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REAL $3x + 1$

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Abstract. The famous $3x+1$ problem involves applying two maps: $T_0(x) = x/2$ and $T_1(x) = (3x+1)/2$ to positive integers. If x is even, one applies T_0, if it is odd, one applies T_1. The conjecture states that each trajectory of the system arrives to the periodic orbit $\{1, 2\}$. In this paper, instead of choosing each time which map to apply, we allow ourselves more freedom and apply both T_0 and T_1 independently of x. That is, we consider the action of the free semigroup with generators T_0 and T_1 on the space of positive real numbers. We prove that this action is minimal (each trajectory is dense) and that the periodic points are dense. Moreover, we give a full characterization of the group of transformations of the real line generated by T_0 and T_1.

1. Introduction

The famous $3x + 1$ problem (known also under the name of the *Collatz problem* and many other names) involves applying two maps $T_0(x) = \frac{x}{2}$ and $T_1(x) = \frac{3x+1}{2}$. They are applied to positive integers. If the integer is even, we apply T_0, if it is odd, we apply T_1. The conjecture states that each trajectory of this system arrives to the periodic orbit $\{1, 2\}$. There is extensive literature about this problem and related questions. The reader is advised to check [6], [7] or [12].

In the original $3x + 1$ problem one has to choose each time whether to apply T_0 or T_1. This choice is made according to the parity of x. We allow ourselves much more freedom. We apply both T_0 and T_1 independently of x. The price we pay for it is that we have to extend the domain of the maps. Thus, we consider the action of the semigroup S generated by T_0 and T_1 on the set of all positive real numbers. This type of dynamical system is most often considered from the point of view of Iterated Function Systems (although in our case the images are fully overlapping and one of the maps is expanding, so there is no nontrivial attractor). However, there is also considerable interest in them from the points of view of Topological Dynamics (see, e.g., [3]), Ergodic Theory (see, e.g., [10], [4], [9]) and Algebra (see, e.g., [11], [2]). Here we adopt mainly the Topological Dynamics point of view, in particular leaving questions concerning measures untouched.

In Section 2 we fix the notation and show that the semigroup S is free. This is related to the problems considered in [11] and [2].

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In Section 3 we show that our system is minimal (each orbit is dense). Moreover, if we fix $x, y > 0$, we can find a piece of the orbit of x ending in a small neighborhood of y that does not get too close to 0 or ∞.

In Section 4 we show that the periodic points are dense (and again the pieces of the trajectories that begin and end at our periodic point can be chosen in such a way that they do not go too close to 0 or ∞).

Let us discuss periodic points in a more detailed way. According to the well-known formula (see [1], [5], [8]), each periodic point is rational with an odd denominator. Thus, if we are moving along a piece of orbit from a periodic point to itself, we encounter only rational numbers with odd denominators. A simple computation shows that then the denominator stays the same and if the numerator is even, we apply T_0, while if it is odd, we apply T_1. This means that our periodic points are exactly the periodic points considered by Lagarias in [8]. However, our approach is completely different. While Lagarias concentrates on the periods of periodic orbits (their temporal features), we are concerned about their location (their spatial feature).

Finally, in Section 5 we look at what happens if instead of the semigroup S we consider the group G generated by T_0 and T_1. It turns out that then the situation is very simple, and we can give a full description of the elements of G.

2. The semigroup

Let T_0 and T_1 be the functions from the positive real half-line to itself defined by

$$T_0(x) = \frac{x}{2}, \qquad T_1(x) = \frac{3x+1}{2}.$$

For a 0-1 finite sequence $A = (\varepsilon_i)_{i=1}^n$ (a parity vector) we set

$$T_A = T_{\varepsilon_n} \circ T_{\varepsilon_{n-1}} \circ \cdots \circ T_{\varepsilon_2} \circ T_{\varepsilon_1},$$

$n(A) = n$, and denote by $m(A)$ the number of 1's in the sequence A. If $k \le n$, then we set $A_k = (\varepsilon_i)_{i=1}^k$ (the initial segment of A of length k).

