PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 4, Pages 1239–1246
S 0002-9939(04)07698-1
Article electronically published on October 18, 2004

UNCORRELATEDNESS SETS FOR RANDOM VARIABLES WITH GIVEN DISTRIBUTIONS

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(Communicated by Richard C. Bradley)

ABSTRACT. Let ξ_1 and ξ_2 be random variables having finite moments of all orders. The set

$$U(\xi_1,\xi_2) := \left\{(j,l)\in\mathbf{N}^2 : \mathbf{E}\left(\xi_1^j\xi_2^l\right) = \mathbf{E}\left(\xi_1^j\right)\mathbf{E}\left(\xi_2^l\right)\right\}$$

is said to be an *uncorrelatedness set* of ξ_1 and ξ_2. It is known that in general, an uncorrelatedness set can be arbitrary. Simple examples show that this is not true for random variables with given distributions. In this paper we present a wide class of probability distributions such that there exist random variables with given distributions from the class having a prescribed uncorrelatedness set. Besides, we discuss the sharpness of the obtained result.

1. Introduction

Since the notion of independence is a basic one in Probability Theory and Mathematical Statistics, various generalizations of independence have been studied. One of the earliest and most used generalizations is *uncorrelatedness* of random variables ξ_1 and ξ_2 defined by the condition

$$\mathbf{E}(\xi_1\xi_2) = \mathbf{E}(\xi_1)\mathbf{E}(\xi_2),$$

provided that all of the mathematical expectations exist. It is commonly known (e.g. [6], vol. I, p. 236) that uncorrelatedness is an essentially weaker condition than independence. In this paper we discuss problems related to uncorrelatedness of powers of random variables. We need the following definition.

Definition. Let ξ_1 and ξ_2 be random variables having finite moments of all orders. The set

$$U(\xi_1,\xi_2) := \left\{(j,l)\in\mathbf{N}^2 : \mathbf{E}\left(\xi_1^j\xi_2^l\right) = \mathbf{E}\left(\xi_1^j\right)\mathbf{E}\left(\xi_2^l\right)\right\}$$

is said to be an *uncorrelatedness set* of ξ_1 and ξ_2.

Clearly, an uncorrelatedness set is uniquely defined for random variables with all finite moments. Obviously, if ξ_1 and ξ_2 are *independent* random variables, then $U(\xi_1,\xi_2) = \mathbf{N}^2$. Note that the converse statement is not true, as was proved in [12].

Received by the editors September 22, 2003 and, in revised form, December 22, 2003.
2000 *Mathematics Subject Classification.* Primary 60E05.
Key words and phrases. Uncorrelatedness, independence, uncorrelatedness set, quasianalytic class, characteristic function.

Uncorrelatedness sets may be regarded as a measure of independence in the following sense: the wider an uncorrelatedness set is, the more independent random variables are. We notice that if

$$U(\xi_1,\xi_2)\supseteq\{(j,l): j+l\le k+1\},$$

then random variables ξ_1 and ξ_2 are *independent of degree k.* Independence of degree k was defined by C. M. Cuadras in [3] (cf. also [4]). For sequences of random variables, the uncorrelatedness condition arises in some limit theorems (cf. [7] and [8]).

Measures of independence for random variables have been intensively studied lately. Information on recent developments concerning measures and generalizations of independence can be found, for example, in [1], [5], [13], [14], [17].

It was proved in [11] that in general an uncorrelatedness set may be arbitrary. That is, for any set $U\subseteq\mathbf{N}^2$ there exist random variables ξ_1 and ξ_2 such that

$$\mathbf{E}\left(\xi_1^j\xi_2^l\right)=\mathbf{E}\left(\xi_1^j\right)\mathbf{E}\left(\xi_2^l\right)\quad\text{for all}\quad (j,l)\in U$$

and

$$\mathbf{E}\left(\xi_1^j\xi_2^l\right)\ne\mathbf{E}\left(\xi_1^j\right)\mathbf{E}\left(\xi_2^l\right)\quad\text{for all}\quad (j,l)\in \mathbf{N}^2\setminus U.$$

This result shows that uncorrelatedness of any positive integer powers of random variables does not imply uncorrelatedness of any other powers. The statement does not remain true if we prescribe distributions of random variables.

