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"BEURLING TYPE" SUBSPACES OF $L^p(\mathbf{T}^2)$ AND $H^p(\mathbf{T}^2)$

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ABSTRACT. In this note we extend the "Beurling type" characterizations of subspaces of $L^2(\mathbf{T}^2)$ and $H^2(\mathbf{T}^2)$ to $L^p(\mathbf{T}^2)$ and $H^p(\mathbf{T}^2)$, respectively.

1. Introduction

In [1], Beurling characterized all of the subspaces (closed linear manifolds) of $H^2(\mathbf{T})$ invariant under multiplication by the coordinate function. Later, Helson and Lowdenslager proved a "Beurling type" result for $L^2(\mathbf{T})$; see [3]. The above mentioned results have been extended to $H^p(\mathbf{T})$ and $L^p(\mathbf{T})$, respectively (see [3]). There is still no characterization of all of the subspace of $H^2(\mathbf{T}^2)$ invariant under multiplication by each of the coordinate functions. In the direction of finding descriptions of all of the subspaces of $H^2(\mathbf{T}^2)$ invariant under multiplication by each of the coordinate functions, Mandrekar [4] found necessary and sufficient conditions for a subspace of $H^2(\mathbf{T}^2)$ invariant under multiplication by each of the coordinate functions to be of "Beurling type". Later, Ghatage and Mandrekar [2] proved a "Beurling type" result in $L^2(\mathbf{T}^2)$. In this note, we extend Ghatage and Mandrekar's "Beurling type" result to $L^p(\mathbf{T}^2)$. As a corollary, we get an $H^p(\mathbf{T}^2)$ result. We follow the procedure given for the one variable case found in [3]. We point out later in this note where the procedure breaks down and how we can fix it.

2. Notation and terminology

We let $\mathbf{C}^2$ denote the cartesian product of two copies of $\mathbf{C}$. The unit bidisc in $\mathbf{C}^2$ is denoted by $\mathbf{U}^2$ and the distinguished boundary by $\mathbf{T}^2$, where $\mathbf{U}$ and $\mathbf{T}$ are the unit disc and unit circle in the complex plane, respectively.

The Hardy space $H^p(\mathbf{U}^2)$ $(1 \leq p < \infty)$ is the Banach space of holomorphic functions over $\mathbf{U}^2$ that satisfy the inequality

$$\sup_{0\leq r<1} \int_{\mathbf{T}^2} |f(r\xi_1, r\xi_2)|^p \, dm_2(\xi_1, \xi_2) < \infty$$

where m_2 denotes normalized Lebesgue measure on $\mathbf{T}^2$. Note, holomorphic here means holomorphic in each variable. The norm $\|f\|_p$ of a function f in $H^p(\mathbf{U}^2)$ is defined by

$$\|f\|_p = \sup_{0\leq r<1} \left(\int_{\mathbf{T}^2} |f(r\xi_1, r\xi_2)|^p \, dm_2(\xi_1, \xi_2) \right)^{1/p}.$$

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The Hardy space $H^\infty(\mathbf{U}^2)$ is the Banach space of holomorphic functions over $\mathbf{U}^2$ that satisfy the inequality

$$\sup_{(z_1,z_2)\in\mathbf{U}^2} |f(z_1,z_2)| < \infty.$$

The norm $\|f\|_\infty$ of a function f in $H^\infty(\mathbf{U}^2)$ is defined by

$$\|f\|_\infty = \sup_{(z_1,z_2)\in\mathbf{U}^2} |f(z_1,z_2)|.$$

It is well known (see [6]) that every function in $H^p(\mathbf{U}^2)$ $(1 \le p \le \infty)$ has a nontangential limit at $[m_2]$ almost every point of $\mathbf{T}^2$. Let f^* denote the boundary function of an f in $H^p(\mathbf{U}^2)$. Then

$$f^* \in H^p(\mathbf{T}^2) \equiv \overline{span}^{L^p(\mathbf{T}^2,m_2)}\Big\{\xi_1^n\xi_2^m : n,m \ge 0\Big\}.$$

It is also known (see [6]) that f can be reconstructed by the Poisson integral as well as the Cauchy integral of f^*. Furthermore,

$$\|f\|_p = \|f^*\|_p$$

where the second norm is the $L^p(\mathbf{T}^2,m_2)$ norm. For this reason, we identify $H^p(\mathbf{U}^2)$ and $H^p(\mathbf{T}^2)$ and no longer distinguish between f and f^*. Therefore, these Banach spaces of holomorphic functions $H^p(\mathbf{U}^2)$ may be viewed as a subspace[1] of $L^p(\mathbf{T}^2,m_2)$.

