

## LOCAL DUAL AND POLY-SCALE REFINABILITY

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ABSTRACT. For a compactly supported function  $f$ , let  $S_n(f)$ ,  $n \geq 0$ , be the space of all finite linear combinations of  $f(M^n \cdot -k)$ ,  $k \in \mathbf{Z}$ . In this paper, we consider the explicit construction of local duals of  $f$  and the poly-scale refinability of functions in  $S_0(f)$  when  $f$  is an  $M$ -refinable function. We show that for any  $M$ -refinable function  $f$ , there exists a local dual of  $f$  in  $S_N(f)$  for some  $N \geq 0$ , and that any function in  $S_0(f)$  is poly-scale refinable.

### 1. INTRODUCTION

Let  $L^2 := L^2(\mathbf{R})$  be the space of all square integrable functions on  $\mathbf{R}$  and  $\langle \cdot, \cdot \rangle$  be the usual inner product on  $L^2$ . For compactly supported functions  $f$  and  $g$  in  $L^2$ , we say that  $g$  is a *local dual* of  $f$  if

$$(1.1) \quad \langle f(\cdot - k), g(\cdot - k') \rangle = \delta_{kk'} \quad \forall k, k' \in \mathbf{Z},$$

where  $\delta$  is the usual Kronecker symbol. Given a compactly supported function  $f$ , we define the corresponding *semi-convolution*  $f *'$  on  $\ell$ , the space of all sequences on  $\mathbf{Z}$ , by

$$f *' : \ell \ni \{c(k)\} \mapsto \sum_{k \in \mathbf{Z}} c(k) f(\cdot - k).$$

We see that for any  $k \in \mathbf{Z}$ , the  $k$ th component of a sequence  $c$  contributes only to  $f *' c$  at the neighborhood  $k + \text{supp} f$  of the location  $k$ , and hence the semi-convolution is locally defined. If there exists a local dual  $g$  of  $f$ , then for any  $k \in \mathbf{Z}$ , the  $k$ -th component  $c(k)$  of the sequence  $c := \{c(k)\}$  can be recovered from the restriction of the semi-convolution  $f *' c$  to a neighborhood of the location  $k$  in a stable way because

$$(1.2) \quad c(k) = \langle f *' c, g(\cdot - k) \rangle \quad \forall k \in \mathbf{Z},$$

and hence the semi-convolution  $f *'$  has a local inverse. From (1.2), we see that the semi-convolution  $f *'$  is one-to-one, which is known as *linear independent shifts* of  $f$  ([10, 13]). Conversely if  $f$  has linear independent shifts, then a local dual  $g$  of  $f$  can be found ([1, 9, 14]). Thus for any given compactly supported  $L^2$  function  $f$ , the existence of its local dual in  $L^2$  is equivalent to its linear independent shifts.

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We say that a compactly supported function  $f$  is  $M$ -refinable, or *refinable* for short, if

$$(1.3) \quad f = \sum_{k \in \mathbf{Z}} a(k) f(M \cdot -k) \quad \text{and} \quad \hat{f}(0) = 1,$$

where the dilation  $M$  is a fixed integer larger than 2, and  $\{a(k)\}$  is a finitely supported sequence on  $\mathbf{Z}$  and satisfies  $\sum_{k \in \mathbf{Z}} a(k) = M$  ([2, 5, 12]). Here the Fourier transform  $\hat{f}$  of an integrable function  $f$  is defined by  $\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-ix\xi} dx$ . The sequence  $\{a(k)\}$  and the function  $H(\xi) = \frac{1}{M} \sum_{k \in \mathbf{Z}} a(k) e^{-ik\xi}$  are called the *mask* and the *symbol* of the refinable function  $f$  respectively. The first topic considered in this paper is the explicit construction of a local dual of a given refinable function. Our motivation is based on the following three easy observations: the first observation is that if  $f$  has orthonormal shifts, that is, (1.1) holds for  $g$  being replaced by  $f$ , then  $f$  itself is the local dual of  $f$ , and hence there is a local dual in the *shift-invariant space*  $S_0(f)$  generated by  $f$ ,

$$S_0(f) = \{f *' c : c \in \ell_0\},$$

where  $\ell_0$  is the space of all finitely supported sequences on  $\mathbf{Z}$ ; the second observation is that if  $f$  is  $M$ -refinable, then the spaces

$$S_n(f) = \{h(M^n \cdot) : h \in S_0(f)\}, \quad n \geq 0,$$

satisfy the following nestedness condition:

$$S_0(f) \subset S_1(f) \subset \cdots \subset S_n(f) \subset \cdots \rightarrow L^2;$$

and the third observation is that for the hat function  $h(x) = \max(1 - |x|, 0)$ , the function  $-\frac{1}{2}(h(2x-1) + h(2x+1)) + 3h(2x)$  in  $S_1(h)$  is a local dual of  $h$ , where we set  $M = 2$ . The above three observations inspire us to consider the problem whether for any refinable function  $f$  we can find its local dual in  $S_N(f)$  for some  $N \geq 0$ . An affirmative answer to the above problem for  $M = 2$  follows from the intertwining multiresolution analysis in [8]. In this paper, we give an affirmative answer to the above problem for any dilation  $M$ , and prove the following result where the local dual can be chosen from the dilation of the shifts of another refinable function.