For example, for discrete random variables ξ_1 and ξ_2 taking two values, uncorrelatedness implies independence. In other words, for such random variables,

$$U(\xi_1,\xi_2)\ni(1,1)\Rightarrow U(\xi_1,\xi_2)=\mathbf{N}^2.$$

For random variables with fixed absolutely continuous distributions an uncorrelatedness set cannot be arbitrary either. Indeed, consider random variables ξ_1 and ξ_2 uniformly distributed on [0,1]. Using the Müntz Theorem (see, for example, [2] Chapter 6) one can prove that if $U(\xi_1,\xi_2)$ contains a lattice $\{(\lambda_i,\mu_j):(i,j)\in\mathbf{N}^2\}$, where

$$\sum_{i=1}^{\infty}\frac{1}{\lambda_i}=\infty\ \text{ and }\ \sum_{j=1}^{\infty}\frac{1}{\mu_j}=\infty,$$

then ξ_1 and ξ_2 are independent. Therefore, $U(\xi_1,\xi_2)=\mathbf{N}^2$.

We note that interesting examples of random variables with unusual uncorrelatedness sets are given in [15], Section 7.

In this paper we present a rather wide class of probability distributions such that there exist random variables with given distributions from the class having a prescribed uncorrelatedness set. Besides, we discuss the sharpness of the obtained result.

2. Statement of results

In this paper we show that for rather slowly decreasing probability densities (however, having finite moments of all orders) and any $U\subseteq\mathbf{N}^2$ there exist random variables ξ_1 and ξ_2 with these densities such that $U(\xi_1,\xi_2)=U$. Our proof assumes that the densities in question satisfy the Krein condition (cf. [9]). Important examples of such densities are given in [16]. Our main result is the following statement.

Theorem. *Let $\rho_1(x)$ and $\rho_2(x)$ be probability densities having finite moments of all orders and satisfying the condition*

$$\rho_i(x) \geq C_i e^{-|x|^\alpha},\ 0 < \alpha < 1\ (i = 1, 2),$$

and let $U \subseteq \mathbf{N}^2$ be given.

There exist random variables ξ_1 and ξ_2 such that

i) ξ_1 and ξ_2 have absolutely continuous distributions with the densities ρ_1 and ρ_2, respectively;

ii) $U(\xi_1, \xi_2) = U$.

Remark. The condition $0 < \alpha < 1$ is essential and, in some sense, it cannot be improved. To be more specific, for any $\alpha \geq 1$ there exists a set $U \subset \mathbf{N}^2$ that is not an uncorrelatedness set for any random variables ξ_1, ξ_2 whose densities ρ_1, ρ_2 satisfy the condition

$$\rho_i(x) \leq C e^{-|x|^\alpha}, \quad \alpha \geq 1 \ \ (i = 1, 2). \tag{1}$$

Moreover, the statement remains true if we replace (1) with the more general condition:

$$\mathbf{E}\left[\exp\{\delta|\xi_i|\}\right] < \infty \ \text{ for some } \ \delta > 0,\ i = 1, 2. \tag{2}$$

Indeed, let $\xi = (\xi_1, \xi_2)$ be a random vector with coordinates ξ_1, ξ_2 satisfying (2), characteristic function $\varphi(t_1, t_2)$ and uncorrelatedness set U. Assume that $\mathbf{N}^2 \backslash U \neq \emptyset$ is finite.

We set $f(t_1, t_2) := \varphi(t_1, t_2) - \varphi(t_1, 0)\varphi(0, t_2)$. Conditions (2) imply analyticity of $f(t_1, t_2)$ in $\{t_1 : |\operatorname{Im} t_1| < \delta/2\} \times \{t_2 : |\operatorname{Im} t_2| < \delta/2\}$ (cf. [10]). Besides,

$$|f(t_1, t_2)| \leq 2 \ \text{ for all } \ (t_1, t_2) \in \mathbf{R}^2. \tag{3}$$

Since $\mathbf{N}^2 \setminus U$ is a nonempty finite set, $f(t_1, t_2)$ is a non-constant polynomial, which contradicts (3).

3. Preliminaries

In this section we introduce notation and state some results needed for the sequel.

