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A NOTE ON THE WEIGHTED HILBERT'S INEQUALITY

XIAN-JIN LI

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Abstract. A finite Hilbert transformation associated with a polynomial is the analogue of a Hilbert transformation associated with an entire function which is a generalization of the classical Hilbert transformation. The weighted Hilbert inequality, which has applications in analytic number theory, is closely related to the finite Hilbert transformation associated with a polynomial. In this note, we study a relation between the finite Hilbert transformation and the weighted Hilbert's inequality. A question is posed about the finite Hilbert transformation, of which an affirmative answer implies the weighted Hilbert inequality.

1. Introduction

Let r be a positive integer. Assume that $P(z)$ is a polynomial of degree at most r such that $|P^*(z)| < |P(z)|$ for $|z| < 1$, where $P^*(z) = z^r \bar{P}(1/\bar{z})$. Let $\mathcal{F}_r(P)$ be the Hilbert space of all polynomials $F(z)$ of degree less than r such that

$$\|F\|^2_{\mathcal{F}_r(P)} = \int_0^1 |F(e^{2\pi i\theta})/P(e^{2\pi i\theta})|^2 d\theta < \infty.$$

The expression

$$K(w,z) = \frac{P(z)\bar{P}(w) - P^*(z)\bar{P}^*(w)}{1-\bar{w}z}, \tag{1.1}$$

considered as a polynomial of z, belongs to $\mathcal{F}_r(P)$ for every complex w. It is the reproducing kernel function for the space $\mathcal{F}_r(P)$ in the sense that the identity

$$F(w) = \langle F(z), K(w,z)\rangle_{\mathcal{F}_r(P)}$$

holds for every element F in $\mathcal{F}_r(P)$ and for all complex w; see [3].

A polynomial S is said to be associated with a space $\mathcal{F}_r(P)$ if

$$\frac{F(z)S(w) - S(z)F(w)}{z-w}$$

belongs to the space for every complex w whenever F belongs to the space. We write

$$P(z) = A(z) - iB(z)$$

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with $A^*(z) = A(z), B^*(z) = B(z)$. It is proved in Theorem 1 and Theorem 3 of [3] that there are polynomials $C(z), D(z)$ with $C^*(z) = C(z), D^*(z) = D(z)$ such that

$$A(z)D(z) - B(z)C(z) = S(z)S^*(z)$$

for all complex z, such that

$$L(w, z) = \frac{2\bar{S}(w)S(z) - 2[A(z)\bar{D}(w) - B(z)\bar{C}(w)]}{1 - \bar{w}z},$$

considered as a polynomial of z, belongs to $\mathcal{F}_r(P)$ for every complex w, and such that the identity

$$\begin{aligned}\frac{1}{2}F(\alpha)\bar{\widetilde{G}}(\beta) + \frac{1}{2}\widetilde{F}(\alpha)\bar{G}(\beta) &= \langle F(z)S(\alpha), G(z)S(\beta)\rangle_{\mathcal{F}_r(P)} \\ &+\alpha \left\langle \frac{F(z)S(\alpha) - S(z)F(\alpha)}{z-\alpha}, G(z)S(\beta)\right\rangle_{\mathcal{F}_r(P)} \\ &+\bar{\beta}\left\langle F(z)S(\alpha), \frac{G(z)S(\beta) - S(z)G(\beta)}{z-\beta}\right\rangle_{\mathcal{F}_r(P)} \\ &-(1-\alpha\bar{\beta})\left\langle \frac{F(z)S(\alpha) - S(z)F(\alpha)}{z-\alpha}, \frac{G(z)S(\beta) - S(z)G(\beta)}{z-\beta}\right\rangle_{\mathcal{F}_r(P)}\end{aligned}$$

holds for all elements F and G in the space and for all complex α and β, where

$$\widetilde{F}(w) = \langle F(z), L(w, z)\rangle_{\mathcal{F}_r(P)}. \tag{1.2}$$

We call $F \to \widetilde{F}$, $F \in \mathcal{F}_r(P)$, the finite Hilbert transformation associated with the polynomial S.

The following explicit formula is given in [4] for the finite Hilbert transformation associated with a polynomial S.

