"BEURLING TYPE" SUBSPACES OF $L^p(T^2)$ AND $H^p(T^2)$

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Abstract. In this note we extend the “Beurling type” characterizations of subspaces of $L^2(T^2)$ and $H^2(T^2)$ to $L^p(T^2)$ and $H^p(T^2)$, respectively.

1. Introduction

In [1], Beurling characterized all of the subspaces (closed linear manifolds) of $H^2(T)$ invariant under multiplication by the coordinate function. Later, Helson and Lowdenslager proved a “Beurling type” result for $L^2(T)$; see [3]. The above mentioned results have been extended to $H^p(T)$ and $L^p(T)$, respectively (see [3]). There is still no characterization of all of the subspace of $H^2(T^2)$ invariant under multiplication by each of the coordinate functions. In the direction of finding descriptions of all of the subspaces of $H^2(T^2)$ invariant under multiplication by each of the coordinate functions, Mandrekar [4] found necessary and sufficient conditions for a subspace of $H^2(T^2)$ invariant under multiplication by each of the coordinate functions to be of “Beurling type”. Later, Ghatage and Mandrekar [2] proved a “Beurling type” result in $L^2(T^2)$. In this note, we extend Ghatage and Mandrekar’s “Beurling type” result to $L^p(T^2)$. As a corollary, we get an $H^p(T^2)$ result. We follow the procedure given for the one variable case found in [3]. We point out later in this note where the procedure breaks down and how we can fix it.

2. Notation and terminology

We let $C^2$ denote the cartesian product of two copies of $C$. The unit bidisc in $C^2$ is denoted by $U^2$ and the distinguished boundary by $T^2$, where $U$ and $T$ are the unit disc and unit circle in the complex plane, respectively.

The Hardy space $H^p(U^2)$ ($1 \leq p < \infty$) is the Banach space of holomorphic functions over $U^2$ that satisfy the inequality

$$\sup_{0 \leq r < 1} \int_{T^2} |f(r\xi_1, r\xi_2)|^p \, dm_2(\xi_1, \xi_2) < \infty$$

where $m_2$ denotes normalized Lebesgue measure on $T^2$. Note, holomorphic here means holomorphic in each variable. The norm $\|f\|_p$ of a function $f$ in $H^p(U^2)$ is defined by

$$\|f\|_p = \sup_{0 \leq r < 1} \left( \int_{T^2} |f(r\xi_1, r\xi_2)|^p \, dm_2(\xi_1, \xi_2) \right)^{1/p}.$$
The Hardy space $H^\infty(U^2)$ is the Banach space of holomorphic functions over $U^2$ that satisfy the inequality

$$\sup_{(z_1, z_2) \in U^2} |f(z_1, z_2)| < \infty.$$  

The norm $\|f\|_\infty$ of a function $f$ in $H^\infty(U^2)$ is defined by

$$\|f\|_\infty = \sup_{(z_1, z_2) \in U^2} |f(z_1, z_2)|.$$  

It is well known (see [6]) that every function in $H^p(U^2)$ ($1 \leq p \leq \infty$) has a nontangential limit at $[m_2]$ almost every point of $T^2$. Let $f^*$ denote the boundary function of an $f$ in $H^p(U^2)$. Then

$$f^* \in H^p(T^2) \equiv \frac{\text{span}_{L^p(T^2,m_2)}}{\ell_{n,m} : n,m \geq 0}.$$  

It is also known (see [6]) that $f$ can be reconstructed by the Poisson integral as well as the Cauchy integral of $f^*$. Furthermore,\[ \|f\|_p = \|f^*\|_p \]
where the second norm is the $L^p(T^2,m_2)$ norm. For this reason, we identify $H^p(U^2)$ and $H^p(T^2)$ and no longer distinguish between $f$ and $f^*$. Therefore, these Banach spaces of holomorphic functions $H^p(U^2)$ may be viewed as a subspace of $L^p(T^2,m_2)$.

For $f$ in $L^p(T^2) = L^p(T^2,m_2)$, $S_1$ and $S_2$ will denote the operators of multiplication by the first and second coordinate functions, respectively. That is,

$$S_1(f)(z_1, z_2) = z_1 f(z_1, z_2)$$

and

$$S_2(f)(z_1, z_2) = z_2 f(z_1, z_2).$$

3. MAIN RESULTS

We start this section by giving the aforementioned theorem of Ghatage and Mandrekar and a corollary which was previously proved by Mandrekar alone.