Let $\rho(x) = C_\alpha e^{-|x|^\alpha}$ be a probability density. Obviously,

$$\rho(x) \leq a_i \rho_i(x) \ \text{ for some } \ a_i > 0\ (i = 1, 2). \tag{4}$$

Consider the function $T(r)$, $r \geq 0$ defined by

$$T(r) = \begin{cases} 1, & \text{if } \ r \leq 1, \\ e^{r^\beta - 1}, & \text{if } \ r > 1, \end{cases}$$

where $\beta \in (\alpha, 1)$. It can be readily seen that the function $T(r)$ satisfies the following conditions:

i) $T(r) \geq 1$;

ii) $\ln T(r)$ is a convex function of $\ln r$;

iii)

$$\int_0^\infty \frac{\ln T(r)}{r^2}\, dr < \infty.$$

Further, we apply the procedure described in [9], Ch. IV, p. 98. We set

$$M_n := \sup_{r>0} \frac{r^n}{T(r)} \ \text{ for } \ n = 0, 1, \ldots. \tag{5}$$

We note that $M_0 = 1$. Since

$$M_n = \sup_{r>0} \frac{r^n}{T(r)} = \sup_{r\geq 1} \frac{r^n}{T(r)} \leq \sup_{r\geq 1} \frac{r^{n+1}}{T(r)} =: M_{n+1},$$

the sequence $\{M_n\}$ is non-decreasing.

Let us construct Ostrowski's function for the sequence $\{M_n\}$:

$$\tilde{T}(r) := \sup_{n\geq 0} \frac{r^n}{M_n}. \tag{6}$$

We have

$$\tilde{T}(r) = \frac{1}{M_0} = 1 \ \text{ for } \ r \leq 1, \tag{7}$$

and it follows from the known assertion (cf. [9], Ch. IV, lemma on p. 98) that

$$\ln \tilde{T}(r) \leq \ln T(r) \leq \ln \tilde{T}(r) + \ln r \ \text{ for } \ r > 1. \tag{8}$$

Inequalities (8) imply that

$$\int_0^\infty \frac{\ln \tilde{T}(r)}{r^2}\, dr < \infty.$$

Hence by the Carleman - Ostrowski Theorem (cf., e.g. [9], Ch. IV, p. 89) the class $C_I(\{M_n\})$ is not quasianalytic for any interval I.

For the sequel we need the following lemma.

Lemma 1. *Let $\{M_n\}$ be a sequence such that the class $C_I(\{M_n\})$ is not quasianalytic for any interval I. Then there exists an infinitely differentiable function $\Delta(x)$ having the following properties:*

i) $\operatorname{supp} \Delta(x) = [-1, 1]$;

ii) $\Delta(x) =$ const $\neq 0$ *for* $x \in [-1/2, 1/2]$;

iii) $\Delta(x)$ *is even;*

iv) $|\Delta^{(n)}(x)| \leq M_n$ *for all* $n = 0, 1, \ldots$.

Proof. Let a sequence $\{\underline{M}_n\}$ be the convex logarithmic regularization of $\{M_n\}$ (cf. [9], Ch. IV, p. 84). Note that $\{\underline{M}_n\}$ is non-decreasing along with $\{M_n\}$. We set

$$L_n := \frac{\underline{M}_n}{2^n},\ n = 0, 1, 2, \ldots.$$

The class $C_I(\{L_n\})$ is not quasianalytic for any interval I (cf. [9], Ch. IV, p. 91). Therefore, there exists a function $f(x)$ such that $\operatorname{supp} f(x) \in [-1, -1/2]$ and $|f^{(n)}(x)| \leq \underline{L}_n$, where $\underline{L}_n$ is the convex logarithmic regularization of $\{L_n\}$. The function $\varphi(x) := f^2(x)$ is nonnegative, $\operatorname{supp} \varphi(x) \in [-1, -1/2]$ and in addition (cf. [9], p. 107)

$$|\varphi^{(n)}(x)| \leq \underline{L}_0 2^n \underline{L}_n \leq \underline{L}_0 2^n L_n = \underline{L}_0 \underline{M}_{n+1},\ n = 0, 1, 2, \ldots.$$

