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ON PEIXOTO'S CONJECTURE FOR FLOWS ON NON-ORIENTABLE 2-MANIFOLDS

CARLOS GUTIERREZ AND BENITO PIRES

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ABSTRACT. Contrary to the case of vector fields on orientable compact 2-manifolds, there is a smooth vector field X on a non-orientable compact 2-manifold with a dense orbit (and therefore without closed orbits) whose phase portrait –up to topological equivalence– remains intact under a one-parameter family of twist perturbations localized in a flow box of X.

1. INTRODUCTION

The importance of the C^r-Closing Lemma Problem lies in the fact that a positive answer to it would lead to very deep positive conclusions in Dynamical Systems, in topics related to the Generic, Stability and Bifurcation Theories. As a consequence of its role, there are several useful C^r-Closing Lemmas. As a sample of some of the important results, we wish to mention Peixoto's C^r-Connecting Lemma [24], Pugh's C^1-Closing Lemma [26], Mañé's C^1-Ergodic Closing Lemma [19], Gutierrez's C^r-counterexample [9], Herman's C^r-Closing Lemma [17, 18], Hayashi's C^1-Connecting Lemma [14, 15, 16]. Besides the articles that will be quoted in this paper, in the same way as above, we wish to mention [12, 13, 20], [22]–[34]. As a sample of some very recent results that use C^1-Closing Lemma results, we wish to mention [3, 5].

Let $\mathfrak{X}^r(M)$, $0 \leq r \leq \infty$, denote the space of C^r–vector fields (with the C^r–topology) on a compact, connected, boundaryless, C^∞, 2-manifold M. Our (compact) flow boxes $V \subset M$ of X will be either the standard ones or those such that $X|_V$ is topogically equivalent to the constant vector field $(1,0)$ on the cylinder $[0,1] \times \mathbb{R}/\mathbb{Z}$. Notice that in the second case, for all $t \in (0,1)$, $\{t\} \times \mathbb{R}/\mathbb{Z}$ is a transversal circle to the vector field $(1,0)$.

To state our main result we shall need the following.

Definition 1.1 (twist perturbation of a vector field)**.** Let M be a 2-manifold, $X \in \mathfrak{X}^r(M)$ and $V \subset M$ be a compact flow box of X. Given $Y \in \mathfrak{X}^r(M)$ with support in V, we say that $X + Y$ is a C^r–twist perturbation of X, localized in V if $X|_{V^o}$ is transversal to $Y|_{V^o}$, where V^o denotes the interior of V.

Points of $\mathbb{R}/\mathbb{Z}$ will be denoted as if they were points of $\mathbb{R}$; in this way, we shall use expressions of the form $x+a$ and $x-a$ when referring to points of $\mathbb{R}/\mathbb{Z}$. Also, if

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$x < y$ are real numbers such that $y - x < 1$, the subinterval (x, y) of $\mathbb{R}$ determines a unique subinterval of $\mathbb{R}/\mathbb{Z}$.

Definition 1.2 (smooth/affine/isometric *iet*)**.** Let Γ be either the the unit interval $[0, 1)$ or the unit circle $\mathbb{R}/\mathbb{Z}$. Given a finite subset $S = \{a_0, a_1, \cdots, a_n\}$ of Γ, with $0 = a_0 < a_1 < \cdots < a_n = 1$, a smooth interval exchange transformation, shortly smooth *iet*, will be an injective transformation $T : \Gamma \setminus S \to \Gamma$ such that $T|_{(a_{i-1},a_i)}$ (i.e. T restricted to the interval (a_{i-1}, a_i)) is a smooth diffeomorphism, $1 \le i \le n$, and the range of T is all Γ but n points. If, moreover, $T|_{(a_{i-1},a_i)}$ is an affine (isometric) transformation for all $1 \le i \le n$, we call T an affine (isometric) *iet*. As usual, the term *iet* will refer to an isometric *iet*.

Definition 1.3 (quasi-minimal vector field)**.** A C^1 vector field on a compact 2–manifold M is quasi-minimal if its set of singularities S is at most finite and any of its orbits in $M \setminus S$ is dense in M.