For f in $L^p(\mathbf{T}^2) = L^p(\mathbf{T}^2,m_2)$, S_1 and S_2 will denote the operators of multiplication by the first and second coordinate functions, respectively. That is,

$$S_1(f)(z_1,z_2) = z_1 f(z_1,z_2)$$

and

$$S_2(f)(z_1,z_2) = z_2 f(z_1,z_2).$$

3. Main results

We start this section by giving the aforementioned theorem of Ghatage and Mandrekar and a corollary which was previously proved by Mandrekar alone.

Theorem 1 (Ghatage & Mandrekar [2]). *Let $\mathcal{M} \ne \{0\}$ be a subspace of $L^2(\mathbf{T}^2)$ invariant under S_1 and S_2. Then, $\mathcal{M} = qH^2(\mathbf{T}^2)$ with q unimodular if and only if S_1 and S_2 are doubly commuting shifts on $\mathcal{M}$.*

Here, S_1 **doubly commuting** with S_2 means S_1 commutes with S_2 and S_1 commutes with S_2^* (S_1 commuting with S_2^* is equivalent to S_1^* commuting with S_2). We say S_1 and S_2 act as **shifts** on $\mathcal{M}$ if $\bigcap_{n=0}^\infty S_k^n(\mathcal{M}) = \{0\}$ for $k = 1,2$.

Corollary 1 (Mandrekar [4]). *Let $\mathcal{M} \ne \{0\}$ be a subspace of $H^2(\mathbf{T}^2)$ invariant under S_1 and S_2. Then, $\mathcal{M} = qH^2(\mathbf{T}^2)$ with q inner if and only if S_1 and S_2 are doubly commuting on $\mathcal{M}$.*

In this note, we prove the following two results.

Theorem 2. *Let $\mathcal{M} \ne \{0\}$ be a subspace of $L^p(\mathbf{T}^2)$, $1 \le p < 2$, invariant under S_1 and S_2. Then $\mathcal{M} = qH^p(\mathbf{T}^2)$ where q is a unimodular function if and only if S_1 and S_2 are doubly commuting shifts on $\mathcal{M} \cap L^2(\mathbf{T}^2)$.*

[1] star-closed when $p = \infty$.

We use the notation $H^p_o(\mathbf{T}^2) = \Big\{f \in H^p(\mathbf{T}^2) : \hat{f}(0,0) = 0\Big\}$ in the next theorem.

Theorem 3. *Let $\mathcal{M} \neq \{0\}$ be a subspace*[2] *of $L^p(\mathbf{T}^2)$, $2 < p \leq \infty$, invariant under S_1 and S_2. Then $\mathcal{M} = qH^p_o(\mathbf{T}^2)$ where q is a unimodular function if and only if S_1 and S_2 are doubly commuting shifts on $A(\mathcal{M}) \cap L^2(\mathbf{T}^2)$.*

Here, $A(\mathcal{M})$ means the annihilator of $\mathcal{M}$. That is, $A(\mathcal{M}) = \{f \in L^{\frac{p}{p-1}}(\mathbf{T}^2) : \int_{\mathbf{T}^2} fg\, dm_2 = 0, \forall g \in \mathcal{M}\}$.

We then get the following two corollaries, which follow directly from these two theorems.