**Theorem 1.1.** *Let  $f$  and  $g$  be compactly supported refinable functions in  $L^2$ . If  $f$  and  $g$  have linear independent shifts, then there is an  $h \in S_N(g)$  for some  $N \geq 0$  so that  $h$  is a local dual of  $f$ .*

From Theorem 1.1, we see that for any refinable function  $f \in L^2$  with linear independent shifts, a spline of order  $L$  with knots on  $\mathbf{Z}/M^N$  can be a local dual of  $f$  when  $N$  is sufficiently large. For the case that both  $f$  and  $g$  are B-splines, a function in  $S_1(g)$  can be chosen to be a local dual of  $f$  from the proof of Theorem 1.1. For instance,  $3\chi_{[0,1/2]} - \chi_{[1/2,1]}$  is a local dual of the hat function  $h$ , where we set  $M = 2$  and we denote the characteristic function on a measurable set  $E$  by  $\chi_E$ .

Given any compactly supported refinable function  $f \in L^2$  with linear independent shifts, the functions in  $S_N(f)$  have been used in the construction of orthonormal wavelets ([2, 5, 12]), tight affine frames ([3, 4, 6]), and also the local dual (Theorem 1.1). The second topic of this paper is the refinability of functions in  $S_0(f)$ . Given a refinable function  $f$ , a function  $g \in S_0(f)$  is not refinable in general. In particular, one may easily verify that  $g = \sum_{k \in \mathbf{Z}} d(k) f(\cdot - k) \in S_0(f)$  is refinable if and only if  $D(z^M)H(z)/D(z)$  is a Laurent polynomial, where  $D(z) = \sum_{k \in \mathbf{Z}} d(k) z^k$

and  $H(z) = \sum_{k \in \mathbf{Z}} a(k)z^k$  with  $\{a(k)\}$  being the mask of the refinable function  $f$ . Recently, poly-scale refinability, a weak concept of refinability, was introduced by Dekel and Dyn ([7]). For a compactly supported function  $f$ , we say that  $f$  is *poly-scale refinable* if

$$(1.4) \quad f = \sum_{n=1}^N \sum_{k \in \mathbf{Z}} c_n(k) f(M^n \cdot -k)$$

for some  $N \geq 1$ , where  $\{c_n(k)\} \in \ell_0, 1 \leq n \leq N$ . Clearly, a refinable function is poly-scale refinable, and a compactly supported function  $f$  is poly-scale refinable if and only if

$$S_0(f) \subset S_1(f) + \dots + S_N(f)$$

for some  $N \geq 1$ . In this paper, we show that given a refinable function  $f$  having linear independent shifts, any function  $g \in S_0(f)$  is poly-scale refinable.

**Theorem 1.2.** *Let  $f$  be a compactly supported refinable function. Then any function in  $S_0(f)$  is poly-scale refinable.*

Given a compactly supported refinable function  $f$  and a function  $g \in S_0(f)$ , we see from Theorem 1.2 that  $S_0(g) \subset S_1(g) + S_2(g) + \dots + S_N(g)$  for some  $N \geq 1$ . Also we notice that  $S_0(g) \subset S_0(f)$ . The above two observations inspire us to consider the following problem: whether

$$(1.5) \quad S_0(f) \subset S_1(g) + \dots + S_N(g)$$

holds for some  $N \geq 1$ .

**Theorem 1.3.** *Let  $f$  be a compactly supported refinable distribution having linear independent shifts, and let  $g = \sum_{k \in \mathbf{Z}} d(k) f(\cdot - k) \in S_0(f)$  for some nonzero sequence  $\{d(k)\} \in \ell_0$ . Assume that there does not exist  $\xi_0 \in \mathbf{R}$  so that  $d(M^n \xi_0) = 0$  for all  $n \geq 0$ , where we set  $d(\xi) := \sum_{k \in \mathbf{Z}} d(k) e^{-ik\xi}$ . Then (1.5) holds for some positive integer  $N$ .*

The condition on  $d$  in Theorem 1.3 cannot be dropped out in general. For instance, if  $g = f - f(\cdot - 1)$ , then any function  $h$  in  $S_N(g), N \geq 1$ , satisfies  $\int_{\mathbf{R}} h(x) dx = 0$ , which implies that (1.5) does not hold for any  $N \geq 1$ .