**Theorem 1** (Ghatage & Mandrekar [2]). Let $\mathcal{M} \neq \{0\}$ be a subspace of $L^2(T^2)$ invariant under $S_1$ and $S_2$. Then, $\mathcal{M} = qH^2(T^2)$ with $q$ unimodular if and only if $S_1$ and $S_2$ are doubly commuting shifts on $\mathcal{M}$.

Here, $S_1$ **doubly commuting** with $S_2$ means $S_1$ commutes with $S_2$ and $S_1$ commutes with $S_2^*$ ($S_1$ commuting with $S_2^*$ is equivalent to $S_1^*$ commuting with $S_2$). We say $S_1$ and $S_2$ act as **shifts** on $\mathcal{M}$ if $\bigcap_{n=0}^\infty S_k^n(\mathcal{M}) = \{0\}$ for $k = 1, 2$.

**Corollary 1** (Mandrekar [4]). Let $\mathcal{M} \neq \{0\}$ be a subspace of $H^2(T^2)$ invariant under $S_1$ and $S_2$. Then, $\mathcal{M} = qH^2(T^2)$ with $q$ inner if and only if $S_1$ and $S_2$ are doubly commuting on $\mathcal{M}$.

In this note, we prove the following two results.

**Theorem 2.** Let $\mathcal{M} \neq \{0\}$ be a subspace of $L^p(T^2)$, $1 \leq p < 2$, invariant under $S_1$ and $S_2$. Then $\mathcal{M} = qH^p(T^2)$ where $q$ is a unimodular function if and only if $S_1$ and $S_2$ are doubly commuting shifts on $\mathcal{M} \cap L^2(T^2)$.

1 star-closed when $p = \infty$.  

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We use the notation $H^p_0(T^2) = \left\{ f \in H^p(T^2) : \hat{f}(0,0) = 0 \right\}$ in the next theorem.

**Theorem 3.** Let $\mathcal{M} \neq \{0\}$ be a subspace\(^2\) of $L^p(T^2)$, $2 < p \leq \infty$, invariant under $S_1$ and $S_2$. Then $\mathcal{M} = qH^p_0(T^2)$ where $q$ is a unimodular function if and only if $S_1$ and $S_2$ are doubly commuting shifts on $A(\mathcal{M}) \cap L^2(T^2)$.

Here, $A(\mathcal{M})$ means the annihilator of $\mathcal{M}$. That is, $A(\mathcal{M}) = \{ f \in \overline{L^\infty(T^2)} : \int_{T^2} fg \, dm = 0, \forall g \in \mathcal{M} \}$.

We then get the following two corollaries, which follow directly from these two theorems.

**Corollary 2.** Let $\mathcal{M} \neq \{0\}$ be a subspace of $H^p(T^2)$, $1 \leq p < 2$, invariant under $S_1$ and $S_2$. Then $\mathcal{M} = qH^p_0(T^2)$ where $q$ is an inner function if and only if $S_1$ and $S_2$ are doubly commuting on $\mathcal{M} \cap H^2(T^2)$.

**Corollary 3.** Let $\mathcal{M} \neq \{0\}$ be a subspace\(^3\) of $H^p(T^2)$, $2 < p \leq \infty$, invariant under $S_1$ and $S_2$. Then $\mathcal{M} = qH^p_0(T^2)$ where $q$ is an inner function if and only if $S_1$ and $S_2$ are doubly commuting shifts on $A(\mathcal{M}) \cap L^2(T^2)$.

4. WHERE THE ONE-_VARIABLE PROOF BREAKS DOWN

If we just try to follow the one-variable proof given in [3], we run into the following problem. The author uses the fact that a real-valued harmonic function in the unit disc is always the real part of a holomorphic function in the unit disc. This is not always the case in the polydisc. There are simple examples given in [6] that show that not every real-valued harmonic function in the polydisc is the real part of a holomorphic function in the polydisc. Our aim in this section is to overcome this problem.

We let $RP(U^2)$ denote the class of all functions in $U^2$ that are the real parts of holomorphic functions.

**Theorem 4** (Rudin [6]). Suppose $f$ is a lower semicontinuous (l.s.c.) positive function on $T^2$ and $f \in L^1(T^2)$. Then there exists a singular (complex Borel) measure $\sigma$ on $T^2$, $\sigma \geq 0$, such that $P[f - d\sigma] \in RP(U^2)$.

In the above theorem, $P[f - d\sigma]$ stands for the Poisson Integral of $f - d\sigma$.

To prove our main results, we need a variation of this theorem, which proves to be a corollary.

**Corollary 4.** Suppose $f$ is real-valued on $T^2$ and $f \in L^1(T^2)$. Then there exists a singular (complex Borel) measure $\sigma$ on $T^2$, such that $P[f - d\sigma] \in RP(U^2)$.