In this paper, we prove the following.

Theorem 1.4 (smooth *iet* version)**.** *There exist a minimal isometric iet* $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *and a family of diffeomorphisms* $\{G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, 0 \le \mu < \epsilon\}$*, with* G_0 *being the identity map, depending smoothly on* $\mu \in [0, \epsilon)$*, such that, for all* $(\mu_0, p_0) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z}$*,*

(1) $\frac{d}{d\mu}\Big|_{\mu=\mu_0} G_\mu(p_0) > 0$ *(in particular for all* $\mu \in (0, \epsilon)$*,* G_μ *has no fixed points);*
(2) $G_{\mu_0} \circ T$ *is a smooth iet* C^∞*–conjugate to* T*.*

In the theorem above, since $T = T_0$ is minimal and every T_μ is topologically conjugate to T, we obtain that every T_μ is also minimal; in particular, all orbits of T_μ are non-trivial recurrent (and so T_μ has no closed orbits).

Given a smooth *iet* T, defined on $\mathbb{R}/\mathbb{Z}$, there exist a compact 2–manifold M, containning $\mathbb{R}/\mathbb{Z}$, and a vector field $X \in \mathfrak{X}^\infty(M)$ transversal to $\mathbb{R}/\mathbb{Z}$ such that the first return Poincaré map $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ induced by X is precisely T (see [8]). Using this, the result above can be extended to vector fields as follows:

Theorem 1.5 (vector field version)**.** *Let* M *be a non-orientable compact* 2*-manifold of genus* 4*. Then there exists a family of quasi-minimal, Kupka-Smale vector fields* $\{X_\mu \in \mathfrak{X}^\infty(M)\}$*, depending smoothly on* $\mu \in [0, \epsilon)$*, such that, for some flow box* $V \subset M$ *of* X_0*, which can be taken to be homeomorphic to either a rectangle or a cylinder, and for all* $\mu, \nu \in [0, \epsilon)$*,*

(1) $X_\mu|_V$ *is a flow box;*
(2) *If* $\mu \ne \nu$*,* X_μ *is a* C^∞*–twist perturbation of* X_ν *localized in* V*;*
(3) X_μ *and* X_ν *are topologically equivalent.*

In the theorem above, since $X = X_0$ is quasi-minimal and every X_μ is topologically equivalent to X, we obtain that every X_μ is also quasi-minimal; in particular, all regular orbits of X_μ are non-trivial recurrent (and so X_μ has no closed orbits).

Now we relate our results to Peixoto's Conjecture. Let Σ^r be the subset of $\mathfrak{X}^r(M)$ formed by the Morse-Smale C^r–vector fields. M. Peixoto states in [24] the following conjecture.

(PC) Let M be a non-orientable 2-manifold. $X \in \mathfrak{X}^r(M)$ is structurally stable if and only if $X \in \Sigma^r$. Moreover, Σ^r is open and dense in $\mathfrak{X}^r(M)$.

Peixoto [24] proved this conjecture for M orientable. As a consequence of Peixoto's work and Pugh's C^1–Closing Lemma [26, 30], it follows that Σ^1 is dense in $\mathfrak{X}^1(M)$. There are some partial results concerning **(PC)** in class C^r, $r \geq 1$: The conjecture is true both for the projective plane $\mathbf{P}^2$ and for the Klein bottle $\mathbf{K}^2$ (see the proof in [21] that flows on $\mathbf{K}^2$ do not have non-trivial recurrence). Gutierrez [6] showed that **(PC)** is true for the torus with one cross-cap.

By [24], **(PC)** is true if, and only if, it is possible to give an affirmative answer for the following C^r–Connecting Lemma question:

(CL) Let $X \in \mathfrak{X}^r(M)$ have finitely many singularities, all hyperbolic (at least one singularity). Suppose that X has a non-trivial recurrent trajectory. Does there exist an arbitrarily small C^r–perturbation of X such that the resulting vector field has one more saddle connection than X?

We recall that, when M is orientable, **(CL)** has a positive answer by arbitrarily small C^r–twist perturbations [24]. Also, when M is non-orientable, there are many cases in which **(CL)** has a positive answer, again by arbitrarily small C^r–twist perturbations [11].

