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CENTRALIZERS OF AREA PRESERVING DIFFEOMORPHISMS ON S^2

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Abstract. It has been conjectured that a generic diffeomorphism on a compact manifold will have trivial centralizer. We give some partial results towards proving this conjecture within the class of area preserving diffeomorphisms of the sphere.

1. Introduction

Let f be a C^r diffeomorphism of a compact manifold M. The *centralizer* of f is the set

$$\mathcal{C}(f) = \{g \in \mathrm{Diff}^{\,r}(M) : fg = gf\}.$$

We say that f has *trivial centralizer* if $\mathcal{C}(f) = \{f^n : n \in \mathbb{Z}\}$. It is not hard to find examples of diffeomorphisms with non-trivial centralizer. For example, if f embeds as the time-1 map of a C^r flow, ϕ_t, then $\mathcal{C}(f)$ contains the set $\{\phi_t : t \in \mathbb{R}\}$, and is therefore non-trivial. A long-standing question of Smale is whether a generic C^r diffeomorphism has trivial centralizer (and therefore does not embed in a flow) [12]. This paper addresses this question in the context of area preserving diffeomorphisms on the 2-sphere.

Many of the ideas presented here come from previous work of Kopell and of Palis and Yoccoz. Kopell showed that there is a C^2 open, C^∞ dense subset of C^2 diffeomorphisms on S^1 with trivial centralizers [5]. Palis and Yoccoz have shown that the generic C^∞ structurally stable diffeomorphism has trivial centralizer [7], and that there is an open and dense subset of C^∞ Anosov diffeomorphisms on the torus with trivial centralizers [6]. Several authors have built on this work, adapting the techniques of [5], [6] and [7] to prove that, within various classes of maps, the centralizer of a generic diffeomorphism will be trivial. These classes include expanding maps on a torus [2], large classes of analytic diffeomorphisms [10], [11], and certain classes of partially hyperbolic diffeomorphisms [3].

In this note, we show that if the centralizer of a generic area preserving diffeomorphism on S^2 is not trivial, then seemingly pathological dynamics occur. We will prove the following theorem.

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Theorem 1.1. *Let $\mathcal{S}$ denote the set of all area preserving, orientation preserving, C^r diffeomorphisms on S^2 (where $r \geq 16$). There is a C^r residual subset $\mathcal{R} \subset \mathcal{S}$ such that if $f \in \mathcal{R}$ and if $g \in \mathcal{C}(f)$, then either*

(1) *g is an integer power of f, or*
(2) *there are regions on S^2 on which g acts as arbitrarily high powers of f. Given any $N \in \mathbb{N}$, there is a non-empty, closed subset $V \subset S^2$ and an integer $n \geq N$ such that either $g|_V = f^n$ or $g|_V = f^{-n}$.*

As an intermediate step in proving Theorem 1.1, we obtain the following characterisation.

Proposition 1.2. *If $f \in \mathcal{R}$, then there is a countable collection of closed, f-invariant sets $\{V_i\}$ such that*

a) *$\overline{\bigcup V_i} = S^2$, and*
b) *if $g \in \mathcal{C}(f)$, then $g|_{V_i} = f^{n_i}$ for some $n_i \in \mathbb{Z}$.*

We will use techniques developed by Palis and Yoccoz in [6] and [7], combined with the following recent result of Franks and LeCalvez, describing the dynamics of generic area preserving diffeomorphisms on the sphere.

Theorem 1.3 ([4], Theorem 9.15)**.** *For all $r \geq 16$, there is a C^r residual subset $\mathcal{U} \subset Diff^{\,r}(S^2)$ such that if $f \in \mathcal{U}$, then*

$$S^2 = \overline{\bigcup W^s(p)}$$

where the union is taken over all hyperbolic periodic points p for f.

An analogous statement is true for area preserving diffeomorphisms on the annulus, but it is not known whether Theorem 1.3 will hold on other manifolds, or for diffeomorphisms with lower differentiability. The high degree of differentiability required in this theorem, $r \geq 16$, is to ensure that all elliptic points are Moser stable. For this reason, the results presented in this paper are valid only for C^r diffeomorphisms on the sphere or the annulus, where $r \geq 16$.