## 2. LOCAL DUAL

In this section, we give a constructive proof to the following slight generalization of Theorem 1.1.

**Theorem 2.1.** *Let  $f$  be a compactly supported refinable function in  $L^2$  having linear independent shifts, and let  $g$  be a compactly supported function in  $L^2$ , have linear independent shifts and satisfy  $\int_{\mathbf{R}} g(x) dx \neq 0$ . Then there exists a function  $h$  in  $S_N(g)$  for some  $N \geq 0$  so that  $h$  is a local dual of  $f$ .*

We remark that the condition  $\int_{\mathbf{R}} g(x) dx \neq 0$  in Theorem 2.1 cannot be dropped out in general. For instance, let  $f = \chi_{[0,1]}$  and  $g = \chi_{[-1/2,0]} - \chi_{[0,1/2]}$ . If we can find  $h \in S_N(g), N \geq 0$ , so that  $h$  is a local dual of  $f$ , then

$$(2.1) \quad \langle f(\cdot - k), h \rangle = \delta_k, \quad k \in \mathbf{Z}.$$

Write  $h = \sum_{k \in \mathbf{Z}} a_N(k) g(2^N \cdot -k)$  for some  $\{a_N(k)\} \in \ell_0$ . Then direct calculation yields

$$(2.2) \quad \langle f(\cdot - k), h \rangle = 2^{-N} a_N(2^N k) - 2^{-N} a_N(2^N(k+1)), \quad k \in \mathbf{Z}.$$

By (2.1) and (2.2), the sequence  $\{a_N(k)\}$  is not finitely supported, which is a contradiction.

We say that  $\xi_0$  is an  $m$ -symmetric root of a trigonometric polynomial  $a$  if  $a(\xi_0 + 2k\pi/m) = 0$  for all  $k \in \mathbf{Z}$ . To prove Theorem 2.1, we need a lemma about  $m$ -symmetric roots of a trigonometric polynomial.

**Lemma 2.2.** *Let  $f$  be a compactly supported refinable function having linear independent shifts, and let  $A$  be a nonzero trigonometric polynomial. Let  $H$  be the symbol of  $f$ , and define*

$$(2.3) \quad H_n(\xi) = H(M^{n-1}\xi) \cdots H(\xi)$$

for  $n \geq 1$ . Denote the set of all  $M^n$ -symmetric roots of  $H_n A$  by  $K_n$  with multiplicity being not counted, where we say that  $a$  and  $b$  are two different  $M^n$ -symmetric roots if  $a - b \notin 2M^{-n}\pi\mathbf{Z}$ . Then  $K_n \subset 2M^{-n}\pi\mathbf{Z}$  for sufficiently large  $n$ .

*Proof.* Let  $f$  satisfy the refinement equation

$$(2.4) \quad f = \sum_{k \in \mathbf{Z}} a(k)f(M \cdot -k),$$

where  $\{a(k)\} \in \ell_0$  satisfies  $\sum_{k \in \mathbf{Z}} a(k) = M$ . Taking the Fourier transform of both sides of the equation (2.4), we get

$$(2.5) \quad \hat{f}(\xi) = H(\xi/M)\hat{f}(\xi/M),$$

and using (2.5) iteratively  $n$  times, we obtain

$$(2.6) \quad \hat{f}(\xi) = H_n(\xi/M^n)\hat{f}(\xi/M^n).$$

Recall that if  $f$  has linear independent shifts, then for any complex number  $\xi_0$  there exists an integer  $k := k(\xi_0)$  so that  $\hat{f}(\xi_0 + 2k\pi) \neq 0$  ([10, 13]). This together with (2.6) concludes that

$$(2.7) \quad H_n(\xi) \text{ does not have any } M^n\text{-symmetric root for any } n \geq 1.$$

Let  $Z(A)$  be the set of all roots of  $A(\xi)$  with multiplicity being not counted. By (2.7), we have

$$(2.8) \quad K_n \subset Z(A) + 2M^{-n}\pi\mathbf{Z},$$

and hence

$$(2.9) \quad \#K_n < \infty$$

for all  $n \geq 1$ . Here  $\#E$  is the cardinality of a finite set  $E$ .