Hence, it is natural to wonder whether, in the non-orientable case, we may make use of an arbitrary C^r–twist perturbation in order to give an affirmative answer for **(CL)**. Theorem 1.5 gives a negative answer to this question.

Finally, Theorems 1.4 and 1.5 above are relevant to the C^r-Closing Lemma problem, because the positive answer to the C^r–Closing Lemma, given in [7], for a large class of flows on orientable two-manifolds, was obtained by means of twist perturbations (see also [1, 2, 4]).

2. A flow on the torus with two cross-caps

Let $\varphi : \mathbb{R} \times M \to M$ be a flow on M and $\gamma = \gamma(t)$ a trajectory of φ. We denote by $\omega(\gamma)(\alpha(\gamma))$ the ω-limit set (α-limit set) of γ. We say that γ is ω–recurrent (α–recurrent) if $\gamma \subset \omega(\gamma)(\gamma \subset \alpha(\gamma))$. A recurrent trajectory is a trajectory that is ω–recurrent or α–recurrent. A fixed point and a closed orbit are called trivial recurrent trajectories.

Given a C^1 flow $\varphi : \mathbb{R} \times M \to M$ with a recurrent trajectory γ and a point $p \in \gamma$, Peixoto proved that there exists a smooth circle Γ, transversal to φ, passing through the point p (see [6, 24]). We shall analyse flows in terms of their action on transverse circles. Let us recall now the example of a flow given by Gutierrez in [10].

Gutierrez constructed in [10] an *iet* $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ (cf. Figures 1 and 2) and a suspension of T that is a quasi-minimal C^∞–flow $\varphi : \mathbb{R} \times M \to M$, on the torus with two cross-caps M, in such a way that:

(1) φ has two hyperbolic saddle points as its only singularities;
(2) $\mathbb{R}/\mathbb{Z}$ is a subset of M, φ is transversal to $\mathbb{R}/\mathbb{Z}$, and the *iet* $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is precisely the Poincaré return map induced by φ on $\mathbb{R}/\mathbb{Z}$; moreover, for some real positive numbers a, b, c, d, e,
 - $\text{dom}(T) = \mathbb{R}/\mathbb{Z} \setminus \{a, a + b, a + b + c, 1\}$, where $\text{dom}(T)$ is the domain of definition of T;
 - the numbers $a, a+e-c, a-c-d+2e, \cdots$, are ordered (modulo 1) according to Figures 1 and 2. In particular,
 $0 < a < b < e < a + b < 1 - e < a + b + c < a + b + c + d = 1\,,\ d < e,$
 $a + b + c + e = 1 + e - d;$