For a given 0-1 sequence A of length n, we define a sequence

$$\sigma(A) = (\sigma_1, \sigma_2, \dots, \sigma_n)$$

by setting $\sigma_i = 0$ if $\varepsilon_i = 0$, and $\sigma_i = 3^j$, where j is the number of 1's among $\varepsilon_{i+1}, \varepsilon_{i+2}, \dots, \varepsilon_n$ if $\varepsilon_i = 1$ (that is, $\sigma_i = \varepsilon_i 3^{\sum_{k=i+1}^n \varepsilon_k}$). Then we set

$$\beta(A) = \sum_{k=1}^n 2^{k-1} \sigma_k.$$

Lemma 2.1. *For every A we have*

$$T_A(x) = \frac{3^{m(A)}}{2^{n(A)}} x + \frac{\beta(A)}{2^{n(A)}}. \tag{2.1}$$

Proof. We use induction. For the unique sequence of length 0, (2.1) clearly holds ($T_\emptyset(x) = x$). Now let us assume that it holds for some $A = (\varepsilon_1, \dots, \varepsilon_n)$ and consider $B = (\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1})$.

If $\varepsilon_{n+1} = 0$, then $\sigma(B) = (\sigma_1, \dots, \sigma_n, 0)$ where $(\sigma_1, \dots, \sigma_n) = \sigma(A)$. We get $m(B) = m(A)$, $n(B) = n + 1 = n(A) + 1$ and $\beta(A) = \beta(B)$, so

$$\begin{aligned} T_B(x) &= T_0(T_A(x)) = \frac{1}{2}\left(\frac{3^{m(A)}}{2^{n(A)}}x + \frac{\beta(A)}{2^{n(A)}}\right) \\ &= \frac{3^{m(A)}}{2^{n(A)+1}}x + \frac{\beta(A)}{2^{n(A)+1}} = \frac{3^{m(B)}}{2^{n(B)}}x + \frac{\beta(B)}{2^{n(B)}}. \end{aligned}$$

Thus, (2.1) holds for B replacing A.

If $\varepsilon_{n+1} = 1$, then $\sigma(B) = (3\sigma_1, \dots, 3\sigma_n, 1)$. We get $m(B) = m(A) + 1$, $n(B) = n(A) + 1$ and $\beta(B) = 3\beta(A) + 2^{n(A)}$, so

$$\begin{aligned} T_B(x) &= T_1(T_A(x)) = \frac{1}{2}\left(3\frac{3^{m(A)}}{2^{n(A)}}x + 3\frac{\beta(A)}{2^{n(A)}} + 1\right) \\ &= \frac{3^{m(A)+1}}{2^{n(A)+1}}x + \frac{3\beta(A) + 2^{n(A)}}{2^{n(A)+1}} = \frac{3^{m(B)}}{2^{n(B)}}x + \frac{\beta(B)}{2^{n(B)}}. \end{aligned}$$

Thus, in this case also (2.1) holds for B replacing A. □

Let us investigate closer the semigroup S generated by T_0 and T_1. It turns out that this semigroup is free. More precisely, we have the following theorem.

Theorem 2.2. *If $T_A = T_B$, then $A = B$. Thus, the semigroup generated by T_0 and T_1 is a free semigroup.*

Proof. Assume that $T_A = T_B$. Then

$$\frac{3^{m(A)}}{2^{n(A)}} = \frac{3^{m(B)}}{2^{n(B)}} \tag{2.2}$$

and

$$\frac{\beta(A)}{2^{n(A)}} = \frac{\beta(B)}{2^{n(B)}}. \tag{2.3}$$

From (2.2) we get $2^{n(B)}3^{m(A)} = 2^{n(A)}3^{m(B)}$, that is, $n(B) = n(A)$ and $m(B) = m(A)$. From this and (2.3) we get $\beta(A) = \beta(B)$.

Let $\sigma(A) = (\sigma_1, \dots, \sigma_n)$ and $\sigma(B) = (\tau_1, \dots, \tau_n)$. Then $\beta(A) = \sum_{k=1}^{n} 2^{k-1}\sigma_k$ and $\beta(B) = \sum_{k=1}^{n} 2^{k-1}\tau_k$, so $\sum_{k=1}^{n} 2^{k-1}(\sigma_k - \tau_k) = 0$. Suppose there is some i such that $\sigma_i \neq \tau_i$. Take the smallest such i. Then

$$\begin{aligned} 0 &= \sum_{k=1}^{n} 2^{k-1}(\sigma_k - \tau_k) = \sum_{k=i}^{n} 2^{k-1}(\sigma_k - \tau_k) = 2^{i-1}\sum_{k=i}^{n-1} 2^{k-i}(\sigma_k - \tau_k) \\ &= 2^{i-1}\left((\sigma_i - \tau_i) + 2\sum_{k=i+1}^{n} 2^{k-(i+1)}(\sigma_k - \tau_k)\right). \end{aligned}$$