Note that Theorem 1.3 implies that for a generic area preserving diffeomorphism on S^2, the union of the unstable manifolds will also be dense. However, this does not immediately imply that there will be a dense set of heteroclinic points (or a dense set of periodic points).

2. Some generic properties of commuting diffeomorphisms

In this section, we sketch the proofs of some results about generic commuting diffeomorphisms on a compact manifold M. The first lemma tells us that for a generic diffeomorphism f, every $g \in \mathcal{C}(f)$ preserves the periodic orbits of f. This is proved in [7], section 12. We include the proof here, for completeness. Note that in Lemma 2.1, we could have assumed that $r \geq 1$. However, we will need $r \geq 2$ for the proof of Proposition 2.2.

Let $r \geq 2$, and let

$$\mathcal{Q} = \{f \in \mathrm{Diff}^r(M) : \text{if } p \in \mathrm{Per}(f) \text{ and } g \in \mathcal{C}(f), \text{then } g(p) = f^i(p) \text{ for some } i \in \mathbb{Z}\}.$$

Lemma 2.1. *$\mathcal{Q}$ is a C^r residual subset of $Diff^{\,r}(M)$.*

Proof. If $g \in \mathcal{C}(f)$, then g permutes the fixed points of f^k, for any $k \in \mathbb{Z}$. Moreover, if p_1 and p_2 are fixed by f^k, and if $g(p_1) = p_2$, then it follows from the chain rule (differentiating the equation $gf^k(p_1) = f^k g(p_1)$) that the matrices $T_{p_1} f^k$ and $T_{p_2} f^k$ are conjugate. In particular, they have the same eigenvalues.

Given any $k \in \mathbb{Z}$, we can find an open, dense subset $\mathcal{Q}_k \subset \operatorname{Diff}^r(M)$ (in the C^r topology) such that for all $f \in \mathcal{Q}_k$,

(1) there is a finite set of fixed points for f^k, and
(2) if p_1, p_2 are fixed by f^k, and if $p_1 \neq f^i(p_2)$ for all $i \in \mathbb{Z}$, then the matrices $T_{p_1} f^k$ and $T_{p_2} f^k$ are not conjugate.

Note that if $f \in \mathcal{Q}_k$ and $f^k(p) = p$, then for every $g \in \mathcal{C}(f)$ we must have $g(p) = f^i(p)$ for some $i \in \mathbb{Z}$. We let $\mathcal{Q} = \bigcap_{k \in Z} \mathcal{Q}_k$. □

The following proposition is a special case of a result of Palis and Yoccoz [7]. They showed that when the stable and unstable manifolds of a hyperbolic periodic point p intersect transversally, the set of homoclinic points can be perturbed to ensure that the centralizer of $f|_{W^s(p) \cup W^u(p)}$ is trivial. The proof sketched below uses arguments found in [7], although, as stated, Proposition 2.2 is slightly stronger. We require that all perturbations can be made within any small neighbourhood of a fundamental domain of $f|_{W^s(p)}$. Moreover, we are working with C^r diffeomorphisms in the C^r topology, for $r \geq 2$, whereas in [7], the diffeomorphisms are assumed to be C^∞. We are able to work with diffeomorphisms of lower differentiability because the stable manifold is one-dimensional. We will use the fact that in the one-dimensional case, every C^r diffeomorphism that commutes with a linear contraction is itself linear, for $r \geq 1$. In higher dimensions, this is only true when $r = \infty$, or with further (non-generic) conditions on the eigenvalues of the contraction ([5], Theorem 6).