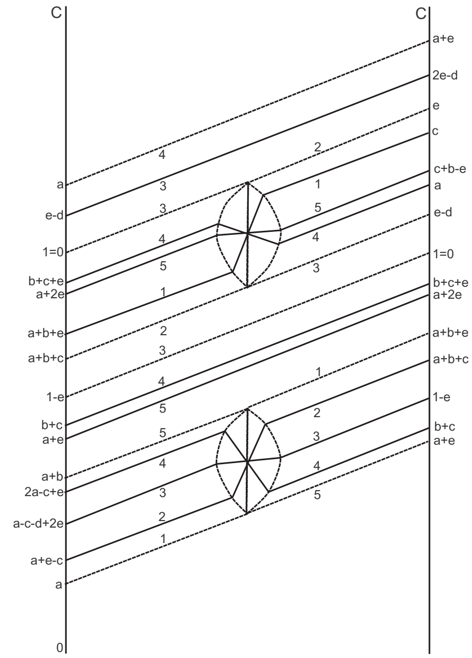

FIGURE 1. The first return map T induced by φ on $\mathbf{R}/\mathbf{Z}$

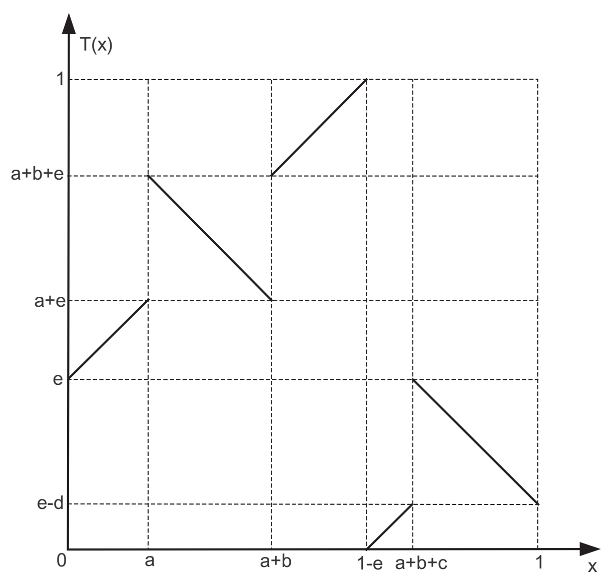

FIGURE 2. The isometric *iet* T

- T operates upon the intervals below, by reversing orientation, in the following way:

$$(a, a+b) \mapsto (a+e, a+b+e),$$
$$(a+b+c, 1) \mapsto (e-d, e);$$

- T operates upon the intervals below, by preserving orientation, in the following way:

$$(0, a) \mapsto (e, a+e);$$
$$(a+b, a+b+c) \mapsto (a+b+e, e-d).$$

In the next section, given a small number $\delta > 0$, we will introduce an affine *iet* T_δ that is topologically conjugate to T and 2δ-close to T, in the uniform C^0-topology.

3. Topological conjugacy

Definition 3.1 (T_δ)**.** Let $\delta > 0$ be small and let T_δ (cf. Figure 3) be the affine *iet* satisfying

- T_δ operates upon the intervals below, linearly, diffeomorphically, and reversing orientation, in the following way:

$$\begin{aligned}
(a, a+e-c+\delta] &\mapsto [a+b+c, a+b+e+\delta),\\
[a+e-c+\delta, a-c-d+2e+2\delta] &\mapsto [1-e-\delta, a+b+c],\\
[a-c-d+2e+2\delta, 2a-c+e] &\mapsto [b+c+\delta, 1-e-\delta],\\
[2a-c+e, a+b) &\mapsto (a+e+\delta, b+c+\delta],\\
(a+b+c, a+b+e+\delta] &\mapsto [c, e+\delta),\\
[a+b+e+\delta, a+2e+2\delta] &\mapsto [c+b-e, c],\\
[a+2e+2\delta, b+c+e+2\delta] &\mapsto [a, c+b-e],\\
[b+c+e+2\delta, 1) &\mapsto (e-d+\delta, a];
\end{aligned}$$

- T_δ operates upon the intervals below, linearly, diffeomorphically and preserving orientation, in the following way:

$$\begin{aligned}
(a+b, a+e+\delta] &\mapsto (a+b+e+\delta, a+2e+2\delta],\\
[a+e+\delta, b+c+\delta] &\mapsto [a+2e+2\delta, b+c+e+2\delta],\\
[b+c+\delta, 1-e-\delta) &\mapsto [b+c+e+2\delta, 1),\\
(1-e-\delta, a+b+c) &\mapsto (0, e-d+\delta),\\
(0, e-d+\delta] &\mapsto (e+\delta, 2e-d+2\delta],\\
[e-d+\delta, a) &\mapsto [2e-d+2\delta, a+e+\delta).
\end{aligned}$$

Proposition 3.2 (topological conjugacy)**.** *Given $\delta > 0$, there exist a fixed-point-free homeomorphism*

$$H = H_\delta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

and a piecewise affine homeomorphism (cf. Figure 4)

$$h = h_\delta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

such that

(1) $T_\delta = H \circ T$;