Since $\sigma_k - \tau_k$ is odd, we get a contradiction. Thus, $\sigma_k = \tau_k$ for all k, that is, $\sigma(A) = \sigma(B)$, and so $A = B$. □

3. Density of orbits

The action of the semigroup S on $(0, \infty)$, which we introduced in the preceding section, defines a (noncommutative) dynamical system (see, e.g., [3], where the name *flow* is used; although in our case the phase space $(0, \infty)$ is not compact, we can still use the same basic definitions). In this section we will investigate its orbits $\mathrm{Orb}(x) = \{T(x) : T \in S\}$.

If $A_n = (1, 1, \dots, 1)$ (n times), then $\lim_{n\to\infty} T_{A_n}(x) = \infty$ for any x. Similarly, if $B_n = (0, 0, \dots, 0)$ (n times), then $\lim_{n\to\infty} T_{B_n}(x) = 0$. Thus, all orbits are unbounded and have 0 as an accumulation point. It turns out that they are even dense in $(0, \infty)$. In other words, our system is minimal (see [3]).

Theorem 3.1. *Given $x, y > 0$ and $\varepsilon > 0$, there is A such that $|T_A(x) - y| < \varepsilon$.*

Proof. We will find A in the form $A = (1, 1, \dots, 1, 0, \dots, 0)$ (with m 1's and $n - m$ 0's). We have $\sigma_1 = 3^{m-1}, \sigma_2 = 3^{m-2}, \dots, \sigma_m = 3^0, \sigma_{m+1} = \dots = \sigma_n = 0$, so

$$\beta(A) = \sum_{k=1}^{m} 2^{k-1}3^{m-k} = 3^{m-1}\sum_{k=0}^{m-1}\left(\frac{2}{3}\right)^k = 3^{m-1}\frac{1-\left(\frac{2}{3}\right)^m}{1-\frac{2}{3}} = 3^m - 2^m.$$

Thus,

$$T_A(x) = \frac{3^m x + 3^m - 2^m}{2^n} = \frac{(x+1)3^m}{2^n} - \frac{1}{2^{n-m}}.$$

Therefore, if

$$\left|\frac{3^m}{2^n}(x+1) - y\right| < \varepsilon - \frac{1}{2^{n-m}},$$

then $|T_A(x) - y| < \varepsilon$.

Since $\log_2 3$ is irrational, for every $\delta > 0$ there exist m, n arbitrarily large such that $n > m$ and

$$\left|m \log_2 3 - n - \log_2 \frac{y}{x+1}\right| < \delta.$$

Then

$$2^{-\delta} - 1 < \frac{3^m/2^n}{y/(x+1)} - 1 < 2^{\delta} - 1,$$

so

$$(2^{-\delta} - 1)y < \frac{3^m}{2^n}(x+1) - y < (2^{\delta} - 1)y.$$

If δ is sufficiently small and m, n sufficiently large, then $n - m$ is also large (since n/m is approximately $\log_2 3$), so

$$(2^{\delta} - 1)y < \varepsilon - \frac{1}{2^{n-m}}$$

and

$$(1 - 2^{-\delta})y < \varepsilon - \frac{1}{2^{n-m}}.$$

Then

$$\left|\frac{3^m}{2^n}(x+1) - y\right| < \varepsilon - \frac{1}{2^{n-m}},$$

as we wanted. □

Corollary 3.2. *The dynamical system $((0, \infty), S)$ is minimal.*

When applying T_A to x, we can think about it as consecutive applications of T_0 or T_1, $n(A)$ times. In other words, we look at $T_{A_0}(x) = x, T_{A_1}(x), T_{A_2}(x), \dots, T_{A_n}(x) = T_A(x)$, where $A_i = (\varepsilon_1, \dots, \varepsilon_i)$. We will refer to it as a *path from x to $T_A(x)$*. In the proof of Theorem 3.1 this path was passing closer and closer to infinity as ε was smaller and smaller. We will show that this is not necessary. Namely, we will construct paths from x to points arbitrarily close to y that are bounded by constants depending only on x and y. Before we do it we prove a lemma which will be essential also for the next section.