We use the following lemma to prove this corollary.

**Lemma 1.** Suppose $f$ is real-valued on $T^2$ and $f \in L^p(T^2)$ for $1 \leq p < \infty$. Then there exist two positive l.s.c. functions $g_1$ and $g_2$ in $L^p(T^2)$ such that $f = g_1 - g_2$ a.e. on $T^2$.

We only need this lemma for the case $p = 1$, but it is no more difficult to prove it for $1 \leq p < \infty$.

\(^2\)Assume further star-closed when $p = \infty$.

\(^3\)Assume further star-closed when $p = \infty$. 
Proof. Since $f$ is real-valued on $\mathbf{T}^2$, $f \in L^p(\mathbf{T}^2)$ and continuous functions are dense in $L^p(\mathbf{T}^2)$ there exists $\phi_1$ continuous such that
\[ \|f - \phi_1\|_p < 2^{-1}, \]
and by the reverse triangle inequality we get
\[ \|\phi_1\|_p < \left(1 + 2\|f\|_p\right) \cdot 2^{-1}. \]
Now we can find $\phi_2$ continuous such that
\[ \left\| (f - \phi_1) - \phi_2 \right\|_p < 2^{-2}, \]
and by the reverse triangle inequality we get
\[ \|\phi_2\|_p < 2^{-2} + \|f - \phi_1\|_p < 3 \cdot 2^{-2}. \]
Continuing in this manner we get the existence of a sequence of real-valued continuous functions $(\phi_n)_n$ such that
\[ f = \sum_{n=1}^{\infty} \phi_n \]
in $L^p(\mathbf{T}^2)$ and
\[ \|\phi_n\|_p < C \cdot 2^{-n} \quad \text{for all } n, \text{ where } C = \max\left\{1 + 2\|f\|_p, 3\right\}. \]
Now, for $\epsilon > 0$, define
\[ \psi^+_n = (\phi_n \lor 0) + \epsilon \cdot 2^{-n} \]
and
\[ \psi^-_n = (-\phi_n \lor 0) + \epsilon \cdot 2^{-n}. \]
Then $\psi^+_n$ and $\psi^-_n$ are positive continuous functions with $\phi_n = \psi^+_n - \psi^-_n$. So
\[ f = \sum_{n=1}^{\infty} (\psi^+_n - \psi^-_n) = \sum_{n=1}^{\infty} \psi^+_n - \sum_{n=1}^{\infty} \psi^-_n \quad \text{in } L^p(\mathbf{T}^2). \]
Since
\[ \sum_{n=1}^{\infty} \|\psi^+_n\|_p \leq \sum_{n=1}^{\infty} (\|\phi_n \lor 0\|_p + \epsilon \cdot 2^{-n}) \leq \sum_{n=1}^{\infty} (\|\phi_n\|_p + \epsilon \cdot 2^{-n}) \]
\[ < \sum_{n=1}^{\infty} (C \cdot 2^{-n} + \epsilon \cdot 2^{-n}) < \infty, \]
we get that there exists a $g_1$ in $L^p(\mathbf{T}^2)$ such that
\[ g_1 = \sum_{n=1}^{\infty} \psi^+_n \quad \text{in } L^p(\mathbf{T}^2). \]
Similarly, we get that there exists a $g_2$ in $L^p(\mathbf{T}^2)$ such that
\[ g_2 = \sum_{n=1}^{\infty} \psi^-_n \quad \text{in } L^p(\mathbf{T}^2). \]
So we have that
\[ f = g_1 - g_2 \quad \text{in } L^p(\mathbf{T}^2). \]
Proof. If $L$ is real-valued on $L^p(T^2)$, there exists a subsequence that converges to $g_1$ a.e. But since $s_n$ is monotone increasing, we get that $s_n$ converges to $g_1$ a.e. and further that sup $s_n = \lim s_n$. We conclude that sup $s_n$ is l.s.c. since the sup of a sequence of continuous functions is l.s.c. It is clear that sup $s_n$ is positive. Therefore, $g_1$ is equal to a positive l.s.c. function a.e. Similarly, we get that $g_2$ is equal to a positive l.s.c. function a.e. So $f$ is equal a.e. to the difference of two positive l.s.c. functions.

We now prove Corollary 4.