Lemma 1. *Let S be a polynomial associated with a nontrivial space $\mathcal{F}_r(P)$. Then we have*

$$D(\alpha) = iB(\alpha)\sum_{j=1}^{R} \frac{|S(\alpha_j)|^2}{K(\alpha_j, \alpha_j)} \frac{\alpha + \alpha_j}{\alpha - \alpha_j} + \frac{1}{2}B(\alpha)\left[\frac{S(0)}{P(0)}\frac{S^*(0)}{B(0)} + \frac{\bar{S}(0)}{\bar{P}(0)}\frac{\bar{S}^*(0)}{\bar{B}(0)}\right]$$

and

$$\begin{aligned}C(\alpha) =& 2i\sum_{j=1}^{R} \frac{\bar{S}(\alpha_j)}{K(\alpha_j, \alpha_j)} \frac{A(\alpha)S(\alpha_j) - S(\alpha)A(\alpha_j)}{\bar{\alpha}_j\alpha - 1} \\ &- S(\alpha)\frac{\bar{S}(0)}{\bar{B}(0)} + A(\alpha)\left[\frac{S(0)}{P(0)}\frac{S^*(0)}{B(0)} + \frac{\bar{S}(0)}{\bar{P}(0)}\frac{\bar{S}^*(0)}{\bar{B}(0)} - \frac{D(0)}{B(0)}\right],\end{aligned}$$

where α_j, $j = 1, \cdots, R$, are the zeros of $B(\alpha)$ that are not zeros of $P(\alpha)$.

Reproducing kernel Hilbert spaces of polynomials are related to Hilbert's inequality. Hilbert demonstrated that

$$|\sum_{r\neq s} \frac{u_r\bar{u}_s}{r-s}| \leq 2\pi \sum_{r=1}^{R} |u_r|^2$$

for any set of complex u_r, and later Schur [6] replaced the 2π by the best possible constant π; for related results see Hellinger and Toeplitz [2]. Suppose that

$\lambda_1, \cdots, \lambda_R$ are distinct real numbers. If $\tau_r = \min_{s \neq r} |\lambda_r - \lambda_s|$, Montgomery and Vaughan [5] showed that

$$|\sum_{r \neq s} \frac{u_r \bar{u}_s}{\lambda_r - \lambda_s}| \leq \frac{3}{2}\pi \sum_{r=1}^{R} |u_r|^2 \tau_r^{-1}.$$

A. Selberg [*unpublished*] has shown that $3\pi/2$ can be replaced by 3.2. But, it is believed that the weighted Hilbert inequality

$$|\sum_{r \neq s} \frac{u_r \bar{u}_s}{\lambda_r - \lambda_s}| \leq \pi \sum_{r=1}^{R} |u_r|^2 \tau_r^{-1} \tag{1.3}$$

holds for any set of complex u_r.

Let $x_1, \cdots, x_R$ be distinct real numbers modulo 1. Denote

$$\delta_r = \min_{s \neq r, 1 \leq s \leq R} \|x_r - x_s\| \tag{1.4}$$

where $\|x\|$ is the distance from x to the nearest integer. The inequality (1.3) is equivalent to

$$|\sum_{r \neq s} u_r \bar{u}_s \csc \pi(x_r - x_s)| \leq \sum_{r=1}^{R} \delta_r^{-1} |u_r|^2$$

for any set of complex u_r.

Let $\alpha_1, \cdots, \alpha_R$ be any distinct complex numbers on the unit circle, and let

$$B(z) = c \prod_{r=1}^{R} (z - \alpha_r) \tag{1.5}$$

where the constant c is given by

$$c^2 = (-1)^R \prod_{r=1}^{R} \bar{\alpha}_r.$$

Then we have

$$B^*(z) = z^R \bar{B}(1/\bar{z}) = B(z).$$

Let A be a polynomial defined by

$$\frac{iA(z)}{B(z)} = -ip + \sum_{r=1}^{R} p_r \frac{1 + \bar{\alpha}_r z}{1 - \bar{\alpha}_r z} \tag{1.6}$$