Lemma 3.3. *Assume that for every $k \le n(A)$ we have $T_{A_k}(x) \le b$. Then*

$$\frac{2^{n(A)}}{3^{m(A)}} \ge (m(A)-1)\frac{x}{3b^2}.$$

Proof. Let $A = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$. Set

$$s_k = T_{A_k}(x)\frac{2^k}{3^{m(A_k)}}.$$

In particular, $s_0 = x$. If $\varepsilon_{k+1} = 0$, then

$$T_{A_{k+1}}(x) = T_0(T_{A_k}(x)) = \frac{1}{2}T_{A_k}(x)$$

and $m(A_{k+1}) = m(A_k)$, so

$$s_{k+1} = T_{A_{k+1}}(x)\frac{2^{k+1}}{3^{m(A_{k+1})}} = \frac{1}{2}T_{A_k}(x)\frac{2^{k+1}}{3^{m(A_k)}} = s_k.$$

If $\varepsilon_{k+1} = 1$, then

$$T_{A_{k+1}}(x) = T_1(T_{A_k}(x)) = \frac{3}{2}T_{A_k}(x) + \frac{1}{2}$$

and $m(A_{k+1}) = m(A_k) + 1$, so

(3.1)
$$s_{k+1} = T_{A_{k+1}}(x)\frac{2^{k+1}}{3^{m(A_{k+1})}} = \frac{3}{2}T_{A_k}(x)\frac{2^{k+1}}{3^{m(A_k)+1}} + \frac{1}{2}\cdot\frac{2^{k+1}}{3^{m(A_k)+1}} = s_k + \frac{1}{3}\cdot\frac{2^k}{3^{m(A_k)}}.$$

Therefore, since $s_k \ge s_0 = x$ for all k, so

$$\frac{2^k}{3^{m(A_k)}} = \frac{s_k}{T_{A_k}(x)} \ge \frac{x}{T_{A_k}(x)} \ge \frac{x}{b}.$$

Thus, by induction (use (3.1) each time $\varepsilon_{i+1} = 1$) we get

$$s_k \ge \frac{m(A_k)-1}{3}\cdot\frac{x}{b},$$

so

$$\frac{2^k}{3^{m(A_k)}} = \frac{s_k}{T_{A_k}(x)} \ge \frac{m(A_k)-1}{3}\cdot\frac{x}{b^2}.$$

In particular, for $k = n(A)$ we get

$$\frac{2^{n(A)}}{3^{m(A)}} \ge (m(A)-1)\frac{x}{3b^2}.$$

□

Now we can strengthen Theorem 3.1.

Theorem 3.4. *Given $x, y > 0$ and $\varepsilon > 0$, there is A such that $\mid T_A(x) - y \mid < \varepsilon$ and*

$$\min(x, y-\varepsilon) \le T_{A_i}(x) \le \max(4y - x, 11x + 4)$$

for all $i \le n(A)$.

Proof. Assume first that $y > \frac{1}{2}$. If we apply T_1^{-1} to $y_0 = y$, we get a point y_1 such that $y_1 < y$. Then we follow a simple rule. If $y_k < y$, then we apply T_0^{-1}; if $y_k > y$, then we apply T_1^{-1} to get y_{k+1}. Repeating this n times, we build $T_{A^{(n)}}^{-1}$ for some sequence $A^{(n)}$ with $n(A^{(n)}) = n$. We set $m(A^{(n)}) = m_n$. Note that our points $y_0, y_1, \dots, y_n$ are all in $[T_1^{-1}(y), T_0^{-1}(y)]$. These points are the same as points $T_{A_k^{(n)}}(y_n)$ for $k = n, n-1, \dots, 0$ (in particular, $T_{A^{(n)}}(y_n) = y$). Hence, by Lemma 3.3,

$$\frac{2^n}{3^{m_n}} \geq (m_n - 1)\frac{y_n}{3(T_0^{-1}(y))^2}.$$

Since $y_n \geq T_1^{-1}(y) = \frac{2y-1}{3}$ and $T_0^{-1}(y) = 2y$, we get

$$\frac{2^n}{3^{m_n}} \geq (m_n - 1)\frac{\frac{2y-1}{3}}{12y^2} = (m_n - 1)\frac{2y-1}{36y^2}. \tag{3.2}$$

Assume additionally that $T_1^{-1}(y) > x$. Then we take the smallest n such that with the construction above, $y_n - \varepsilon\frac{2^n}{3^{m_n}} < x$. Such an n exists by (3.2) and since $m_n \to \infty$ as $n \to \infty$ (this follows immediately from our construction).