For any  $\xi_0 \in K_n$ , let  $k_1$  be an integer so chosen that  $H(M^{n-1}\xi_0 + 2k_1\pi/M) \neq 0$ , and set  $\xi'_0 = \xi_0 + 2M^{-n}k_1\pi$ . Then, since  $H_n(\xi_0 + 2M^{-n}k\pi)A(\xi_0 + 2M^{-n}k\pi) = 0$  for all  $k \in \mathbf{Z}$  and  $H_n(\xi) = H(M^{n-1}\xi)H_{n-1}(\xi)$ , we have  $\xi'_0 \in K_{n-1}$ . Also, one may verify that if  $\xi_0, \xi'_0 \in K_n$  are two different  $M^n$ -symmetric roots of  $H_n A$ , then  $\xi_0 + 2M^{-n}k_1\pi$  and  $\xi'_0 + 2M^{-n}k'_1\pi$  are two different  $M^{n-1}$ -symmetric roots of  $H_{n-1}A$ , where  $k_1, k'_1 \in \mathbf{Z}$ . Therefore we conclude that

$$(2.10) \quad \#K_n \leq \#K_{n-1} \quad \forall n \geq 1.$$

Moreover, we see that  $\#K_n = \#K_{n-1}$  if and only if

$$(2.11) \quad K_{n-1} = \{\xi_0 + 2M^{-n}k(\xi_0)\pi : \xi_0 \in K_n\},$$

where  $k(\xi_0) \in \mathbf{Z}$  is chosen so that

$$(2.12) \quad H(M^{n-1}\xi_0 + 2k\pi/M) = 0 \quad \forall k \notin k(\xi_0) + M\mathbf{Z},$$

and

$$(2.13) \quad H(M^{n-1}\xi_0 + 2k\pi/M) \neq 0 \quad \forall k \in k(\xi_0) + M\mathbf{Z}.$$

By (2.10) and (2.11), there exist an integer  $N_0$  and a finite set  $K$  such that

$$(2.14) \quad K_n = K \quad \text{modulo } 2M^{-n}\pi\mathbf{Z}$$

for all  $n \geq N_0$ .

For any  $\xi_0 \in K$  and  $n \geq N_0$ ,  $H(M^n\xi_0 + 2k\pi/M) = 0$  for  $k \in \mathbf{Z} \setminus M\mathbf{Z}$  and  $H(M^n\xi_0 + 2k\pi/M) \neq 0$  for  $k \in M\mathbf{Z}$  by (2.12) and (2.13). Then for any  $1 \leq k \leq M-1$ ,  $\{M^n\xi_0 + 2k\pi/M : 1 \leq k \leq M-1, n \geq N_0\}$  is a finite set modulo  $2\pi\mathbf{Z}$ , which implies that  $M^{n_1}\xi_0 + 2k\pi/M = M^{n_2}\xi_0 + 2k\pi/M \pmod{2\pi\mathbf{Z}}$  for some integers  $N_0 \leq n_1 < n_2$ . Therefore  $\eta_0 := M^{n_1}\xi_0$  is rational and we may assume that

$$(2.15) \quad \eta_0 = 2j_0\pi/(m-1)$$

for some  $0 \leq j_0 \leq m-2$ , where  $m = M^{n_2-n_1}$ , and

$$(2.16) \quad H(M^n\eta_0 + 2k\pi/M) = 0 \quad \forall k \in \mathbf{Z} \setminus M\mathbf{Z} \quad \text{and} \quad n \geq 0.$$

From (2.16), it follows that

$$(2.17) \quad H_{n_2-n_1}(\eta_0 + 2k\pi/m) = H(M^{n_2-n_1-1}(\eta_0 + 2k\pi/m)) \cdots H(\eta_0 + 2k\pi/m) = 0$$

for all  $k \in \mathbf{Z} \setminus m\mathbf{Z}$ , which implies that for any  $k \in \mathbf{Z}$  with  $k = j_0 + \cdots + m^{l-1}j_0 + m^l k'$  for some  $l \geq 0$  and  $k' - j_0 \in m\mathbf{Z}$ , we have

$$\begin{aligned} \hat{f}(\eta_0 + 2k\pi) &= \hat{f}(m\eta_0 + 2(k - j_0)\pi) \\ &= H_{n_2-n_1}(\eta_0 + 2(k - j_0)\pi/m) \hat{f}(\xi_0 + 2(k - j_0)\pi/m) \\ &= H_{n_2-n_1}(\eta_0) \hat{f}(\xi_0 + 2(k - j_0)\pi/m) \\ &= \cdots = H_{n_2-n_1}(\eta_0)^l \hat{f}(\xi_0 + 2k'\pi) \\ &= H_{n_2-n_1}(\eta_0)^l H_{n_2-n_1}(\eta_0 + 2(k' - j_0)\pi/m) \hat{f}(\eta_0 + 2(k' - j_0)\pi/m) \\ (2.18) \quad &= 0. \end{aligned}$$