Corollary 2. *Let $\mathcal{M} \neq \{0\}$ be a subspace of $H^p(\mathbf{T}^2)$, $1 \leq p < 2$, invariant under S_1 and S_2. Then $\mathcal{M} = qH^p(\mathbf{T}^2)$ where q is an inner function if and only if S_1 and S_2 are doubly commuting on $\mathcal{M} \cap H^2(\mathbf{T}^2)$.*

Corollary 3. *Let $\mathcal{M} \neq \{0\}$ be a subspace*[3] *of $H^p(\mathbf{T}^2)$, $2 < p \leq \infty$, invariant under S_1 and S_2. Then $\mathcal{M} = qH^p_o(\mathbf{T}^2)$ where q is an inner function if and only if S_1 and S_2 are doubly commuting shifts on $A(\mathcal{M}) \cap L^2(\mathbf{T}^2)$.*

4. Where the one-variable proof breaks down

If we just try to follow the one-variable proof given in [3], we run into the following problem. The author uses the fact that a real-valued harmonic function in the unit disc is always the real part of a holomorphic function in the unit disc. This is not always the case in the polydisc. There are simple examples given in [6] that show that not every real-valued harmonic function in the polydisc is the real part of a holomorphic function in the polydisc. Our aim in this section is to overcome this problem.

We let $RP(\mathbf{U}^2)$ denote the class of all functions in $\mathbf{U}^2$ that are the real parts of holomorphic functions.

Theorem 4 (Rudin [6])**.** *Suppose f is a lower semicontinuous* (*l.s.c.*) *positive function on $\mathbf{T}^2$ and $f \in L^1(\mathbf{T}^2)$. Then there exists a singular* (*complex Borel*) *measure σ on $\mathbf{T}^2$, $\sigma \geq 0$, such that $P[f - d\sigma] \in RP(\mathbf{U}^2)$.*

In the above theorem, $P[f - d\sigma]$ stands for the Poisson Integral of $f - d\sigma$.

To prove our main results, we need a variation of this theorem, which proves to be a corollary.

Corollary 4. *Suppose f is real-valued on $\mathbf{T}^2$ and $f \in L^1(\mathbf{T}^2)$. Then there exists a singular* (*complex Borel*) *measure σ on $\mathbf{T}^2$, such that $P[f - d\sigma] \in RP(\mathbf{U}^2)$.*

We use the following lemma to prove this corollary.

Lemma 1. *Suppose f is real-valued on $\mathbf{T}^2$ and $f \in L^p(\mathbf{T}^2)$ for $1 \leq p < \infty$. Then there exists two positive l.s.c. functions g_1 and g_2 in $L^p(\mathbf{T}^2)$ such that $f = g_1 - g_2$ a.e. on $\mathbf{T}^2$.*

We only need this lemma for the case $p = 1$, but it is no more difficult to prove it for $1 \leq p < \infty$.

[2]Assume further star-closed when $p = \infty$.
[3]Assume further star-closed when $p = \infty$.

Proof. Since f is real-valued on $\mathbf{T}^2$, $f \in L^p(\mathbf{T}^2)$ and continuous functions are dense in $L^p(\mathbf{T}^2)$ there exists ϕ_1 continuous such that

$$\|f - \phi_1\|_p < 2^{-1},$$

and by the reverse triangle inequality we get

$$\|\phi_1\|_p < \Big(1 + 2\|f\|_p\Big) \cdot 2^{-1}.$$

Now we can find ϕ_2 continuous such that

$$\Big\|(f - \phi_1) - \phi_2\Big\|_p < 2^{-2},$$

and by the reverse triangle inequality we get

$$\|\phi_2\|_p < 2^{-2} + \|f - \phi_1\|_p < 3 \cdot 2^{-2}.$$

Continuing in this manner we get the existence of a sequence of real-valued continuous functions $(\phi_n)_n$ such that

$$f = \sum_{n=1}^{\infty} \phi_n$$

in $L^p(\mathbf{T}^2)$ and

$$\|\phi_n\|_p < C \cdot 2^{-n} \quad \text{for all } n, \text{ where } C = \max\Big\{1 + 2\|f\|_p \ , \ 3\Big\}.$$