for a real number p, where

$$p_r = \frac{\delta_r}{4|B'(\alpha_r)|^2}$$

for $r = 1, \cdots, R$. Then we have

$$A^*(z) = z^R \bar{A}(1/\bar{z}) = A(z).$$

Let

$$P(z) = A(z) - iB(z), \tag{1.7}$$

where B and A are defined by (1.5) and (1.6). Since p is real and $p_r > 0$, $r = 1, 2, \cdots, R$, we have

$$|P(z)| > |P^*(z)|$$

for $|z| < 1$, where

$$P^*(z) = z^R \bar{P}(1/\bar{z}).$$

Therefore, a space $\mathcal{F}_R(P)$ exists. Assume that S is a polynomial associated with the space $\mathcal{F}_R(P)$. Let

$$D(\alpha) = iB(\alpha)\sum_{j=1}^{R} \frac{|S(\alpha_j)|^2}{K(\alpha_j,\alpha_j)}\frac{\alpha+\alpha_j}{\alpha-\alpha_j} + \frac{1}{2}B(\alpha)\left[\frac{S(0)}{P(0)}\frac{S^*(0)}{B(0)} + \frac{\bar{S}(0)}{\bar{P}(0)}\frac{\bar{S}^*(0)}{\bar{B}(0)}\right] \tag{1.8}$$

and

$$\begin{aligned} C(\alpha) =& 2i\sum_{j=1}^{R} \frac{\bar{S}(\alpha_j)}{K(\alpha_j,\alpha_j)}\frac{A(\alpha)S(\alpha_j)-S(\alpha)A(\alpha_j)}{\bar{\alpha}_j\alpha-1} \\ & - S(\alpha)\frac{\bar{S}(0)}{\bar{B}(0)} + A(\alpha)\left[\frac{S(0)}{P(0)}\frac{S^*(0)}{B(0)} + \frac{\bar{S}(0)}{\bar{P}(0)}\frac{\bar{S}^*(0)}{\bar{B}(0)} - \frac{D(0)}{B(0)}\right] \end{aligned} \tag{1.9}$$

where B, A and P are given in (1.5)–(1.7), respectively. Put

$$L(w,z) = \frac{2\bar{S}(w)S(z) - 2[A(z)\bar{D}(w) - B(z)\bar{C}(w)]}{1-\bar{w}z} \tag{1.10}$$

and

$$Q(w,\alpha) = 2\frac{D(\alpha)\bar{C}(w) - C(\alpha)\bar{D}(w)}{i(1-\bar{w}\alpha)}. \tag{1.11}$$

For any set of complex numbers $u_1, \cdots, u_R$, if

$$F(z) = B(z)\sum_{r=1}^{R}\frac{u_r/B'(\alpha_r)}{z-\alpha_r}, \tag{1.12}$$

then F belongs to the space $\mathcal{F}_R(P)$ and satisfies $F(\alpha_r) = u_r$ for $r = 1, 2, \cdots, R$.

Question. *Let $u_1, \cdots, u_R$ be any set of complex numbers, and let $x_1, \cdots, x_R$ be distinct real numbers modulo 1. Do there exist distinct complex numbers $\alpha_1, \cdots, \alpha_R$ on the unit circle and a polynomial $S(z)$ of degree R, which is not a linear combination of $A(z)$ and $B(z)$, such that*

$$\langle F(z), L(w_r, z)\rangle_{\mathcal{F}_R(P)} = \sum_{s\neq r, s=1}^{R} \bar{u}_s \csc \pi(x_r - x_s)$$

and

$$Q(w_r, w_r) \le \frac{1}{\delta_r}$$

for $r = 1, \cdots, R$, where δ_r is given in (1.4), where $w_1, \cdots, w_R$ are zeros of $C(z)$, and where B, A, P, D, C, L, Q, F are given in (1.5)–(1.12), respectively?

Theorem. *If the question has an affirmative answer, then we have*

$$|\sum_{r\neq s} u_r\bar{u}_s \csc\pi(x_r - x_s)| \le \sum_{r=1}^{R}\delta_r^{-1}|u_r|^2$$

for all complex numbers $u_1, \cdots, u_R$.

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2. Preliminary results

Lemma 2 ([3, Theorem 2]). *Let P be given in (1.7). Then the identity*

$$\|F\|^2_{\mathcal{F}_R(P)} = \sum_{j=1}^{R} |F(\alpha_j)|^2 K(\alpha_j, \alpha_j)^{-1}$$

holds for every element F in $\mathcal{F}_R(P)$.