Clearly, for $\psi(x) := \varphi(x)/\underline{L}_0$ we have

$$\left|\psi^{(n)}(x)\right| \leq \underline{M}_{n+1} \leq M_{n+1}, \ \ n = 0, 1, 2, \ldots. \tag{9}$$

Let $\tilde{\psi}(x)$ be an *odd* function coinciding with $\psi(x)$ on $(-\infty, 0]$. Obviously, estimate (9) is valid for $\tilde{\psi}(x)$, that is, $\left|\tilde{\psi}^{(n)}(x)\right| \leq \underline{M}_{n+1}$. We set

$$\Delta(x) := c \int_{-1}^{x} \tilde{\psi}(t) dt,$$

where $c \in (0,1]$ is chosen in such a way that $\Delta(0) \le M_0$. Evidently, the function $\Delta(x)$ satisfies the conditions *i) - iii)* of the lemma. Since

$$\left|\Delta^{(n)}(x)\right| = \left|c\tilde{\psi}^{n-1}(x)\right| \le M_n \text{ and } |\Delta(x)| \le M_0,$$

we conclude that $\Delta(x)$ is a required function. □

4. Proof of the Theorem

Let $U \subset \mathbf{N}^2$ be given. Consider an entire function

$$\zeta(t_1,t_2) := \sum_{j,l=0}^{\infty} \frac{a_{jl}}{j!\,l!}\, t_1^j t_2^l,$$

where the coefficients a_{jl} are chosen in such a way that

a) $a_{jl} = 0$ if $(j,l) \in U$ or $\min(j,l) = 0$;
b) $a_{jl} \ne 0$ is real if $(j,l) \notin U$ and $j+l$ is even;
c) $a_{jl} \ne 0$ is imaginary if $(j,l) \notin U$ and $j+l$ is odd;
d) $\sum_{j,l=0}^{\infty} |a_{jl}| < 1$.

Condition *d)* implies that

$$\left|\frac{\partial^{m+n}\zeta(t_1,t_2)}{\partial t_1^m \partial t_2^n}\right| \le 1 \text{ for } |t_1|,\, |t_2| \le 1 \text{ and all } m,\, n \in \mathbf{N}. \tag{10}$$

It follows from *a) - c)* that

$$\zeta(-t_1,-t_2) = \overline{\zeta(t_1,t_2)}. \tag{11}$$

Let $\{M_n\}$ be the numbers given by (5), and let $\Delta(t)$ be a function from Lemma 1. We set

$$\eta(t_1,t_2) := \zeta(t_1,t_2)\Delta(t_1)\Delta(t_2).$$

Note that

$$\eta(t_1,0) = \eta(0,t_2) = \eta(0,0) = 0. \tag{12}$$

Consider the inverse Fourier transform of $\eta(t_1,t_2)$:

$$q(x_1,x_2) := \frac{1}{(2\pi)^2}\int\!\!\int_{\mathbf{R}^2} e^{-i(t_1x_1+t_2x_2)}\eta(t_1,t_2)\,dt_1dt_2 =$$

$$\frac{1}{(2\pi)^2}\int_{-1}^{1} e^{-it_1x_1}\Delta(t_1)\,dt_1 \int_{-1}^{1} e^{-it_2x_2}\zeta(t_1,t_2)\Delta(t_2)\,dt_2. \tag{13}$$

We note that the function $q(x_1,x_2)$ is real due to (11). Integrating by parts n times we get

$$\int_{-1}^{1} e^{-it_2x_2}\zeta(t_1,t_2)\Delta(t_2)\,dt_2 = \frac{1}{i^n\,x_2^n}\int_{-1}^{1} e^{-it_2x_2}\frac{\partial^n}{\partial t_2^n}\Big(\zeta(t_1,t_2)\Delta(t_2)\Big)\,dt_2.$$

By the Leibnitz formula,

$$\frac{\partial^n}{\partial t_2^n}\Big(\zeta(t_1,t_2)\Delta(t_2)\Big) = \sum_{k=0}^{n}\binom{n}{k}\Delta^{(k)}(t_2)\frac{\partial^{n-k}\zeta(t_1,t_2)}{\partial t_2^{n-k}}.$$