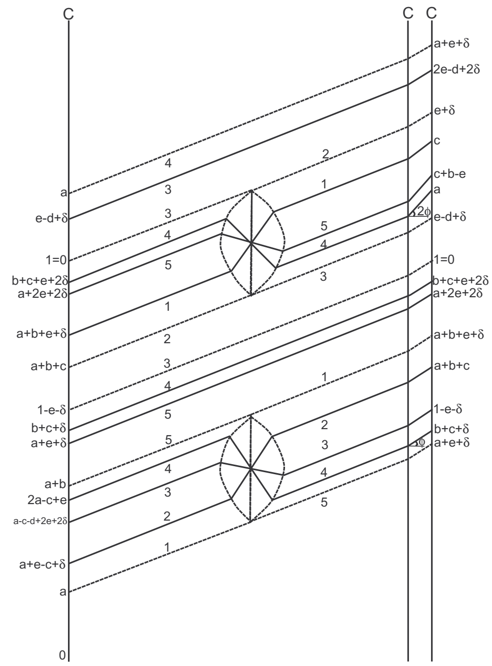

FIGURE 3. The affine *iet* T_δ

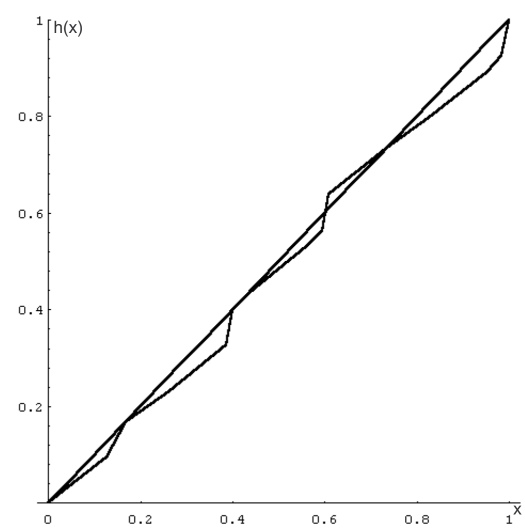

FIGURE 4. The homeomorphism h in local coordinates

(2) $\text{fix}(h) \supset \text{dis}(T_\delta) = \text{dis}(T) = \mathbb{R}/\mathbb{Z} \setminus \text{dom}(T_\delta) = \{a, a+b, a+b+c, 1\}$, *where* $\text{fix}(h)$ *(resp.* $\text{dis}(T)$*) denotes the set of fixed points of* h *(resp. the discontinuity point set of* T*);*

(3) $(h^{-1}\circ T\circ h)(x) = T_\delta(x) = (H\circ T)(x), \forall x\in \mathrm{dom}(T_\delta)$; *in particular, T and T_δ are topologically conjugate;*
(4) $h = h_\delta \to I$, *in the uniform C^0–topology, as δ goes to 0, where $I:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ denotes the identity transformation; and*
(5) *the derivative $(h_\delta)'$ of h_δ is a piecewise constant map that converges uniformly to the constant map 1 as $\delta\to 0$.*

Proof. The existence of h follows from the definition of T_δ. Put

$$\begin{aligned}
h(a) &= a,\\
h(a+e-c+\delta) &= a+e-c,\\
h(a-c-d+2e+2\delta) &= a-c-d+2e,\\
h(2a-c+e) &= 2a-c+e,\\
h(a+b) &= a+b,\\
h(a+e+\delta) &= a+e,\\
h(b+c+\delta) &= b+c,\\
h(1-e-\delta) &= 1-e,\\
h(a+b+c) &= a+b+c,\\
h(a+b+e+\delta) &= a+b+e,\\
h(a+2e+2\delta) &= a+2e,\\
h(b+c+e+2\delta) &= b+c+e,\\
h(1) &= 1,\\
h(e-d+\delta) &= e-d.
\end{aligned}$$

We may extend h linearly to the other points so that h becomes a piecewise affine homeomorphism of the unit circle $\mathbb{R}/\mathbb{Z}$. It follows at once that h satisfies (2)–(5). □

4. C^∞–CONJUGACY

In this section we show that the homeomorphism h conjugating T and T_δ and the fixed-point-free homeomorphism H can be substituted by a C^∞–diffeomorphism g and a fixed-point-free C^∞–diffeomorphism G, respectively, in such a way that the relation $g^{-1}\circ T\circ g = G\circ T$ remains true.