Proof. Let $K(w, z)$ be the reproducing kernel function for the space $\mathcal{F}_R(P)$. Then we can write

$$K(w, z) = 2\frac{B(z)\bar{A}(w) - A(z)\bar{B}(w)}{i(1 - \bar{w}z)}$$

by (1.1) and (1.7). Since $\alpha_1, \cdots, \alpha_R$ are distinct complex numbers on the unit circle, we have

$$K(\alpha_r, \alpha_s) = 0 \tag{2.1}$$

for $r \neq s$,

$$K(\alpha_r, \alpha_r) = 2i\alpha_r B'(\alpha_r)\bar{A}(\alpha_r), \tag{2.2}$$

and

$$K(\alpha_r, z) = -2i\bar{A}(\alpha_r)\frac{B(z)}{1 - \bar{\alpha}_r z}. \tag{2.3}$$

We have

$$\left\langle \frac{B(z)}{1 - \bar{\alpha}_r z}, \frac{B(z)}{1 - \bar{\alpha}_r z} \right\rangle_{\mathcal{F}_R(P)} = \frac{K(\alpha_r, \alpha_r)}{4|A(\alpha_r)|^2} \tag{2.4}$$

by the definition of the reproducing kernel function $K(w, z)$ for the space $\mathcal{F}_R(P)$. Since

$$F(z) = -\sum_{r=1}^{R} \frac{F(\alpha_r)}{\alpha_r B'(\alpha_r)} \frac{B(z)}{1 - \bar{\alpha}_r z}$$

for every element F in $\mathcal{F}_R(P)$, by (2.1)–(2.4) we have

$$\begin{aligned}\|F(z)\|^2_{\mathcal{F}_r(P)} &= \sum_{r=1}^{R} \frac{|F(\alpha_r)|^2}{|B'(\alpha_r)|^2} \frac{K(\alpha_r, \alpha_r)}{4|A(\alpha_r)|^2} \\ &= \sum_{r=1}^{R} |F(\alpha_r)|^2 K(\alpha_r, \alpha_r)^{-1}\end{aligned}$$

for every element F in $\mathcal{F}_R(P)$. □

Remark. The above proof of Lemma 2 was suggested by the referee.

Lemma 3 ([3, Theorem 3]). *Let P be given in (1.7), and let C and D be given in (1.9) and (1.8). Then we have*

$$\|\widetilde{F}(z)\|_{\mathcal{F}_R(Q)} \leq \|F(z)\|_{\mathcal{F}_R(P)}$$

for every element F in $\mathcal{F}_R(P)$, where $\widetilde{F}$ is the finite Hilbert transformation of F associated with a polynomial S and $Q(z) = D(z) + iC(z)$.

Proof. By the proof of Theorem 3 in [3], we have

$$Q(w,w) = \langle L(w,z), L(w,z)\rangle_{\mathcal{F}_R(P)}.$$

It then follows from Lemma 2 that

$$Q(w,w) = \sum_{j=1}^{R} |L(w,\alpha_j)|^2 K(\alpha_j,\alpha_j)^{-1}.$$

We know that

$$L(w,\alpha_j) = 2\frac{\bar{S}(w)S(\alpha_j) - \bar{D}(w)A(\alpha_j)}{1-\bar{w}\alpha_j}$$

for $j = 1, \cdots, R$. If $Q(w,w) \equiv 0$ for all w, then $L(w,\alpha_j) \equiv 0$ for all j. This implies that $S(z) = \eta D(z)$ for a nonzero constant η, and hence we have

$$L(w,\alpha_j) = 2\bar{D}(w)\frac{\bar{\eta}S(\alpha_j) - A(\alpha_j)}{1-\bar{w}\alpha_j}$$

for all complex w and for $j = 1, \cdots, R$. Since $D(w) \not\equiv 0$, we have $\bar{\eta}S(\alpha_j) = A(\alpha_j)$ for $j = 1, \cdots, R$. It follows that $S(z) = \frac{1}{\bar{\eta}}A(z) + \mu B(z)$ for a constant μ. This contradicts the choice of $S(z)$, which is not a linear combination of $A(z)$ and $B(z)$. Therefore, we have $Q(w,w) \not\equiv 0$. If C and D were linearly dependent, then by (1.11) we would have $Q(w,w) \equiv 0$ for all w. This derives a contradiction, and therefore C and D are linearly independent. By Theorem 3 of [3], a space $\mathcal{F}_R(Q)$ exists. The stated inequality then follows from Theorem 3 of [3]. □