On the other hand, we see that if  $0 < j_0 \leq m-2$ , then the union of the sets  $U_l(\eta_0) := j_0 + \cdots + m^l j_0 + m^l \mathbf{Z} \setminus m^{l+1} \mathbf{Z}$ ,  $l \geq 0$ , is the whole integer set  $\mathbf{Z}$ , because  $\mathbf{Z} \setminus (\bigcup_{l=0}^L U_l) = j_0 + \cdots + m^L j_0 + m^{L+1} \mathbf{Z}$ , and  $\min\{|k| : k \in \mathbf{Z} \setminus (\bigcup_{l=0}^L U_l)\} \geq \min(m^{L+1} - (j_0 + \cdots + m^L j_0), (j_0 + \cdots + m^L j_0)) \rightarrow \infty$  as  $L \rightarrow \infty$ . Therefore  $j_0 = 0$  by (2.18) and the linear independent assumption on  $f$ . This proves that for any  $\xi_0 \in K$  there exists an integer  $n$  so that  $M^n \xi_0 \in 2\pi\mathbf{Z}$ . Hence  $K \subset 2M^{-N_1}\pi\mathbf{Z}$  for some sufficiently large integer  $N_1$ . This together with (2.14) proves that  $K_n \subset 2M^{-n}\mathbf{Z}$  for all  $n \geq \max(N_0, N_1)$ , and hence the result follows.  $\square$

Now we reach the stage of the proof of Theorem 2.1.

*Proof of Theorem 2.1.* Define the correlation of two compactly supported  $L^2$  functions  $f$  and  $g$  by

$$(2.19) \quad A_{f,g}(\xi) := \sum_{k \in \mathbf{Z}} \langle f, g(\cdot - k) \rangle e^{-ik\xi}.$$

Recall that  $f$  is a refinable function in  $L^2$ , which implies the unit partition property  $\sum_{k \in \mathbf{Z}} f(\cdot - k) \equiv 1$ . Thus

$$(2.20) \quad A_{f,g}(0) = \int_{\mathbf{R}} g(x) dx \neq 0.$$

Denote the set of all  $M^n$ -symmetric roots of  $H_n A_{f,g}$  by  $K_n$  with multiplicity being not counted, where as before we say that  $a$  and  $b$  are two different  $M^n$ -symmetric roots if  $a - b \notin 2M^{-n}\pi\mathbf{Z}$ . By (2.20) and Lemma 2.2,  $K_N \subset 2M^{-N}\pi\mathbf{Z}$  for some integer  $N$ . Thus  $K_N$  is an empty set, since otherwise  $0 \in K_N$ , and then  $A_{f,g}(0) = H_N(0)A_{f,g}(0) = 0$ , which contradicts (2.20). From the definition of  $K_N$  and the assertion  $K_N = \emptyset$ , it follows that  $H_N(\xi)A_{f,g}(\xi)$  does not have any  $M^N$ -symmetric root. By the Bézout identity, there exists a trigonometric polynomial  $B(\xi)$  so that

$$(2.21) \quad \sum_{k=0}^{M^N-1} H_N\left(\frac{\xi + 2k\pi}{M^N}\right) A_{f,g}\left(\frac{\xi + 2k\pi}{M^N}\right) \overline{B\left(\frac{\xi + 2k\pi}{M^N}\right)} = 1.$$

Define  $h$  by  $\hat{h}(\xi) = B(\xi/M^N)\hat{g}(\xi/M^N)$ . Obviously  $h \in S_N(g)$ . By (2.6) and (2.21), we have

$$\begin{aligned} A_{f,h}(\xi) &= \sum_{k \in \mathbf{Z}} \hat{f}(\xi + 2k\pi) \overline{\hat{h}(\xi + 2k\pi)} \\ &= \sum_{k \in \mathbf{Z}} H_N\left(\frac{\xi + 2k\pi}{M^N}\right) \overline{B\left(\frac{\xi + 2k\pi}{M^N}\right)} \hat{f}\left(\frac{\xi + 2k\pi}{M^N}\right) \overline{\hat{g}\left(\frac{\xi + 2k\pi}{M^N}\right)} \\ &= \sum_{k=0}^{M^N-1} H_N\left(\frac{\xi + 2k\pi}{M^N}\right) A_{f,g}\left(\frac{\xi + 2k\pi}{M^N}\right) \overline{B\left(\frac{\xi + 2k\pi}{M^N}\right)} \\ &= 1, \end{aligned}$$

and hence  $h$  is a local dual of  $f$ . □

### 3. POLY-SCALE REFINABILITY

In this section, we give the proofs of Theorems 1.2 and 1.3. To prove Theorems 1.2 and 1.3, we need a lemma about trigonometric polynomials.