Therefore,

$$q(x_1,x_2) = \frac{1}{i^n\,x_2^n}\sum_{k=0}^{n}\binom{n}{k}\int_{-1}^{1} e^{-it_2x_2}\Delta^{(k)}(t_2)dt_2\int_{-1}^{1} e^{-it_1x_1}\frac{\partial^{n-k}\zeta(t_1,t_2)}{\partial t_2^{n-k}}\,\Delta(t_1)\,dt_1.$$

Using m times integration by parts and the Leibnitz formula, we obtain

$$\begin{aligned}&\int_{-1}^{1} e^{-it_1x_1}\frac{\partial^{n-k}\zeta(t_1,t_2)}{\partial t_2^{n-k}}\Delta(t_1)\,dt_1\\ &\quad= \frac{1}{i^m\,x_1^m}\int_{-1}^{1} e^{-it_1x_1}\frac{\partial^m}{\partial t_1^m}\left(\frac{\partial^{n-k}\zeta(t_1,t_2)}{\partial t_2^{n-k}}\Delta(t_1)\right)dt_1\\ &\quad= \frac{1}{i^m\,x_1^m}\sum_{j=0}^{m}\binom{m}{j}\int_{-1}^{1} e^{-it_1x_1}\frac{\partial^{m-j+n-k}\zeta(t_1,t_2)}{\partial t_1^{m-j}\partial t_2^{n-k}}\Delta^{(j)}(t_1)\,dt_1.\end{aligned}$$

Therefore,

$$\begin{aligned}&q(x_1,x_2)\\ &\quad= \frac{1}{i^{m+n}\,x_1^m x_2^n}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}\int_{-1}^{1}\int_{-1}^{1} e^{-i(t_1x_1+t_2x_2)}\Delta^{(j)}(t_1)\Delta^{(k)}(t_2)\\ &\qquad\qquad\times\frac{\partial^{m-j+n-k}\zeta(t_1,t_2)}{\partial t_1^{m-j}\partial t_2^{n-k}}dt_1dt_2\end{aligned}$$

for all $m,n\in\mathbf{N}$.

We may estimate the modulus of $q(x_1,x_2)$ using (10) and Lemma 1(iv) as follows:

$$\begin{aligned}|q(x_1,x_2)| &\le \frac{4}{|x_1|^m|x_2|^n}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j}M_jM_k\\ &\le \frac{4M_mM_n}{|x_1|^m|x_2|^n}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n}{k}\binom{m}{j} = 4\,\frac{2^mM_m}{|x_1|^m}\cdot\frac{2^nM_n}{|x_2|^n} = 4\,\frac{M_m}{\left|\frac{x_1}{2}\right|^m}\cdot\frac{M_n}{\left|\frac{x_2}{2}\right|^n}\end{aligned}$$

for all $m,n\in\mathbf{N}$.

Hence

$$|q(x_1,x_2)|\le 4\inf_{m,n}\frac{M_m}{\left|\frac{x_1}{2}\right|^m}\cdot\frac{M_n}{\left|\frac{x_2}{2}\right|^n} = 4\,\frac{1}{\tilde T\left(\frac{|x_1|}{2}\right)}\frac{1}{\tilde T\left(\frac{|x_2|}{2}\right)},$$

where $\tilde T(r)$ is given by (6). It follows from (7) and (8) that

$$\tilde T(r)=1 \ \text{ for } \ r\le 1 \ \text{ and } \ \tilde T(r)\ge\frac{T(r)}{r} \ \text{ for } \ r>1.$$

Hence

$$|q(x_1,x_2)|\le 4\,\frac{|x_1|\,|x_2|}{T\left(\frac{|x_1|}{2}\right)T\left(\frac{|x_2|}{2}\right)}\le C\,\omega(x_1)\omega(x_2),$$

where C is a positive constant and

$$\omega(x):=\begin{cases}|x|, & \text{if } \ |x|\le 2,\\ |x|\exp\left(-\left|\frac{x}{2}\right|^\beta\right), & \text{if } \ |x|>2.\end{cases}$$

Since $\alpha<\beta$, this implies that $\varepsilon|q(x_1,x_2)|\le\rho(x_1)\rho(x_2)$ for $\varepsilon>0$ small enough. It follows from (4) that

$$\varepsilon|q(x_1,x_2)|\le a_1a_2\rho_1(x_1)\rho_2(x_2). \tag{14}$$

We set

$$f(x_1,x_2):=\rho_1(x_1)\rho_2(x_2)+\delta q(x_1,x_2),$$

where $\delta\in(0,\varepsilon/a_1a_2)$. Clearly, for this choice of δ we have $f(x_1,x_2)\ge 0$.