Lemma 4. *Let D, C and F be given in (1.8), (1.9) and (1.12). Assume that S is a polynomial associated with the space $\mathcal{F}_R(P)$ with P being given in (1.7). Then*

$$\widetilde{F}(w_r) = 2\sum_{j=1}^{R} \delta_j^{-1} u_j \frac{S(w_r)\bar{S}(\alpha_j) - D(w_r)\bar{A}(\alpha_j)}{1-\bar{\alpha}_j w_r}$$

for $r = 1, \cdots, R$, where $w_1, \cdots, w_R$ are the zeros of $C(z)$.

Proof. By (1.2), (1.10) and Lemma 2 we have

$$\widetilde{F}(w_r) = \sum_{j=1}^{R} u_j \frac{\bar{L}(w_r,\alpha_j)}{K(\alpha_j,\alpha_j)}.$$

Since

$$p_r = \frac{\delta_r}{4|B'(\alpha_r)|^2},$$

we have

$$K(\alpha_r,\alpha_r) = \delta_r$$

for $r = 1, \cdots, R$, and hence we have

$$\widetilde{F}(w_r) = \sum_{j=1}^{R} \delta_j^{-1} u_j \bar{L}(w_r,\alpha_j).$$

Since

$$\bar{L}(w_r,\alpha_j) = 2\frac{S(w_r)\bar{S}(\alpha_j) - D(w_r)\bar{A}(\alpha_j)}{1-\bar{\alpha}_j w_r},$$

the stated identity follows. □

Lemma 5. *Let S be a polynomial of degree R. Then we can write*

$$S(\alpha) = 2i \sum_{j=1}^{R} \delta_j^{-1} S(\alpha_j) \bar{A}(\alpha_j) \frac{\alpha B(\alpha)}{\alpha - \alpha_j} + \frac{B(\alpha)}{B(0)} S(0).$$

Proof. Since both sides of the stated identity are polynomials of degree R, and since they are equal at $z = 0, \alpha_1, \cdots, \alpha_R$, they must be the same polynomial. □

Lemma 6. *Let C be given in (1.9). Then all zeros of $C(z)$ are simple and lie on the unit circle.*

Proof. Let $Q(z) = D(z) + iC(z)$. Since $|Q^*(z)| < |Q(z)|$ for $|z| < 1$, zeros of $D(z)$ and $C(z)$ must lie on the unit circle. In fact, if $C(\beta) = 0$, then $Q^*(\beta) = D^*(\beta) - iC^*(\beta) = D(\beta) - iC(\beta) = D(\beta) = Q(\beta)$. By the inequality $|Q^*(z)| < |Q(z)|$ for all $|z| < 1$, we have $|\beta| \geq 1$. Suppose that $|\beta| > 1$. Then we must have $Q(\beta, \beta) = 0$ because, otherwise, we would have

$$0 < Q(\beta, \beta) = \frac{|Q(\beta)|^2 - |Q^*(\beta)|^2}{1 - |\beta|^2} = 0$$

where $Q(w, z)$ is given in (1.11). Since

$$Q(\beta, \beta) = \langle Q(\beta, z), Q(\beta, z) \rangle_{\mathcal{F}_R(Q)},$$

the identity $Q(\beta, \beta) = 0$ implies that $Q(\beta, z) \equiv 0$ for all z. It follows from the identity

$$Q(\beta, z) = \frac{-2C(z)\bar{D}(\beta)}{i(1 - \bar{\beta} z)}$$

that $D(\beta) = 0$, and hence $Q(\beta) = 0$. Since $Q^*(\beta) = Q(\beta)$, we have $Q^*(\beta) = 0$. Since $Q^*(\beta) = \beta^R \bar{Q}(1/\bar{\beta})$, we have $Q(1/\bar{\beta}) = 0$. Since $1/|\bar{\beta}| < 1$, it follows from the inequality $|Q^*(z)| < |Q(z)|$ for all $|z| < 1$ that $Q(1/\bar{\beta}) \neq 0$. This derives a contradiction, and therefore we must have $|\beta| = 1$. Similarly, we can show that all zeros of $D(z)$ lie on the unit circle. Let $w_1, \cdots, w_R$ be the zeros of $C(z)$.