Now, for $\epsilon > 0$, define

$$\psi_n^+ = (\phi_n \vee 0) + \epsilon \cdot 2^{-n}$$

and

$$\psi_n^- = (-\phi_n \vee 0) + \epsilon \cdot 2^{-n}.$$

Then ψ_n^+ and ψ_n^- are positive continuous functions with $\phi_n = \psi_n^+ - \psi_n^-$. So

$$f = \sum_{n=1}^{\infty}(\psi_n^+ - \psi_n^-) = \sum_{n=1}^{\infty} \psi_n^+ - \sum_{n=1}^{\infty} \psi_n^- \quad \text{in } L^p(\mathbf{T}^2).$$

Since

$$\begin{aligned} \sum_{n=1}^{\infty} \|\psi_n^+\|_p \quad &\leq \quad \sum_{n=1}^{\infty}(\|\phi_n \vee 0\|_p + \epsilon \cdot 2^{-n}) \ \leq \ \sum_{n=1}^{\infty}(\|\phi_n\|_p + \epsilon \cdot 2^{-n}) \\ &< \quad \sum_{n=1}^{\infty}(C \cdot 2^{-n} + \epsilon \cdot 2^{-n}) \ < \ \infty, \end{aligned}$$

we get that there exists a g_1 in $L^p(\mathbf{T}^2)$ such that

$$g_1 = \sum_{n=1}^{\infty} \psi_n^+ \quad \text{in } L^p(\mathbf{T}^2).$$

Similarly, we get that there exists a g_2 in $L^p(\mathbf{T}^2)$ such that

$$g_2 = \sum_{n=1}^{\infty} \psi_n^- \quad \text{in } L^p(\mathbf{T}^2).$$

So we have that

$$f = g_1 - g_2 \quad \text{in } L^p(\mathbf{T}^2).$$

It is left to show that g_1 and g_2 are equal to positive *l.s.c.* functions *a.e.* Let

$$s_n = \sum_{k=1}^{n} \psi_k^+.$$

Since s_n converges to g_1 in $L^p(\mathbf{T}^2)$, there exists a subsequence that converges to g_1 *a.e.* But since s_n is monotone increasing, we get that s_n converges to g_1 *a.e.* and further that $\sup s_n = \lim s_n$. We conclude that $\sup s_n$ is *l.s.c.* since the sup of a sequence of continuous functions is *l.s.c.* It is clear that $\sup s_n$ is positive. Therefore, g_1 is equal to a positive *l.s.c.* function *a.e.* Similarly, we get that g_2 is equal to a positive *l.s.c.* function *a.e.* So f is equal *a.e.* to the difference of two positive *l.s.c.* functions. □

We now prove Corollary 4.

Proof. If f is real-valued on $\mathbf{T}^2$ and $f \in L^1(\mathbf{T}^2)$, then Lemma 1 asserts the existence of two positive *l.s.c.* functions g_1 and g_2 in $L^1(\mathbf{T}^2)$ such that $f = g_1 - g_2$ *a.e.* By Theorem 4 there exist nonnegative singular measures σ_1 and σ_2 such that $P[g_1 - d\sigma_1]$ and $P[g_2 - d\sigma_2]$ are in $RP(\mathbf{U}^2)$. Letting $\sigma = \sigma_1 - \sigma_2$ we get a singular measure such that

$$\begin{aligned} P[f - d\sigma] &= P[(g_1 - g_2) - d(\sigma_1 - \sigma_2)] \\ &= P[(g_1 - d\sigma_1) - (g_2 - d\sigma_2)] \\ &= P[g_1 - d\sigma_1] - P[g_2 - d\sigma_2]. \end{aligned}$$

So, $P[f - d\sigma]$ is in $RP(\mathbf{U}^2)$. □

5. Proof of main results

We now prove Theorem 2.