**Lemma 3.1.** *For any given nonzero trigonometric polynomial  $d(\xi)$ , there exist an integer  $N$  and trigonometric polynomials  $c_l(\xi), 1 \leq l \leq N$ , so that*

$$(3.1) \quad d(M^N \xi) = c_1(M^{N-1} \xi)d(M^{N-1} \xi) + c_2(M^{N-2} \xi)d(M^{N-2} \xi) + \dots + c_N(\xi)d(\xi).$$

*Furthermore, if there does not exist  $\xi_0 \in \mathbf{C}$  so that  $d(M^n \xi_0) = 0$  for all  $n \geq 0$ , then there exist an integer  $N$  and trigonometric polynomials  $c_l^*(\xi), 1 \leq l \leq N$ , so that*

$$(3.2) \quad 1 = c_1^*(M^{N-1} \xi)d(M^{N-1} \xi) + c_2^*(M^{N-2} \xi)d(M^{N-2} \xi) + \dots + c_N^*(\xi)d(\xi).$$

*Proof.* We let  $d_0(\xi) = d(\xi)$ , and inductively for  $n \geq 1$ , we let  $d_n(\xi)$  be the trigonometric polynomial with minimal degree so that any common factor between  $d(\xi)$  and  $d_{n-1}(\xi)$  is a factor of  $d_n(M\xi)$ . Here the degree of a trigonometric polynomial  $p(\xi) := \sum_{k=k_1}^{k_2} p(k)e^{-ik\xi}$  with  $p(k_2)p(k_1) \neq 0$  is defined by  $\deg p = k_2 - k_1$ . Note that  $d_n(\xi)$  is a factor of  $d_{n-1}(\xi/M) \cdots d_{n-1}(\xi/M + 2(M-1)\pi/M)$  since  $d_{n-1}(\xi)$  is a factor of the trigonometric polynomial  $d_{n-1}(\xi) \cdots d_{n-1}(\xi + 2(M-1)\pi/M)$ . Then the degree of  $d_n$  is less than that of  $d_{n-1}$ , which implies that there exists an integer  $N$  so that the degree of  $d_n$  is the same as the degree of  $d_N$  for all  $n \geq N$ . Recall that  $d_{n+1}(M\xi)$  is a factor of  $d_n(\xi) \cdots d_n(\xi + 2(M-1)\pi/M)$ . Then comparing the degrees of the above two trigonometric polynomials, we get

$$(3.3) \quad d_{n+1}(M\xi) = c_n(\xi)d_n(\xi) \cdots d_n(\xi + 2(M-1)\pi/M),$$

where  $c_n(\xi), n \geq N$ , are nonzero monomials. This together with the construction of  $d_{n+1}$  implies that

$$(3.4) \quad d_n(\xi) \text{ is a factor of } d(\xi) \text{ for all } n \geq N.$$

Therefore there exists a trigonometric polynomial  $c_0$  so that

$$(3.5) \quad d(\xi) = c_0(\xi)d_N(\xi).$$

From the construction of  $d_n$ , there exist trigonometric polynomials  $a_n$  and  $b_n$  so that

$$(3.6) \quad d_n(M\xi) = a_n(\xi)d(\xi) + b_n(\xi)d_{n-1}(\xi)$$

for all  $1 \leq n \leq N$ . The assertion (3.1) follows from (3.5) and (3.6).

Now we prove (3.2). Let  $d_n, n \geq 0$ , and  $N$  be as above. By (3.6), it suffices to show that  $d_N$  is a monomial when there does not exist  $\xi_0 \in \mathbf{C}$  so that  $M^n \xi_0, n \geq 0$ , are roots of  $d(\xi)$ . Suppose, on the contrary, that  $d_N$  is not a monomial. Then  $d_N(\eta_1) = 0$  for some  $\eta_1 \in \mathbf{C}$ . Iteratively using (3.3), we obtain  $d_{n+N}(M^n \eta_1) = 0$  for all  $n \geq 0$ . This together with (3.4) implies that  $d(M^n \eta_1) = 0$  for all  $n \geq 0$ , which is a contradiction.  $\square$

Now we start to prove Theorems 1.2 and 1.3.