We first show that those zeros of $C(z)$ that are not zeros of $Q(z)$ are simple. Since $\mathcal{F}_R(Q)$ is a nontrivial space, we have $Q(w_r, w_r) > 0$ for those w_r's that are not zeros of $Q(z)$. Otherwise, if $Q(w_r, w_r) = 0$ for such a w_r, then the identity

$$Q(w_r, w_r) = \langle Q(w_r, z), Q(w_r, z) \rangle_{\mathcal{F}_R(Q)}$$

implies that $Q(w_r, z) \equiv 0$ for all complex z. Since

$$Q(w_r, z) = \frac{Q(z)\bar{Q}(w_r) - Q^*(z)\bar{w}_r^R Q(w_r)}{1 - \bar{w}_r z}$$

if $Q(w_r) \neq 0$, then the identity $Q(w_r, z) \equiv 0$ implies $Q^*(z) = \ell Q(z)$ for constant ℓ of absolute value one. It follows that

$$Q(w, z) = \frac{Q(z)\bar{Q}(w) - Q^*(z)\bar{Q}^*(w)}{1 - \bar{w} z} \equiv 0$$

for all complex w and z. This contradicts that $\mathcal{F}_R(Q)$ is a nontrivial space, and therefore we must have $Q(w_r) = 0$. This again contradicts that w_r is not a zero of $Q(z)$. Hence we have $Q(w_r, w_r) > 0$. Since $Q(w_r, w_r) > 0$, it follows from (1.11) that $C'(w_r) \neq 0$, and hence w_r is a simple zero of $Q(z)$.

We claim that zeros of $A(z)$ are simple. Let z_0 be any zero of $A(z)$. Then $|z_0| = 1$. We must have $K(z_0, z_0) > 0$. Otherwise, it follows from the identity

$$K(z_0, z_0) = \langle K(z_0, z), K(z_0, z)\rangle_{\mathcal{F}_R(P)}$$

that $K(z_0, z) \equiv 0$ for all z. Since

$$K(z_0, z) = \frac{P(z)\bar{P}(z_0) - P^*(z)\bar{z}_0^R P(z_0)}{1 - \bar{z}_0 z}$$

if $P(z_0) \neq 0$, then the identity $K(z_0, z) \equiv 0$ implies $P^*(z) = \epsilon P(z)$ for a constant ϵ of absolute value one. It follows that

$$K(w, z) = \frac{P(z)\bar{P}(w) - P^*(z)\bar{P}^*(w)}{1 - \bar{w}z} \equiv 0$$

for all complex w and z. This contradicts that $\mathcal{F}_R(P)$ is a nontrivial space. Therefore we must have $P(z_0) = 0$, and hence $B(z_0) = 0$. This means that $z_0 = \alpha_{j_0}$ for some index j_0. But, we know that $A(\alpha_{j_0}) \neq 0$. This derives a contradiction. Thus we have $K(z_0, z_0) > 0$. By using the expression

$$K(w, z) = 2\frac{B(z)\bar{A}(w) - A(z)\bar{B}(w)}{i(1 - \bar{w}z)},$$

we find that $A'(z_0) \neq 0$. Hence, zeros of $A(z)$ are simple.