Recall $\mathrm{dis}(T)\subset \mathrm{fix}(h)$ and that

$$\begin{aligned}
\mathrm{dom}(T) &= \mathbb{R}/\mathbb{Z}\backslash\{a, a+b, a+b+c, 1\},\\
\mathrm{dis}(T) &= \{a, a+b, a+b+c, 1\},\\
\mathrm{dom}(T^{-1}) &= \mathbb{R}/\mathbb{Z}\backslash\{e, a+e, a+b+e, e-d\},\\
\mathrm{dis}(T^{-1}) &= \{e, a+e, a+b+e, e-d\}.
\end{aligned}$$

Let $\mathcal{G}$ be the set of C^∞–diffeomorphisms $g:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ such that
(P1) $g|_U = I|_U$, where U is a small neighborhood of $\mathrm{fix}(h)$,
(P2) $g'(x)\ge \frac{1}{2}$, $\forall x\in\mathbb{R}/\mathbb{Z}$;

Lemma 4.1. *If $g\in\mathcal{G}$, then*
(P3) $T(g(x)) - T(x) = T'(x)(g(x)-x), \quad \forall x\in\mathrm{dom}(T)$

(where $T'(x)\in\{-1,1\}$).

Proof. An immediate consequence of (P1) is that for any $x \in \mathrm{dom}(T)$, x and $g(x)$ lie in the same interval of the partition associated to the *iet* T; that is, for each $x \in \mathrm{dom}(T)$, there exists $1 \leq i \leq n$ such that x, $g(x) \in (a_{i-1}, a_i)$. This implies the lemma. □

We prove now the main result of this section.

Proposition 4.2. *Let $g \in \mathcal{G}$.*

(1) *The map*

$$G = g^{-1} \circ T \circ g \circ T^{-1}$$

is well defined in $\mathrm{dom}(T^{-1})$ *and extends to a C^∞–diffeomorphism G : $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that $\forall x \in \mathrm{dom}(T)$, $(g^{-1} \circ T \circ g)(x) = (G \circ T)(x)$. In particular, the isometric iet T and the smooth iet $G \circ T$ are C^∞–conjugate.*

(2) *Fix $\delta > 0$ small, and let $h = h_\delta$ be as in Proposition 3.2. If $g \in \mathcal{G}$ is C^0-close enough to h and G is as above, then $G : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a fixed-point-free C^∞–diffeomorphism.*

Proof. Extend G to the whole $\mathbb{R}/\mathbb{Z}$ by defining $G(q) = g^{-1}(q), \forall q \in \mathrm{dis}(T^{-1})$. Notice that G is bijective, $G|_{\mathrm{dom}(T^{-1})}$ is smooth, and $\left(G|_{\mathrm{dom}(T^{-1})}\right)^{-1}$ is smooth. If x is in a small neighborhood W_q of $q \in \mathrm{dis}(T^{-1})$, $x \neq q$, then $T^{-1}(x)$ is in a neighborhood U_p of p for some $p \in \mathrm{dis}(T) \subset \mathrm{fix}(h)$ and by (P1) and by definition of G, we get

$$G(x) = g^{-1}(x), \forall x \in W_q.$$

Therefore, G is smooth at any point $q \in \mathrm{dis}(T^{-1})$ and as, by (P2), $G'(q) \neq 0$, we obtain that G^{-1} is a C^∞–diffeomorphism. By definition of G, $(g^{-1} \circ T \circ g)(x) = (G \circ T)(x)$, $\forall x \in \mathrm{dom}(T)$. This proves (1). If g is C^0-close to h, then G will also be C^0-close to H; therefore, since H is a fixed-point-free homeomorphism, we will obtain that G is also fixed-point-free. □

5. Main results

In this section we prove Theorems 1.4 and 1.5. We start by proving Theorem 1.4.

Theorem 1.4 (smooth *iet* version)**.** There exist a minimal isometric *iet* $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and a family of diffeomorphisms $\{G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, 0 \leq \mu < \epsilon\}$, with G_0 being the identity map, depending smoothly on $\mu \in [0, \epsilon)$, such that, for all $(\mu_0, p_0) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z}$,

(1) $\frac{d}{d\mu}\big|_{\mu=\mu_0} G_\mu(p_0) > 0$;

(2) $G_{\mu_0} \circ T$ is a smooth *iet* C^∞–conjugate to T.