Observe that for any i the derivative of $T_{A^{(i)}}$ is $\frac{3^{m_i}}{2^i}$, so

$$T^{-1}_{A^{(i)}}((y-\varepsilon, y+\varepsilon)) = \left(y_i - \varepsilon\frac{2^i}{3^{m_i}}, y_i + \varepsilon\frac{2^i}{3^{m_i}}\right).$$

Thus, since

$$y_n - \varepsilon\frac{2^n}{3^{m_n}} < x < T_1^{-1}(y) \leq y_n < y_n + \varepsilon\frac{2^n}{3^{m_n}},$$

we get $T_{A^{(n)}}(x) \in (y-\varepsilon, y+\varepsilon)$ and for $i \leq n$ we have

$$T_{A_i^{(n)}}(x) \in \left(y_{n-i} - \varepsilon\frac{2^{n-i}}{3^{m_{n-i}}}, y_{n-i} + \varepsilon\frac{2^{n-i}}{3^{m_{n-i}}}\right).$$

Therefore $T_{A_i^{(n)}}(x) > x$ (by the minimality of n) and

$$T_{A_i^{(n)}}(x) < T_0^{-1}(y) + \varepsilon \max_{i \leq n}\frac{2^i}{3^{m_i}}.$$

If $i \leq n$, then $x \leq y_i - \varepsilon\frac{2^i}{3^{m_i}}$, so

$$\varepsilon \max_{i \leq n}\frac{2^i}{3^{m_i}} \leq \max_{i \leq n}(y_i - x) \leq T_0^{-1}(y) - x = 2y - x.$$

Thus, $T_{A_i^{(n)}}(x) < 2y + (2y - x) = 4y - x$.

Now we have to remove the assumption that $y > \frac{1}{2}$ and $T_1^{-1}(y) > x$. We do this by replacing y by $\tilde{y} = y \cdot 2^j$ which satisfies these two assumptions, constructing $A^{(n)}$ as above and then appending $A^{(n)}$ with a string of j 0's to get A. Since we have $|T_{A^n}(x) - \tilde{y}| < \varepsilon$, we get $|T_A(x) - y| < \frac{\varepsilon}{2^j} \leq \varepsilon$.

We can take the minimal non-negative j when choosing $\tilde{y}$. If $j = 0$, then, as we proved, $x < T_{A_i^{(n)}}(x) < 4y - x$ for all i. If $j > 0$, then $y - \varepsilon < T_{A_i} < 4\tilde{y} - x$. The conditions $\tilde{y} > \frac{1}{2}$ and $T_1^{-1}(\tilde{y}) > x$ can be written as one condition $\tilde{y} > \max\left(\frac{1}{2}, T_1(x)\right)$. Since j is minimal, we get

$$\tilde{y} \leq 2\max\left(\frac{1}{2}, T_1(x)\right) = \max(1, 3x+1) = 3x+1.$$

Thus, $4\tilde{y} - x \leq 4(3x+1) - x = 11x + 4$.

Putting the above estimates together, we get for all $i \leq n(A)$,

$$\min(x, y-\varepsilon) \leq T_{A_i}(x) \leq \max(4y - x, 11x + 4).$$

□

4. Periodic points

Following the usual terminology we will call a point x *periodic* for the action of the semigroup S if there is a $T_A \in S$, $A \neq \emptyset$, such that $T_A(x) = x$. In this section we will prove that periodic points are dense in $(0, \infty)$. Moreover, acting in the spirit of Theorem 3.4, we will construct paths from periodic points to themselves with explicit lower and upper bounds.

We start with the following simple lemma.

Lemma 4.1. *Suppose* $|T_A(x) - x| < \varepsilon$ *and* $\frac{3^{m(A)}}{2^{n(A)}} < \delta < 1$. *Then* T_A *has a fixed point* z *such that* $|x - z| < \frac{\varepsilon}{1-\delta}$.

Proof. By Lemma 2.1, z is a fixed point of T_A if and only if

$$z = \frac{\beta(A)}{2^{n(A)} - 3^{m(A)}}$$

(this formula is well known; see [1], [5], [8]).