Proof. If $f$ is real-valued on $T^2$ and $f \in L^1(T^2)$, then Lemma 4 asserts the existence of two positive l.s.c. functions $g_1$ and $g_2$ in $L^1(T^2)$ such that $f = g_1 - g_2$ a.e. By Theorem 4 there exist nonnegative singular measures $\sigma_1$ and $\sigma_2$ such that $P[g_1 - d\sigma_1]$ and $P[g_2 - d\sigma_2]$ are in $RP(U^2)$. Letting $\sigma = \sigma_1 - \sigma_2$ we get a singular measure such that
\[
P(f - d\sigma) = P[(g_1 - g_2) - d(\sigma_1 - \sigma_2)] = P[(g_1 - d\sigma_1) - (g_2 - d\sigma_2)] = P[g_1 - d\sigma_1] - P[g_2 - d\sigma_2].
\]
So, $P(f - d\sigma)$ is in $RP(U^2)$. □

5. PROOF OF MAIN RESULTS

We now prove Theorem 2.

Proof. Let $N$ denote $\mathcal{M} \cap L^2(T^2)$. Then $N$ is a (closed) invariant subspace of $L^2(T^2)$ and by hypothesis $S_1$ and $S_2$ are doubly commuting shifts on $N$. Therefore, by Theorem 1 $N = qH^2(T^2)$ where $q$ is a unimodular function. Now since $N$ is contained in $\mathcal{M}$ and $\mathcal{M}$ is closed, the closure of $N$ in $L^p(T^2)$, which is $qH^p(T^2)$, is contained in $\mathcal{M}$. So we need to show that $N$ is dense in $\mathcal{M}$. To do this, let $f \in \mathcal{M}$, $f$ not identically zero. Then define
\[
u_n = \begin{cases} 0, & |f| \leq n, \\ \log |f|^{-1}, & |f| > n. \end{cases}
\]
Note that $\nu_n \in L^p(T^2)$ for all $n$ since
\[
\int |\nu_n|^p \, dm = \int_{|f| > n} |\log |f|^{-1}|^p \, dm = \int_{|f| > n} |\log |f||^p \, dm \\
\leq \int_{|f| > n} |f|^p \, dm \leq \|f\|^p < \infty.
\]
So in particular, $\nu_n \in L^1(T^2)$ and is real valued for all $n$. So by Corollary 4 there exists a sequence $\{\sigma_n\}_{n \geq 0}$ of singular measures such that $P[\nu_n - d\sigma_n] \in RP(U^2)$ for all $n$. So there exists a sequence of analytic functions $(F_n)_n$ such that $Re(F_n) = P[\nu_n - d\sigma_n]$. By the M. Riesz theorem, which holds on the polydisc (see [3]), we have $\|F_n\|_p \leq C_p \|\nu_n\|_p$ for all $n$. Now since $\nu_n \in L^p(T^2)$ and $\nu_n$ converges to $0$ in...
$L^p(T^2)$, we get that $F_n$ converges to 0 in $L^p(T^2)$, and hence at least a subsequence converges to zero a.e. Let $\phi_n = \exp\{F_n\}$. Then

$$|\phi_n| = \begin{cases} 1, & |f| \leq n, \\ |f|^{-1}, & |f| > n, \end{cases}$$

and $\phi_n$ tends to the constant function 1. By construction, $\phi_n f$ is a bounded function dominated by $f$ for all $n$. Also, $\phi_n f \in M$ because $\phi_n$ is bounded analytic and hence is boundedly the limit of analytic trigonometric polynomials. Since $\phi_n f$ is bounded, it is in $N$. As $n$ goes to infinity, $\phi_n f$ converges to $f$ in $L^p(T^2)$ by the dominated convergence theorem. So each $f$ in $M$ is the limit of functions from $N$. So $N$ is dense in $M$ as desired.

Conversely, if $M = qH^p(T^2)$ with $q$ unimodular, then $M \cap L^2(T^2) = qH^2(T^2)$. So $S_1$ and $S_2$ are doubly commuting shifts on $M \cap L^2(T^2)$ by Theorem 1. We finally prove Theorem 3.

Proof. If $M = qH^p(T^2)$, where $q$ is a unimodular function, then $A(M) = \overline{q}H^{p-1}(T^2)$. Therefore, $A(M) \cap L^2(T^2) = \overline{q}H^2(T^2)$. It then follows from Theorem 1 that $S_1$ and $S_2$ are doubly commuting shifts on $A(M) \cap L^2(T^2)$. Conversely, if $S_1$ and $S_2$ are doubly commuting shifts on $A(M) \cap L^2(T^2)$, then by Theorem 2 we get that $A(M) = qH^{p-1}(T^2)$ where $q$ is a unimodular function. Therefore, $M = \overline{q}H^p(T^2)$ where $q$ is a unimodular function. When $p = \infty$ we need that $M$ is star-closed to make our final conclusion.

References


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