Proof. By Proposition 4.2, there exist a diffeomorphism g with Properties (P1) to (P3) and a fixed-point-free diffeomorphism G such that $g^{-1} \circ T \circ g = G \circ T$ at every point of $\mathrm{dom}(T)$. Given $\mu \in [0, 1]$, let $g_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined by

$$g_\mu(x) = \mu g(x) + (1 - \mu)x. \tag{5.1}$$

Notice that $\{g_\mu : \mu \in [0, 1]\}$ provides a smooth isotopy between the identity map $g_0 = I$ and $g_1 = g$; also by (P2),

$$\frac{dg_\mu}{dx}(x) = \mu g'(x) + 1 - \mu \geq \frac{\mu}{2} + 1 - \mu = 1 - \frac{\mu}{2} \geq \frac{1}{2}, \forall x \in \mathbb{R}/\mathbb{Z}.$$

Thus, by construction, g_μ is a diffeomorphism of the unit circle satisfying (P1)-(P3), for each $\mu \in [0,1]$, that is, $g_\mu \in \mathcal{G}$. Given $\mu \in [0,1]$, by Proposition 4.2, there exists a diffeomorphism $G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ satisfying

$$(G_\mu \circ T)(x) = \big((g_\mu)^{-1} \circ T \circ g_\mu\big)(x),\ \forall x \in \operatorname{dom}(T). \tag{5.2}$$

The proof of this theorem follows at once from the following lemma.

Lemma 5.1. *Given $x \in \mathbb{R}/\mathbb{Z}$, the map $\mu \in [0,1] \mapsto G_\mu(x) \in \mathbb{R}/\mathbb{Z}$ is differentiable. Moreover, there exist $\epsilon > 0$ and $\sigma > 0$ such that $\forall (\mu_0, x) \in [0,\epsilon) \times \mathbb{R}/\mathbb{Z}$, $\frac{d}{d\mu}\Big|_{\mu=\mu_0} G_\mu(x) > \sigma$.*

Proof. Let $u \in (0,1]$ be a real number. Then, from (5.2),

$$\big(T \circ g_u\big)(x) = \big(g_u \circ G_u \circ T\big)(x),\ \forall x \in \operatorname{dom}(T). \tag{5.3}$$

From (5.1), we reach

$$g_u(y) = y + u(g(y) - y),\ \forall y \in \mathbb{R}/\mathbb{Z}. \tag{5.4}$$

From (5.4) and from property (P3) of g_u, we obtain

$$\frac{(T \circ g_u)(x) - (T \circ g_0)(x)}{u} = \frac{T'(x) \cdot \big(g_u(x) - x\big)}{u} = T'(x) \cdot \big(g(x) - x\big). \tag{5.5}$$

From equations (5.3)-(5.5), we get

$$\begin{aligned}
T'(x) \cdot \big(g(x) - x\big) &\overset{(5.5)}{=} \{(T \circ g_u)(x) - (T \circ g_0)(x)\}/u \\
&\overset{(5.3)}{=} \{(g_u \circ G_u \circ T)(x) - (g_0 \circ G_0 \circ T)(x)\}/u \\
&\overset{(5.4)}{=} \frac{G_u(T(x)) - G_0(T(x))}{u} + g(G_u(T(x))) - G_u(T(x)).
\end{aligned}$$

When $u \to 0$, we get

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(T(x)) = T'(x) \cdot \big(g(x) - x\big) - \big(g(T(x)) - T(x)\big), \tag{5.6}$$

and from Property (P3) of g, we reach

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(T(x)) = T(g(x)) - g(T(x)), \quad \forall x \in \operatorname{dom}(T). \tag{5.7}$$

Since $G = g^{-1} \circ T \circ g \circ T^{-1}$ has no fixed points, we have that $T(g(x)) - g(T(x)) \neq 0$, $\forall x \in \operatorname{dom}(T)$. Therefore,