We show that $f(x_1, x_2)$ is a probability density. Indeed, the definition of $q(x_1, x_2)$ given by (13) implies that

$$\eta(t_1, t_2) = \int\int_{\mathbf{R}^2} e^{i(t_1 x_1 + t_2 x_2)} q(x_1, x_2) dx_1 dx_2.$$

Hence by (12)

$$\int\int_{\mathbf{R}^2} q(x_1, x_2)\, dx_1 dx_2 = \eta(0, 0) = 0,$$

and, therefore,

$$\int\int_{\mathbf{R}^2} f(x_1, x_2)\, dx_1 dx_2 = \int\int_{\mathbf{R}^2} \rho_1(x_1)\rho_2(x_2)\, dx_1 dx_2 + \delta \int\int_{\mathbf{R}^2} q(x_1, x_2) = 1.$$

Thus, $f(x_1, x_2)$ is a probability density.

Let $\xi = (\xi_1, \xi_2)$ be a random vector having absolutely continuous distribution with the density $f(x_1, x_2)$. The characteristic function of the vector ξ equals

$$\varphi(t_1, t_2) = \psi(t_1, t_2) + \delta\eta(t_1, t_2),$$

where $\psi(t_1, t_2)$ is a characteristic function of a random vector with independent coordinates whose densities are ρ_1 and ρ_2. Since by (12) we have $\eta(t_1, 0) = \eta(0, t_2) = 0$, it follows that

$$\varphi(t_1, 0) = \psi(t_1, 0); \quad \varphi(0, t_2) = \psi(0, t_2).$$

This means that the coordinates ξ_1 and ξ_2 have absolutely continuous distributions with the given densities ρ_1 and ρ_2.

Besides,

$$\frac{\partial^{j+l}}{\partial t_1^j \partial t_2^l}\, \varphi(t_1, t_2)\Big|_{(0,0)} = \frac{\partial^{j+l}}{\partial t_1^j \partial t_2^l}\, \psi(t_1, t_2)\Big|_{(0,0)} + \delta \frac{\partial^{j+l}}{\partial t_1^j \partial t_2^l}\, \eta(t_1, t_2)\Big|_{(0,0)}$$
$$= \frac{\partial^{j+l}}{\partial t_1^j \partial t_2^l}\, \varphi(t_1, 0)\varphi(0, t_2)\Big|_{(0,0)} + \delta a_{jl} = \frac{\partial^{j+l}}{\partial t_1^j \partial t_2^l}\, \psi(t_1, 0)\psi(0, t_2)\Big|_{(0,0)} + \delta\, a_{jl}.$$

Therefore,

$$\mathbf{E}\left(\xi_1^j\, \xi_2^l\right) = \mathbf{E}\left(\xi_1^j\right) \mathbf{E}\left(\xi_2^l\right) + \delta a_{jl}.$$

Since $a_{jl} = 0$ for $(j, l) \in U$ and $a_{jl} \neq 0$ for $(j, l) \in \mathbf{N}^2 \setminus U$, it follows that

$$\mathbf{E}\left(\xi_1^j\, \xi_2^l\right) = \mathbf{E}\left(\xi_1^j\right) \mathbf{E}\left(\xi_2^l\right) \quad \text{for} \quad (j, l) \in U$$

and

$$\mathbf{E}\left(\xi_1^j\, \xi_2^l\right) \neq \mathbf{E}\left(\xi_1^j\right) \mathbf{E}\left(\xi_2^l\right) \quad \text{for} \quad (j, l) \in \mathbf{N}^2 \setminus U.$$

Thus, U is an uncorrelatedness set for ξ_1 and ξ_2. □

Acknowledgements

I would like to express my gratitude to the anonymous referee for his/her thorough reading of the manuscript and valuable suggestions, and to Dr. Paul Danesh for his help in preparation of the manuscript.

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