Set $z = \frac{2^{n(A)} T_A(x) - 3^{m(A)} x}{2^{n(A)} - 3^{m(A)}}$. By Lemma 2.1, we have

$$z = \frac{2^{n(A)} \left(\frac{3^{m(A)}}{2^{n(A)}} x + \frac{\beta(A)}{2^{n(A)}} \right) - 3^{m(A)} x}{2^{n(A)} - 3^{m(A)}} = \frac{\beta(A)}{2^{n(A)} - 3^{m(A)}},$$

so z is the fixed point of T_A. Now,

$$\begin{aligned} |x - z| &= \left| x - \frac{2^{n(A)} T_A(x) - 3^{m(A)} x}{2^{n(A)} - 3^{m(A)}} \right| = \left| \frac{2^{n(A)} (x - T_A(x))}{2^{n(A)} - 3^{m(A)}} \right| \\ &= \left| \frac{2^{n(A)}}{2^{n(A)} - 3^{m(A)}} \right| \cdot |x - T_A(x)| = \frac{1}{1 - \frac{3^{m(A)}}{2^{n(A)}}} \cdot |x - T_A(x)| < \frac{\varepsilon}{1 - \delta}. \end{aligned}$$

□

Now we can prove the main result of the section.

Theorem 4.2. *Given* $x > 0$ *and* $\varepsilon > 0$, *there are* $z > 0$ *and* A *such that* $|z - x| < \varepsilon$, $T_A(z) = z$ *and*

$$\frac{x}{4} \leq T_{A_i}(z) \leq \frac{33x + 20}{2}$$

for all $i \leq n(A)$.

Proof. We make a similar construction as in the proof of Theorem 3.4, except that now we move forward. That is, we set $x = x_0, x_1 = T_1(x)$ and $x_{n+1} = T_0(x)$ if $x_n > x$ and $x_{n+1} = T_1(x)$ if $x_n < x$ (if $x_n = x$, then we set $z = x$ and we are done). We do this as many times as we need in order to build a sufficiently large $m(A')$ for the corresponding sequence A'. Then we apply Theorem 3.4 to get from x_n to a small neighborhood of x. In such a way we get a sequence A such that $|T_A(x) - x|$ is as small as we wish (in particular, less than ε) and $m(A)$ is as large as we wish.

Clearly,

$$\frac{x}{2} \leq x_i \leq \frac{3x + 1}{2} \tag{4.1}$$

for all $i \leq n$. In particular, $\frac{x}{2} \leq x_n \leq \frac{3x+1}{2}$. By Theorem 3.4 we have

$$\min(x_n, x - \varepsilon) \leq T_{A_i}(x_n) \leq \max(4x - x_n, 11x_n + 4)$$

for $n \le i \le n(A)$. We have

$$\min(x_n, x-\varepsilon) \ge \min\left(\frac{x}{2}, x-\varepsilon\right)$$

and

$$\max(4x - x_n, 11x_n + 4) \le \max\left(4x - \frac{x}{2}, 11 \cdot \frac{3x+1}{2} + 4\right) = \frac{33x+19}{2}.$$

Thus,

$$\min\left(\frac{x}{2}, x-\varepsilon\right) \le T_{A_i}(x) \le \frac{33x+19}{2}$$

for all $n \le i \le n(A)$.

Together with (4.1) we get

$$\min\left(\frac{x}{2}, x-\varepsilon\right) \le T_{A_i}(x) \le \frac{33x+19}{2} \tag{4.2}$$

for all $i \le n(A)$.

Now by Lemma 3.3 we know that

$$\frac{3^m}{2^n} \le \frac{3\left(\frac{33x+19}{2}\right)^2}{(m(A)-1)x}, \tag{4.3}$$

and since $m(A)$ is as large as we want, the right-hand side of (4.3) is as small as we want. Now we can apply Lemma 4.1, and we get a fixed point z of T_A with $|z-x| < \varepsilon$.

We have by Lemma 3.3 for $i \le n(A)$,

$$\frac{T_{A_i}(z) - T_{A_i}(x)}{z-x} = \frac{3^{m(A_i)}}{2^{n(A_i)}} \le \frac{3\left(\frac{33x+19}{2}\right)^2}{(m(A_i)-1)x}.$$

Thus, if $m(A_i) \ge 2$, then

$$|T_{A_i}(z) - T_{A_i}(x)| \le |z-x| \frac{3\left(\frac{33x+19}{2}\right)^2}{(m(A_i)-1)x} \le \varepsilon \frac{3(33x+19)^2}{4x}.$$

If $m(A_i) \le 1$, then

$$\frac{T_{A_i}(z) - T_{A_i}(x)}{z-x} = \frac{3^{m(A_i)}}{2^{n(A_i)}} \le 3,$$

so $|T_{A_i}(z) - T_{A_i}(x)| \le 3\varepsilon$.

