(\xi^2 + \zeta^2)q^{2k}}{(1 - \rho_1^2\rho_2^2q^{2k})^2 - 4\rho_1\rho_2q^k(1 + \rho_1^2\rho_2^2q^{2k})\xi\zeta + 4\rho_1^2\rho_2^2(\xi^2 + \zeta^2)q^{2k}}.
\]
3. Determinants of Hankel Matrices

In this section we are interested in calculating the determinants of the Hankel matrices

$$M_n = [m_{i+j}]_{i,j=0,...,n-1},$$

where $m_i = \int x^i \mu(dx)$ are the moments of a certain (perhaps signed) measure $\mu$. It is well known that for positive measures we must have $\det M_n \geq 0$, and that these determinants can be read out from the three-term recurrence for the corresponding monic orthogonal polynomials.

Consider first the moments $m_k(y) = \int x^k \mu(dx)$ of the (perhaps signed) measure $\mu = \mu_{y,\rho}$, which solves (2.1). Then $m_k(y)$ are polynomials of degree $k$ in the variable $y$ and can be written as follows. Let $a_{n,2i}$, $i \leq \lfloor n/2 \rfloor$ be the coefficients in the expansion of the monomial $x^n$ into the $q$-Hermite polynomials,

$$x^n = \sum_{i=0}^{\lfloor n/2 \rfloor} a_{n,2i} H_{n-2i}(x|q), \ n \geq 0.$$ 

Then

$$m_n(y) = \int x^n d\mu(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \rho^{n-2k} a_{n,2k} H_{n-2k}(y|q).$$

Let $S_n$ be the Hankel matrix of moments $m_k(y)$,

$$S_n(y|q, \rho) = \begin{bmatrix} m_0(y) & m_1(y) & \cdots & m_{n-1}(y) \\ m_1(y) & m_2(y) & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ m_{n-1}(y) & \cdots & m_{2n-2}(y) \end{bmatrix}.$$ 

It is well known that $\det S_n$ is the product of the coefficients at the third term of (2.2), which implies the following.

**Corollary 4.** $\det S_{n+1}/\det S_n = [n]_q! \prod_{i=1}^{\lfloor n/2 \rfloor} (1 - \rho^2 q^{i-1}).$

Our second Hankel matrix has an even simpler form. As indicated in [IS97], [IS02] the $q$-Hermite polynomials can be viewed as moments of a signed measure, $H_n(x|q) = \int u^n \mu(du|x, q)$. It turns out that if $q \neq 0$, the measure $\mu(du|x, q)$ cannot be positive even for a single value of $x$. To see this, consider the following $n \times n$ matrices:

$$M_n(x|q) = \begin{bmatrix} H_0(x|q) & H_1(x|q) & H_2(x|q) & \cdots & H_{n-1}(x|q) \\ H_1(x|q) & H_2(x|q) & H_3(x|q) & \cdots & H_n(x|q) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ H_{n-1}(x|q) & H_n(x|q) & \cdots & H_{2n-2}(x|q) \end{bmatrix}.$$ 

The following $q$-generalization of [Kra99 (3.55)] shows that the determinants $\det M_n(x|q)$ are free of the variable $x$ and take negative values.

**Theorem 3.**

$$\frac{\det M_{n+1}}{\det M_n} = (-1)^n q^{n(n-1)/2}[n]_q!.$$
Proof. Using (1.3), we row-reduce the first column of the matrix. Namely, from the second row of $M_{n+1}$, we subtract the first one multiplied by $x$. Similarly, for $i \geq 3$, we subtract $x$ times row $i - 1$ and add the $(i - 2)$-th row multiplied by $[i - 1]_q$. Taking (1.3) into account, det $M_{n+1}(x|q)$ becomes

$$
\begin{vmatrix}
H_0 & H_1 & H_2 & \ldots & H_n \\
0 & ([0] - [1])H_0 & ([0] - [2])H_1 & ([0] - [n])H_{n-1} \\
0 & ([1] - [2])H_1 & ([1] - [3])H_2 & ([1] - [n + 1])H_n \\
\vdots & \ddots & \ddots & \ddots \\
0 & ([n - 1] - [n])H_{n-1} & ([n - 1] - [n + 1])H_n & ([n - 1] - [2n - 1])H_{2n-2} \\
\end{vmatrix}
$$

Now, we use the fact that for $m \leq n$ we have $[n]_q - [m]_q = q^n [n - m]_q$. Thus det $M_{n+1}(x|q)$ becomes

$$
\begin{vmatrix}
H_0 & H_1 & H_2 & \ldots & H_{n-1} \\
0 & -[1]H_0 & -[2]H_1 & -[n]H_{n-1} \\
\vdots & \ddots & \ddots & \ddots \\
0 & -q^{n-1}[1]H_{n-1} & -q^{n-1}[2]H_n & -q^{n-1}[n]H_{2n-2} \\
\end{vmatrix}
$$

Expanding det $M_{n+1}$ with respect to the first column, and factoring out the common factors $-q^{i-1}$ from the $i$-th row and $[j]_q$ from the $j$-th column of the resulting matrix, we get

$$
\text{det} \ M_{n+1} = (-1)^n q^{\sum_{i=1}^{n} i} \prod_{j=1}^{n} [j]_q \text{det} \ M_n = (-1)^n q^{n(n-1)/2} [n]_q! \text{det} \ M_n. \quad \Box
$$

The formula stated in Corollary 3 was originally discovered through symbolic computations and motivated this paper. We were unable to find a direct algebraic proof along the lines of the proof of Theorem 3 and our search for the explanation of why det $S_n(y)$ does not depend on $y$ led us to Al-Salam–Chihara polynomials and identity (1.8).

The fact that Hankel determinants formed of certain linear combinations of the $q$-Hermite polynomials do not depend on the argument of these polynomials as exposed in Theorem 3 and Corollary 1 is striking and unexpected to us. A natural question arises whether other linear combinations have this property.

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