Proposition 2.2. *Let p_0 be a hyperbolic periodic point for $f \in \mathcal{Q}$, with period k, and suppose that $W^s(p_0)$ and $W^u(p_0)$ intersect transversally. Let $I \subset W^s(p_0)$ be any fundamental domain of $f|_{W^s(p_0)}$. Then given any neighbourhood $U \subset M$ of I, and any $\epsilon > 0$, there is a diffeomorphism $\tilde{f} \in \mathcal{Q}$, ϵ-close to f in the C^r topology, such that*

(1) *p_0 is a hyperbolic periodic point for $\tilde{f}$, with period k,*
(2) *$\tilde{f} = f$ on $M - U$, and*
(3) *if $g \in \mathcal{C}(\tilde{f})$, then $g|_{W^s_{\tilde{f}}(p_0)} = \tilde{f}^n$ for some integer n.*

Moreover, property (3) is open in $\mathcal{Q}$. There is a neighbourhood $\mathcal{V} \subset \mathcal{Q}$ of $\tilde{f}$ such that if $\hat{f} \in \mathcal{V}$, then

(1) *there is a hyperbolic periodic point $p_0(\hat{f})$, close to p_0, and*
(2) *if $g \in \mathcal{C}(\hat{f})$, then $g|_{W^s_{\hat{f}}(p_0(\hat{f}))} = \hat{f}^n$ for some integer n.*

Sketch of the proof. For simplicity of notation, we will assume that p_0 is a hyperbolic fixed point of f, and that if $g \in \mathcal{C}(f)$, then $g(p_0) = p_0$. The proof in the general case is identical; p_0 will be a fixed point of f^k for some $k \in \mathbb{Z}$, and if $g \in \mathcal{C}(f)$, then p_0 is a fixed point for gf^{-i} for some $i \in \mathbb{Z}$. Note that if $gf^{-i}|_{W^s(p_0)}$ is an integer power of f^k, then $g|_{W^s(p_0)}$ is an integer power of f.

If $g \in \mathcal{C}(f)$ and if $g(p_0) = p_0$, then $g(W^s(p_0)) = W^s(p_0)$ and $g(W^u(p_0)) = W^u(p_0)$. In particular, g preserves the set $W^s(p_0) \cap W^u(p_0)$ of homoclinic points for f at p_0. By [13] (Theorem 2), there are C^{r-1} diffeomorphisms $\alpha : W^s(p_0) \to \mathbb{R}$ and $\beta : W^u(p_0) \to \mathbb{R}$ such that

$$\alpha f \alpha^{-1}(x) = \lambda x \quad \text{and} \quad \beta f \beta^{-1}(x) = \frac{1}{\lambda} x \tag{2.1}$$

where $\lambda < 1$ and $\frac{1}{\lambda}$ are the eigenvalues of $T_{p_0} f$. Both α and β depend continuously on f [1], and by [5] (Lemma 1), the diffeomorphisms α and β also linearise g at p_0;

$$\alpha g \alpha^{-1}(x) = ax \quad \text{and} \quad \beta g \beta^{-1}(x) = bx \tag{2.2}$$

where a and b are the eigenvalues of $T_{p_0} g$.

Let $A = A(f) = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$, and let $B = B(g) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Note that $A(f)$ and $B(g)$ depend continuously on f and g, respectively, and if $f = \tilde{f}$ in a neighbourhood of p_0, then $A(f) = A(\tilde{f})$. By equations (2.1) and (2.2), $B(g) = A(f)^n$ if and only if $g|_{W^s(p_0)} = f^n$ and $g|_{W^u(p_0)} = f^n$.

Fix any fundamental domain I for $f|_{W^s(p_0)}$, any small neighbourhood $U \subset M$ of I, and any $\epsilon > 0$. The idea of the proof is to show that we can find a diffeomorphism $\tilde{f}$ such that

(1) $\tilde{f}$ is ϵ-close to f in the C^r topology,
(2) $\tilde{f} = f$ outside of U, and
(3) if $\tilde{g} \in \mathcal{C}(\tilde{f})$, then $B(\tilde{g})$ is an integer power of $A(\tilde{f})$.

If $\tilde{f}$ satisfies these properties and if $\tilde{g} \in \mathcal{C}(\tilde{f})$, then $\tilde{g}|_{W^s_{\tilde{f}}(p_0)} = \tilde{f}^n$ for some $n \in \mathbb{Z}$ (and $\tilde{g}|_{W^u_{\tilde{f}}(p_0)} = \tilde{f}^n$).