Together with (4.2) we get for all $i \le n(A)$,

$$\begin{aligned}\min\left(\frac{x}{2}, x-\varepsilon\right) - \max\left(\frac{3(33x+19)^2}{4x}, 3\right)\varepsilon &\le T_{A_i}(z) \\ &\le \frac{33x+19}{2} + \max\left(\frac{3(33x+19)^2}{4x}, 3\right)\varepsilon.\end{aligned}$$

We can take ε as small as we wish and then $x - \varepsilon > \frac{x}{2}$ and

$$\max\left(\frac{3(33x+19)^2}{4x}, 3\right)\varepsilon \le \min\left(\frac{1}{2}, \frac{x}{4}\right).$$

Then

$$\frac{x}{4} \le T_{A_i}(z) \le \frac{33x+20}{2}.$$

□

5. The full group

In Section 2 we considered the semigroup S generated by T_0 and T_1. Let us consider now the group G generated by T_0 and T_1 (acting on the real line). It turns out that we can easily identify all the elements of this group.

Theorem 5.1. *The group G consists of all maps of the form*

$$x \mapsto 2^s 3^t x + \frac{k}{2^i 3^j},$$

where $k, s, t \in \mathbf{Z}$ and $i, j \in \mathbf{N}$.

Proof. Let $f = T_0^{-1} T_1^{-1} T_0 T_1$. Then

$$f(x) = 2 \cdot \frac{\left(2\frac{1}{2} \cdot \frac{3x+1}{2} - 1\right)}{3} = \frac{2}{3} \cdot \frac{3x-1}{2} = x - \frac{1}{3}.$$

Now let $g = T_0^{-1} T_1 f$. Then

$$g(x) = \frac{2\left(3\left(x - 1/3\right) + 1\right)}{2} = 3x - 1 + 1 = 3x.$$

For $s, t \in \mathbf{Z}$ we have $T_0^{-s} g^t(x) = 2^s 3^t x$. Observe that $f^{-3}(x) = x + 1$. Now,

$$h(x) = g^{-j} T_0^{-i} f^{-3} T_0^{-i} g^j(x) = \frac{1}{2^i 3^j}(2^i 3^j x + 1) = x + \frac{1}{2^i 3^j}$$

and

$$h^k T_0^{-s} g^t(x) = 2^s 3^t x + \frac{k}{2^i 3^j}.$$

This proves that all functions of the form

$$x \mapsto 2^s 3^t x + \frac{k}{2^i 3^j}$$

are in G. On the other hand, applying any of $T_0, T_0^{-1}, T_1, T_1^{-1}$ to such a function gives a function of the same form. □

Note that while S was a free semigroup, G is not a free group (for instance, T_0 and g commute). However, G is finitely presented.

Theorem 5.2. *The group G is isomorphic to the group*

$$H = \langle a, b, c \mid ab = ba,\ ac = c^2 a,\ bc = c^3 b \rangle.$$

Therefore G is a finitely presented group.

Proof. From Theorem 5.1 it follows easily that the maps R_a, R_b, R_c, given by $R_a(x) = 2x$, $R_b(x) = 3x$ and $R_c(x) = x + 1$ generate G. Clearly, $R_a R_b = R_b R_a$, $R_a R_c = R_c^2 R_a$ and $R_b R_c = R_c^3 R_b$. Thus, $\varphi(a) = R_a$, $\varphi(b) = R_b$, $\varphi(c) = R_c$ generate an epimorphism $\varphi : H \to G$. The relations in H and the relations $ca^{-1} = a^{-1}c^2$, $cb^{-1} = b^{-1}c^3$, which can be derived from them, allow us to represent every element of H in the form $d = a^{-i} b^{-j} c^k a^m b^n$, with $i, j, m, n \geq 0$. If such a d is in the kernel of φ, then $x \equiv (\varphi(d))(x) = 2^{m-i} 3^{n-j} x + k 2^{-i} 3^{-j}$, so $m = i$, $n = j$ and $k = 0$, and therefore d is the unit of H. This shows that φ is an isomorphism. □

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