*Proof of Theorem 1.2.* Write  $g = \sum_{k \in \mathbf{Z}} d(k)f(\cdot - k)$  for some  $\{d(k)\} \in \ell_0$ , and assume that  $f$  satisfies the refinement equation  $f = \sum_{k \in \mathbf{Z}} a(k)f(M \cdot - k)$  for some  $\{a(k)\} \in \ell_0$ . Then setting  $H(\xi) = \frac{1}{M} \sum_{k \in \mathbf{Z}} a(k)e^{-ik\xi}$  and  $d(\xi) = \sum_{k \in \mathbf{Z}} d(k)e^{-ik\xi}$  yields

$$(3.7) \quad \hat{f}(M\xi) = H(\xi)\hat{f}(\xi) \quad \text{and} \quad \hat{g}(\xi) = d(\xi)\hat{f}(\xi).$$

By Lemma 3.1, there exist a positive integer  $N$  and trigonometric polynomials  $c_n(\xi), 1 \leq n \leq N$ , so that

$$(3.8) \quad d(M^N \xi) = \sum_{n=1}^N c_n(M^{N-n} \xi)d(M^{N-n} \xi).$$

Multiplying  $\hat{f}(M^N \xi)$  at both sides of (3.8) and using (3.7), we obtain

$$(3.9) \quad \begin{aligned} \hat{g}(M^N \xi) &= \sum_{n=1}^N c_n(M^{N-n} \xi)d(M^{N-n} \xi)\hat{f}(M^N \xi) \\ &= \sum_{n=1}^N c_n(M^{N-n} \xi)H(M^{N-1} \xi) \cdots H(M^{N-n} \xi)\hat{g}(M^{N-n} \xi). \end{aligned}$$

Hence taking the inverse Fourier transform at both sides of (3.9) proves that  $g$  is poly-scale refinable.  $\square$

*Proof of Theorem 1.3.* We use the same argument as in the proof of Theorem 1.2 except for the equation (3.8) being replaced by (3.2). We omit the details of the proof here.  $\square$

4. GENERALIZATION, APPLICATIONS AND PROBLEMS

In this section, we give a few remarks about generalization, applications and open problems related to Theorems 1.1, 1.2 and 1.3.

For a vector-valued compactly supported function  $F = (f_1, \dots, f_r)^T$  on  $\mathbf{R}^d$ , we define the *semi-convolution*  $F *'$  by

$$F *' : (\ell)^r \ni \{c(k)\} \longmapsto \sum_{k \in \mathbf{Z}^d} c(k)^T F(\cdot - k),$$

where  $(\ell)^r$  denotes the  $r$  copies of the space of all sequences on  $\mathbf{Z}^d$ , and the *shift-invariant spaces*  $S_n(F)$  by

$$S_n(F) = \{F *' c(M^n \cdot) : c \in (\ell_0)^r\}.$$

We say that  $F$  is *M-refinable* if

$$S_0(F) \subset S_1(F),$$

and *poly-scale refinable* if

$$S_0(F) \subset S_1(F) + \dots + S_N(F)$$

for some  $N \geq 1$ . For two vector-valued compactly supported functions  $F = (f_1, \dots, f_r)^T \in L^2$  and  $G = (g_1, \dots, g_r)^T \in L^2$ , we say that  $G$  is a *local dual* of  $F$  if  $\langle f_i(\cdot - k), g_{i'}(\cdot - k') \rangle = \delta_{ii'} \delta_{kk'}$  for all  $1 \leq i, i' \leq r$  and  $k, k' \in \mathbf{Z}^d$ .

First we consider the generalization of Theorem 1.1 to the vector-valued case.

**Theorem 4.1.** *Let  $F = (f_1, \dots, f_r)^T$  and  $G = (g_1, \dots, g_r)^T$  be vector-valued compactly supported functions in  $L^2$ . Assume that  $F$  is M-refinable and has linear independent shifts, the symbol  $H(\xi)$  of the refinable function  $F$  satisfies  $\det H(\xi) \neq 0$ ,  $G$  satisfies  $\int_{\mathbf{R}} G(x) dx \neq 0$ , and the correlation matrix  $\sum_{k \in \mathbf{Z}} \widehat{F}(\xi + 2k\pi) \overline{\widehat{G}(\xi + 2k\pi)}^T$  is nonsingular for some  $\xi_0 \in \mathbf{C}$ . Then there exist functions  $h_1, \dots, h_r \in S_N(g)$  for some  $N \geq 0$  so that  $(h_1, \dots, h_r)^T$  is a local dual of  $F$ .*

As an easy consequence of Theorem 4.1, we see that for a vector-valued refinable function  $F \in L^2$  having linear independent shifts, if the corresponding symbol has nonzero determinant, then a local dual to  $F$  can be found in  $S_N(F)$  for some  $N \geq 1$ . We do not include the proof of Theorem 4.1 here since the proof is similar to the one of Theorem 2.1, except for more delicate analysis about symmetric roots of a matrix-valued trigonometric polynomial.