Proof. Let N denote $\mathcal{M} \cap L^2(\mathbf{T}^2)$. Then N is a (closed) invariant subspace of $L^2(\mathbf{T}^2)$ and by hypothesis S_1 and S_2 are doubly commuting shifts on N. Therefore, by Theorem 1, $N = qH^2(\mathbf{T}^2)$ where q is a unimodular function. Now since N is contained in $\mathcal{M}$ and $\mathcal{M}$ is closed, the closure of N in $L^p(\mathbf{T}^2)$, which is $qH^p(\mathbf{T}^2)$, is contained in $\mathcal{M}$. So we need to show that N is dense in $\mathcal{M}$. To do this, let $f \in \mathcal{M}$, f not identically zero. Then define

$$u_n = \begin{cases} 0, & |f| \le n, \\ \log|f|^{-1}, & |f| > n. \end{cases}$$

Note that $u_n \in L^p(\mathbf{T}^2)$ for all n since

$$\begin{aligned} \int |u_n|^p \, dm &= \int_{|f|>n} |\log|f|^{-1}|^p \, dm = \int_{|f|>n} |\log|f||^p \, dm \\ &\le \int_{|f|>n} |f|^p \, dm \le \|f\|_p^p < \infty. \end{aligned}$$

So in particular, $u_n \in L^1(\mathbf{T}^2)$ and is real valued for all n. So by Corollary 4, there exists a sequence $\{\sigma_n\}_{n\ge 0}$ of singular measures such that $P[u_n - d\sigma_n] \in RP(U^2)$ for all n. So there exists a sequence of analytic functions $(F_n)_n$ such that $Re(F_n) = P[u_n - d\sigma_n]$. By the M. Riesz theorem, which holds on the polydisc (see [5]), we have $\|F_n\|_p \le C_p \|u_n\|_p$ for all n. Now since $u_n \in L^p(\mathbf{T}^2)$ and u_n converges to 0 in

$L^p(\mathbf{T}^2)$, we get that F_n converges to 0 in $L^p(\mathbf{T}^2)$, and hence at least a subsequence converges to zero *a.e.* Let $\phi_n = \exp\{F_n\}$. Then

$$|\phi_n| = \begin{cases} 1, & |f| \le n, \\ |f|^{-1}, & |f| > n, \end{cases}$$

and ϕ_n tends to the constant function 1. By construction, $\phi_n f$ is a bounded function dominated by f for all n. Also, $\phi_n f \in \mathcal{M}$ because ϕ_n is bounded analytic and hence is boundedly the limit of analytic trigonometric polynomials. Since $\phi_n f$ is bounded, it is in N. As n goes to infinity, $\phi_n f$ converges to f in $L^p(\mathbf{T}^2)$ by the dominated convergence theorem. So each f in $\mathcal{M}$ is the limit of functions from N. So N is dense in $\mathcal{M}$ as desired.

Conversely, if $\mathcal{M} = qH^p(\mathbf{T}^2)$ with q unimodular, then $\mathcal{M} \cap L^2(\mathbf{T}^2) = qH^2(\mathbf{T}^2)$. So S_1 and S_2 are doubly commuting shifts on $\mathcal{M} \cap L^2(\mathbf{T}^2)$ by Theorem 1. □

We finally prove Theorem 3.

Proof. If $\mathcal{M} = qH^p_o(\mathbf{T}^2)$, where q is a unimodular function, then $A(\mathcal{M}) = \overline{q}H^{\frac{p}{p-1}}(\mathbf{T}^2)$. Therefore, $A(\mathcal{M}) \cap L^2(\mathbf{T}^2) = \overline{q}H^2(\mathbf{T}^2)$. It then follows from Theorem 1 that S_1 and S_2 are doubly commuting shifts on $A(\mathcal{M}) \cap L^2(\mathbf{T}^2)$. Conversely, if S_1 and S_2 are doubly commuting shifts on $A(\mathcal{M}) \cap L^2(\mathbf{T}^2)$, then by Theorem 2 we get that $A(\mathcal{M}) = qH^{\frac{p}{p-1}}(\mathbf{T}^2)$ where q is a unimodular function. Therefore, $\mathcal{M} = \overline{q}H^p_o(\mathbf{T}^2)$ where q is a unimodular function. When $p = \infty$ we need that $\mathcal{M}$ is star-closed to make our final conclusion. □

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