For every $q \in W^s(p_0) \cap W^u(p_0)$, we will associate a point $\overline{q} = (\alpha(q), \beta(q)) \in \mathbb{R}^2$. We have seen that both f and g preserve the set of homoclinic points for f at p_0. So the linear maps $A(f)$ and $B(g)$ will both preserve the set

$$J(f) = \{\overline{q} = (\alpha(q), \beta(q)) : q \in W^s(p_0) \cap W^u(p_0)\}.$$

$J(f)$ is a countable subset of $\mathbb{R}^2$. Each point in $J(f)$ is isolated, since $W^s(p_0)$ and $W^u(p_0)$ intersect transversally. Since α and β depend continuously on f, the set $J(f)$ varies continuously with f on compact subsets of $\mathbb{R}^2$.

If $B \neq A^t$ for every $t \in \mathbb{R}$, then there are integers n and m such that $A^n B^m$ is a contraction. Since $J(f)$ is invariant under $A^n B^m$, this contradicts the discreteness of the set. So it must be true that for all $g \in \mathcal{C}(f)$, $B(g) = A(f)^t$ for some $t \in \mathbb{R}$.

Suppose that the fundamental domain I has endpoints q_0 and $f(q_0)$, and let $\overline{q}_0 = (\alpha(q_0), \beta(q_0))$. Since every point in $J(f)$ is isolated in $\mathbb{R}^2$, there are at most a finite number of points of $J(f)$ in the set $\{A^s(\overline{q}_0) : 0 < s < 1\}$, and inside any bounded neighbourhood of this set.

If in fact $J(f) \cap \{A^s(\overline{q}_0) : 0 < s < 1\} = \emptyset$, then A^t preserves the set $J(f)$ if and only if $t \in \mathbb{Z}$. This is because if $A^t(\overline{q}) \in J(f)$, then for some $n \in \mathbb{Z}$, $A^{t-n}(\overline{q}) \in \{A^s(\overline{q}) : 0 \leq s < 1\}$, and therefore $t = n$. So in this case $g \in \mathcal{C}(f)$ if and only if $g|_{W^s(p_0)} = f^n$ for some $n \in \mathbb{Z}$.

So suppose that $\overline{q} = (\alpha(q), \beta(q)) \in J(f) \cap \{A^s(\overline{q}_0) : 0 < s < 1\}$. We can make a small perturbation to f in a neighbourhood of q (disjoint from q_0 and $f(q_0)$) in order to change the stable coordinate, $\alpha(q)$, while leaving $\beta(q)$ unchanged. This will give us a new diffeomorphism with a homoclinic point at q_0 and another at a point q', close to q, such that $\overline{q}' \notin \{A^s(q_0) : 0 < s < 1\}$. Note that since there are at most a finite number of points of $J(f)$ in any compact neighbourhood of

$\{A^s(\overline{q}_0) : 0 < s < 1\}$, and since $J(f)$ varies continuously with f on compact sets, we can perform this perturbation without creating any new intersections of $J(f)$ with $\{A^s(\overline{q}_0) : 0 < s < 1\}$.

By making a finite number of such perturbations, it is possible to find a diffeomorphism $\tilde{f}$ such that

(1) $f = \tilde{f}$ outside of U. In particular, $A(f) = A(\tilde{f})$,
(2) $\tilde{f}$ is ϵ-close to f in the C^r topology,
(3) q_0 is a homoclinic point for $\tilde{f}$, and
(4) $J(\tilde{f}) \cap \{A^s(\overline{q}_0) : 0 < s < 1\} = \emptyset$.

As discussed above, $\tilde{g} \in \mathcal{C}(\tilde{f})$ if and only if $\tilde{g}|_{W^s_{\tilde{f}}(p_0)} = \tilde{f}^n$ for some $n \in \mathbb{Z}$.