Next we consider the compactly supported distributional solution of a poly-scale refinement equation. As mentioned in [7], for a poly-scale refinement equation, there is a corresponding vector-valued refinement equation. In particular, let  $F = (f_1, \dots, f_r)^T$  satisfy the following *poly-scale refinement equation*,

$$(4.1) \quad F = \sum_{n=1}^N \sum_{k \in \mathbf{Z}} c_n(k) F(M^n \cdot - k),$$

where  $\{c_n(k)\} \in \ell_0^{r \times r}$ ,  $1 \leq n \leq N$ , set

$$H_n(\xi) = M^{-n} \sum_{k \in \mathbf{Z}} c_n(k) e^{-ik\xi}, \quad 1 \leq n \leq N,$$

and define

$$\tilde{H}(\xi) = \begin{pmatrix} H_1(M^{N-1}\xi) & H_2(M^{N-2}\xi) & \cdots & H_{N-1}(M\xi) & H_N(\xi) \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & I & 0 \end{pmatrix}.$$

Then the vector-valued function  $\tilde{F}$  defined by

$$\hat{\tilde{F}}(\xi) = \begin{pmatrix} \hat{F}(M^{N-1}\xi) \\ \vdots \\ \hat{F}(M\xi) \\ \hat{F}(\xi) \end{pmatrix}$$

satisfies the refinement equation

$$\hat{\tilde{F}}(\xi) = H(\xi/M)\hat{\tilde{F}}(\xi/M).$$

As an application of Theorem 2.1 in [11], we have the following result about the existence of a nonzero compactly supported distributional solution of a poly-scale refinement equation.

**Theorem 4.2.** *Let  $\{c_l(k)\} \in \ell_0^{r \times r}, 1 \leq l \leq N$ , and set*

$$H(z, w) = \sum_{l=1}^N \sum_{k \in \mathbf{Z}} c_l(k) z^k w^l.$$

*Then there exists a nonzero compactly supported distributional solution  $F$  of the poly-scale refinement equation*

$$(4.2) \quad F = \sum_{n=1}^N \sum_{k \in \mathbf{Z}} c_n(k) \hat{F}(M^n \cdot -k)$$

*if and only if  $H(1, M^{-l}) - I$  is singular for some nonnegative integer  $l$ .*

Now we consider the possible application of Theorem 1.3. Let  $f$  be a compactly supported refinable function on the real line, and let  $g \in S_0(f)$ . By Theorem 1.3, there exist a positive integer  $N$  and sequences  $\{c_n(k)\} \in \ell_0, 1 \leq n \leq N$ , so that  $f = \sum_{n=1}^N \sum_{k \in \mathbf{Z}} c_n(k) g(M^n \cdot -k)$ . Then taking the inner product with an  $L^2$  function  $h$  leads to

$$\langle h, f(\cdot - k) \rangle = \sum_{n=1}^N \sum_{l \in \mathbf{Z}} c_n(l) \langle h, g(M^n \cdot -M^n k - l) \rangle \quad \text{for all } k \in \mathbf{Z},$$

which implies that the local information of the sequence  $\{\langle h, f(\cdot - k) \rangle\}$  can be obtained from the local information of  $\{\langle h, g(M^n \cdot -k') \rangle\}$  at the finer level  $n$  between 1 and  $N$ . The above formula is useful in the reconstruction of the original function or signal in the shift-invariant space generated by  $f$ , especially when it is more costly for designing a good average sampler  $g$  to obtain the data at an initial level than for using an easy average sampler  $g$  to obtain the data at a finer level.

Finally, we propose some problems about local dual and poly-scale refinability:

**Problem 1.** Let  $F$  and  $G \in L^2$  be vector-valued compactly supported refinable functions on  $\mathbf{R}^d$ ,  $d \geq 2$ , and have linear independent shifts. Can we find a local dual of  $F$  in  $S_N(G)$  for some  $N \geq 1$ .

**Problem 2.** Let  $F = (f_1, \dots, f_r)^T$  be a vector-valued compactly supported (poly-scale) refinable function having linear independent shifts. Characterize all functions  $g_1, \dots, g_r \in S_0(F)$  that are poly-scale refinable.

**Problem 3.** Let  $F = (f_1, \dots, f_r)^T$  be a vector-valued compactly supported refinable function having linear independent shifts. Characterize all functions  $g_1, \dots, g_r \in S_0(F)$  so that

$$S_0(F) \subset S_1(G) + \dots + S_N(G)$$

for some  $N \geq 1$ , where  $G = (g_1, \dots, g_r)^T$ .

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