Next, we show that those zeros of $C(z)$ that are zeros of $Q(z)$ are simple. Suppose that w_r is such a zero of $C(z)$ of multiplicity at least two. Let θ be any real number. By the proof of Theorem 3 in [3], the identity

$$Re\,\frac{e^{i\theta}[D(z) + iC(z)] + e^{-i\theta}[D(z) - iC(z)]}{e^{i\theta}[A(z) - iB(z)] - e^{-i\theta}[A(z) + iB(z)]} = \sum q_j \frac{1 - \bar{z}z}{|1 - \bar{\rho}_j z|^2}\left|\frac{S(\rho_j)}{P(\rho_j)}\right|^2$$

holds for all complex z, where $q_j = |P(\rho_j)|^2/K(\rho_j, \rho_j)$ and where the sum runs over all complex ρ_j satisfying $e^{i\theta}P(\rho_j) = e^{-i\theta}P^*(\rho_j) \neq 0$. Choose $\theta = \pi/2$. Then the above identity becomes

$$Re\,\frac{iC(z)}{A(z)} = \sum_{j=1}^{R} \frac{1 - \bar{z}z}{|1 - \bar{\rho}_j z|^2}\frac{|S(\rho_j)|^2}{K(\rho_j, \rho_j)}$$

for all complex z, where the sum runs over all zeros ρ_j of $A(z)$. It follows from this identity that $A(z)$ and $C(z)$ have no common zeros. We rewrite the identity as

$$\frac{1}{1 - |z|^2} Re\,\frac{iC(z)}{A(z)} = \sum_{j=1}^{R} \frac{1}{|1 - \bar{\rho}_j z|^2}\frac{|S(\rho_j)|^2}{K(\rho_j, \rho_j)}.$$

Let $z = w_r$. Since $A(z)$ has R distinct zeros, and since $S(z)$ is not a constant multiple of $A(z)$, we have

$$\lim_{z\to w_r} \frac{1}{1 - |z|^2} Re\,\frac{iC(z)}{A(z)} = \sum_{j=1}^{R} \frac{1}{|1 - \bar{\rho}_j w_r|^2}\frac{|S(\rho_j)|^2}{K(\rho_j, \rho_j)} > 0.$$

On the other hand, since w_r is a zero of $C(z)$ of multiplicity at least two and since $A(w_r) \neq 0$, we have

$$\lim_{z\to w_r} \frac{1}{1 - |z|^2} Re\,\frac{iC(z)}{A(z)} = 0.$$

This derives a contradiction. Therefore, those zeros of $C(z)$ that are zeros of $Q(z)$ are also simple. This completes the proof of the lemma. □

3. Proof of Theorem

Proof of Theorem. Let F be given in (1.12). Since

$$p_r = \frac{\delta_r}{4|B'(\alpha_r)|^2},$$

we have

$$K(\alpha_r, \alpha_r) = \delta_r$$

for $r = 1, 2, \cdots, R$. By Lemma 2 we have

$$\|F\|^2_{\mathcal{F}_R(P)} = \sum_{r=1}^{R} \delta_r^{-1} |u_r|^2. \tag{3.1}$$

By Cauchy's inequality we have

$$|\sum_{r\neq s} u_r \bar{u}_s \csc \pi(x_r - x_s)|^2 \leq (\sum_{r=1}^{R} \delta_r^{-1}|u_r|^2)(\sum_{r=1}^{R} \delta_r | \sum_{s\neq r, s=1}^{R} \bar{u}_s \csc \pi(x_r - x_s)|^2). \tag{3.2}$$

If the question has an affirmative answer, then we should have

$$\begin{aligned} \sum_{r=1}^{R} \delta_r | \sum_{s\neq r, s=1}^{R} \bar{u}_s \csc \pi(x_r - x_s)|^2 &= \sum_{r=1}^{R} \delta_r |\widetilde{F}(w_r)|^2 \\ \leq \sum_{r=1}^{R} \frac{|\widetilde{F}(w_r)|^2}{Q(w_r, w_r)} = \|\widetilde{F}\|^2_{\mathcal{F}_R(Q)} &\leq \|F\|^2_{\mathcal{F}_R(P)} = \sum_{r=1}^{R} \delta_r^{-1}|u_r|^2 \end{aligned} \tag{3.3}$$

by Lemma 2, Lemma 3, and (3.1). It follows from (3.2) and (3.3) that

$$|\sum_{r\neq s} u_r \bar{u}_s \csc \pi(x_r - x_s)| \leq \sum_{r=1}^{R} \delta_r^{-1}|u_r|^2$$

for any set of complex numbers $u_1, u_2, \cdots, u_R$. This completes the proof of the theorem. □

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Department of Mathematics, Brigham Young University, Provo, Utah 84602
E-mail address: xianjin@math.byu.edu