Note that if $\tilde{f}$ satisfies these conditions, then there is a neighbourhood W of $\{A^s(\overline{q}_0) : 0 < s < 1\}$ such that $J(\tilde{f}) \cap W = \{q_0, \tilde{f}(q_0)\}$. Since $J(\tilde{f})$ varies continuously with $\tilde{f}$ on compact sets, this condition will persist under perturbations of $\tilde{f}$. There is a neighbourhood $\mathcal{V} \subset \mathcal{Q}$ of $\tilde{f}$ such that if $\hat{f} \in \mathcal{V}$, then

(1) there is a hyperbolic periodic point $p_0(\hat{f})$, close to p_0, with a homoclinic point $q_0(\hat{f})$ close to q_0, and
(2) if $g \in \mathcal{C}(\hat{f})$, then $g|_{W^s_{\hat{f}}(p_0(\hat{f}))} = \hat{f}^n$ for some integer n.

$\square$

3. Proof of Proposition 1.2

Proposition 1.2. *Let $\mathcal{S}$ denote the set of all area preserving, orientation preserving, C^r diffeomorphisms on S^2 (where $r \geq 16$). There is a C^r residual subset $\mathcal{R} \subset \mathcal{S}$ such if $f \in \mathcal{R}$, then there is a countable collection of closed, f-invariant sets $\{V_i\}$ such that*

a) $\overline{\bigcup V_i} = S^2$, *and*
b) *if $g \in \mathcal{C}(f)$, then $g|_{V_i} = f^{n_i}$ for some $n_i \in \mathbb{Z}$.*

In particular, every diffeomorphism in $\mathcal{C}(f)$ is area preserving.

Proof. Let $\mathcal{R}' \subset \mathcal{S} \cap \mathcal{Q}$ be the set of all diffeomorphisms in $\mathcal{S} \cap \mathcal{Q}$ that satisfy the following properties.

(1) *Every periodic point is either hyperbolic or elliptic.* (A fixed point p for f^k is hyperbolic if T_pf^k has two real eigenvalues, $\lambda > 1$ and $\mu = \frac{1}{\lambda} < 1$. The periodic point p is elliptic if T_pf^k has complex conjugate eigenvalues, λ and μ, where $|\lambda| = |\mu| = 1$.)
(2) *Stable and unstable manifolds intersect transverally.* For every hyperbolic periodic point p, if Γ is any branch of $W^s(p) - \{p\}$ and Γ' is any branch of $W^u(p) - \{p\}$, then $\Gamma \cap \Gamma'$ is non-empty and the intersection is transverse.
(3) $S^2 = \overline{\bigcup W^s(p)}$, where the union is taken over all hyperbolic periodic points p for f.

Note that condition (1) could also be stated as "there are no degenerate periodic points", where a periodic point is degenerate if the eigenvalues of T_pf^k are either both 1 or both -1. This condition is clearly open and dense in the C^r topology, for any $r \geq 1$. The second condition is generic in the C^r topology, for all $r \geq 1$ ([9], [8]), and the third is C^r generic, for $r \geq 16$, by Theorem 1.3. So $\mathcal{R}'$ is a residual subset of $\mathcal{S}$.

For $f \in \mathcal{R}'$, let

$$T(f) = \{p : p \text{ is a hyperbolic periodic point for } f, \text{ and } f|_{W^s(p)} \text{ has trivial centralizer}\}.$$

For $\epsilon > 0$, let

$$\mathcal{R}_\epsilon = \{f \in \mathcal{R}' : \bigcup_{p \in T(f)} W^s(p) \text{ is } \epsilon\text{-dense in } S^2\}.$$

Lemma 3.1. *For all $\epsilon > 0$, the set $\mathcal{R}_\epsilon$ contains an open and dense subset of $\mathcal{R}'$ (in the C^r topology).*

Proof. Let $W^s(p, N)$ be the set $\{q \in W^s(p) : d_s(p, q) < N\}$, where d_s is distance measured along the stable manifold, and let

$$\mathcal{P}_\epsilon = \{f \in \mathcal{R}' : \text{ there is a finite subset } \{p_1, \ldots, p_n\} \subset T(f) \text{ and some } N \in \mathbb{N} \text{ such that } \bigcup_{i=1}^{n} W^s(p_i, N) \text{ is } \epsilon\text{-dense in } S^2\}.$$

Clearly $\mathcal{P}_\epsilon \subset \mathcal{R}_\epsilon$. We will show that $\mathcal{P}_\epsilon$ is open and dense in $\mathcal{R}'$.