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(T(x)) \neq 0,\ \forall x \in \operatorname{dom}(T).$$

That is,

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(y) \neq 0, \quad \forall y \in \operatorname{dom}(T^{-1}). \tag{5.8}$$

If $q \in \operatorname{dis}(\mathrm{T}^{-1})$, then $G_\mu(q) = (g_\mu)^{-1}(q)$, $\forall \mu$. Besides, we have by Figure 4 that $q - g(q) > 0$ (the points $(q, g(q))$ with $q \in \operatorname{dis}(T^{-1})$ lie below the diagonal). Observe that

$$\begin{aligned}
\big(g_\mu \circ (g_\mu)^{-1}\big)(x) &= x, \quad \forall x \in \mathbb{R}/\mathbb{Z}, \\
\mu \cdot g((g_\mu)^{-1}(x)) + (1-\mu)(g_\mu)^{-1}(x) &= x, \quad \forall x \in \mathbb{R}/\mathbb{Z}.
\end{aligned}$$

Therefore, by differentiating the previous equation with respect to μ at $\mu = 0$, we get

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(q) = \frac{d}{d_\mu}\Big|_{\mu=0} (g_\mu)^{-1}(q) = q - g(q) > 0. \tag{5.9}$$

We remark that equations (5.7) and (5.9) are compatible; that is, for any $q \in \mathrm{dis}(T^{-1})$,

$$\lim_{y\to q} \frac{d}{d\mu}\Big|_{\mu=0} G_\mu(y) = q - g(q).$$

Hence, the map $(\mu, x) \in [0,1] \times \mathbb{R}/\mathbb{Z} \mapsto \frac{d}{d\mu}\Big|_{\mu=0} G_\mu(x)$ is continuous. This implies the lemma. □

Corollary 5.2. *For all $\mu \in (0, \epsilon)$, G_μ has no fixed points.*

Proof. This follows immediately from Theorem 1.4. □

We now prove Theorem 1.5.

Theorem 1.5 (Vector field version)**.** Let M be a non-orientable compact 2-manifold of genus 4. Then there exists a family of quasi-minimal, Kupka-Smale flows $\{X_\mu \in \mathfrak{X}^\infty(M)\}$, depending smoothly on $\mu \in [0, \epsilon)$, such that, for some flow box $V \subset M$ of X_0, which can be taken to be homeomorphic to either a rectangle or a cylinder, and for all $\mu, \nu \in [0, \epsilon)$,

(1) $X_\mu|_V$ is a flow box;
(2) if $\mu \neq \nu$, X_μ is a C^∞-twist perturbation of X_ν localized in V;
(3) X_μ and X_ν are topologically equivalent.

Proof. Let T be as in Theorem 1.4. From Theorem 1.4, we know that $\{G_\mu \circ T\}$ is a family of smooth *iet*'s conjugate to T. Each *iet* $\{G_\mu \circ T\}$ may be suspended to obtain a smooth vector field X_μ on a non-orientable compact manifold M of genus 4 (see [8]). We may assume that M contains $\mathbb{R}/\mathbb{Z}$ and does not depend on μ. By definition of suspension, each X_μ is tranversal to $\mathbb{R}/\mathbb{Z}$ and $G_\mu \circ T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the forward Poincaré map induced by X_μ. This family $\{X_\mu\}$ can be constructed to satisfy the condition of the theorem in the case in which V is a cylinder. The other case is similar. In both cases, the fact that T is minimal and every $G_\mu \circ T$ is conjugate to T ensures that the family $\{X_\mu\}$ has the required properties of quasi-minimality and topological equivalence. □

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Departamento de Matemática, Instituo de Ciências Matemáticas e de Computaçâo, Universidade de São Paulo, Av. do Trabalhador São Carlense, 400, Centro, CEP 13560-970 São Carlos - SP, Brazil

E-mail address: gutp@icmc.usp.br

Departamento de Matemáticas, Universidad Autonoma de Barcelona, Edificio C, Bellaterra, Cerdanyola del Valles, Spain

E-mail address: bpires@icmc.usp.br