Given any $f \in \mathcal{P}_\epsilon$, there is a neighbourhood $\mathcal{U} \subset \mathcal{R}'$ of f such that if $\tilde{f} \in \mathcal{U}$, then

(1) by Proposition 2.2, for $1 \leq i \leq n$, there is a hyperbolic periodic point $p_i(\tilde{f}) \in T(\tilde{f})$, close to p_i, and
(2) $\bigcup_{i=1}^n W^s_{\tilde{f}}(p_i(\tilde{f}), N)$ is ϵ-dense in S^2, for some $N \in \mathbb{N}$.

So $\mathcal{P}_\epsilon$ is open in $\mathcal{R}'$.

To prove that $\mathcal{P}_\epsilon$ is dense in $\mathcal{R}'$, choose any $f \in \mathcal{R}'$. By Theorem 1.3, we can find a finite subset $\{p_1, \ldots, p_n\}$ of hyperbolic periodic points such that $\bigcup_{i=1}^n W^s(p_i, N)$ is ϵ-dense in S^2. If $p_j \notin T(f)$ for some $1 \leq j \leq n$, then by Proposition 2.2, we can make a small perturbation to f to give some $\tilde{f}$ such that

(1) for $0 \leq i \leq n$, p_i is a hyperbolic periodic point for $\tilde{f}$,
(2) $\bigcup_{i=1}^n W^s_{\tilde{f}}(p_i, N)$ is ϵ-dense in S^2, and
(3) $p_j \in T(\tilde{f})$.

At most a finite number of similar perturbations will be required to obtain a diffeomorphism $\hat{f} \in \mathcal{R}'$, C^r close to f, such that $\hat{f} \in \mathcal{P}_\epsilon$. □

Let

$$\mathcal{R} = \bigcap_{n \in \mathbb{N}} \mathcal{R}_{\frac{1}{n}}.$$

By Lemma 3.1, $\mathcal{R}$ is a residual subset of $\mathcal{S}$. Suppose that $f \in \mathcal{R}$. For each $p_i \in T(f)$, let

$$V_i = \overline{\bigcup_{j \in \mathbb{Z}} f^j(W^s(p_i))}.$$

Then

a) each V_i is closed and f-invariant,
b) $\bigcup V_i$ is dense in S^2, and
c) if $g \in \mathcal{C}(f)$, then $g|_{V_i} = f^{n_i}$ for some $n_i \in \mathbb{Z}$. In particular, g is area preserving. □

4. Proof of Theorem 1.1

Theorem 1.1. *If $f \in \mathcal{R}$ and $g \in \mathcal{C}(f)$, then either*

(1) *g is an integer power of f, or*

(2) *there are regions on S^2 on which g acts as arbitrarily high powers of f. Given any $N \in \mathbb{N}$, there is a non-empty, closed set $V \subset S^2$ and an integer $n \geq N$ such that either $g|_V = f^n$ or $g|_V = f^{-n}$.*

Proof. Suppose that $f \in \mathcal{R}$ and $g \in \mathcal{C}(f)$. So there are closed, f-invariant sets $\{V_i\}$ such that

a) $\overline{\bigcup V_i} = S^2$, and

b) $g|_{V_i} = f^{n_i}$ for some $n_i \in \mathbb{Z}$.

Suppose that there is some $N \in \mathbb{N}$ such that for every i, $|n_i| \leq N$. We will show that under this assumption, g is an integer power of f.

For each $n \leq N$, let

$$U_n = \overline{\bigcup_{g|_{V_i} = f^n} V_i}.$$

If $n \neq m$ and $q \in U_n \cap U_m$, then $f^{n-m}(q) = q$. There are at most a finite number of periodic points of any given period (since all periodic points are hyperbolic or elliptic). So $U_n \cap U_m$ contains at most a finite set of points. But for all n, the set U_n is closed, and

$$\bigcup_{n=-N}^{N} U_n = S^2.$$

This is only possible if only one of the U_n's is nonempty. So for some $n \in \mathbb{Z}$, $g = f^n$ on all of S^2. □

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