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ON THE TOPOLOGY OF NESTED SET COMPLEXES

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Abstract. Nested set complexes appear as the combinatorial core of De Concini-Procesi arrangement models. We show that nested set complexes are homotopy equivalent to the order complexes of the underlying meet-semilattices without their minimal elements. For atomic semilattices, we consider the realization of nested set complexes by simplicial fans proposed by the first author and Yuzvinsky and we strengthen our previous result showing that in this case nested set complexes in fact are homeomorphic to the mentioned order complexes.

1. Introduction

In the same way as intersection lattices capture the combinatorial essence of arrangements of hyperplanes, building sets and nested set complexes encode the combinatorics of De Concini-Procesi arrangement models: They prescribe the model construction by sequences of blowups, they describe the incidence combinatorics of the divisor stratification, and they naturally appear in presentations of cohomology algebras for arrangement models in terms of generators and relations (cf. [2]).

Nested set complexes have been defined in various generalities. The notion of nested sets goes back to the model construction for configuration spaces of algebraic varieties by Fulton and MacPherson [5]; the underlying poset in this special case is the lattice of set partitions. De Concini and Procesi [2] defined building sets and nested set complexes for intersection lattices of subspace arrangements in real or complex linear space; in this setting they have the broad geometric significance outlined above.

In joint work of the first author with Kozlov [3], purely order-theoretic definitions of building sets and nested set complexes for arbitrary meet-semilattices were given. Together with the notion of a combinatorial blowup in a meet-semilattice, a complete combinatorial counterpart to the resolution process of De Concini and Procesi was established. Having these purely combinatorial notions at hand, Yuzvinsky and the first author [4] studied abstract algebras that generalize arrangement model cohomology and solely depend on nested set complexes. In this context, nested set complexes attain yet another geometric meaning as the defining data for certain toric varieties.

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In this article, we study nested set complexes from the viewpoint of topological combinatorics. Relying on techniques from the homotopy theory of partially ordered sets due to Quillen [6], we show that, for any building set $\mathcal{G}$ in a meet-semilattice $\mathcal{L}$, the nested set complex $\mathcal{N}(\mathcal{L},\mathcal{G})$ is homotopy equivalent to the order complex of the underlying meet-semilattice without its minimal element, $\mathcal{L}\setminus\{\hat{0}\}$.

For atomic meet-semilattices we can strengthen this result. We consider the realization of nested set complexes $\mathcal{N}(\mathcal{L},\mathcal{G})$ by simplicial fans $\Sigma(\mathcal{L},\mathcal{G})$ proposed in [4], and we show that, for building sets $\mathcal{H}\subseteq\mathcal{G}$ in $\mathcal{L}$, the simplicial fan $\Sigma(\mathcal{L},\mathcal{G})$ is obtained from $\Sigma(\mathcal{L},\mathcal{H})$ by a sequence of stellar subdivisions. This in particular implies that, for a given atomic meet-semilattice $\mathcal{L}$, the nested set complex for any building set is homeomorphic to the order complex of $\mathcal{L}\setminus\{\hat{0}\}$.

After a brief review of the definitions for building sets, nested set complexes, and combinatorial blowups in Section 2, we present our result on the homotopy type of nested set complexes in Section 3. The strengthening in the case of atomic meet-semilattices is given in Section 4.

2. Preliminaries on building sets and nested sets

For the sake of completeness we here review the definitions of building sets and nested sets for finite meet-semilattices as proposed in [3].

All posets occurring in this article are finite. We mostly assume that the posets are meet-semilattices (semilattices, for short), i.e., greatest lower bounds exist for any subset of elements in the poset. Any finite meet-semilattice $\mathcal{L}$ has a minimal element, which we denote by $\hat{0}$. We frequently use the notation $\mathcal{L}_{>\hat{0}}$ to denote $\mathcal{L}$ without its minimal element. For any subset $\mathcal{S}$ in $\mathcal{L}$ we denote the set of maximal elements in $\mathcal{S}$ by $\max\mathcal{S}$. For any $X\in\mathcal{L}$, we set $\mathcal{S}_{\leq X}=\{Y\in\mathcal{S}\,|\,Y\leq X\}$, and we use the standard notation for intervals in $\mathcal{L}$, $[X,Y]:=\{Z\in\mathcal{L}\,|\,X\leq Z\leq Y\}$. The standard simplicial complex built from a poset $\mathcal{L}$ is the *order complex* of $\mathcal{L}$, which we denote by $\Delta(\mathcal{L})$; it is the abstract simplicial complex on the elements of $\mathcal{L}$ with simplices corresponding to linearly ordered subsets in $\mathcal{L}$. As a general reference on posets we refer to [7, Chapter 3].

Definition 2.1. Let $\mathcal{L}$ be a finite meet-semilattice. A subset $\mathcal{G}$ in $\mathcal{L}_{>\hat{0}}$ is called a *building set* if for any $X\in\mathcal{L}_{>\hat{0}}$ and $\max\mathcal{G}_{\leq X}=\{G_1,\ldots,G_k\}$ there is an isomorphism of posets

$$\varphi_X:\ \prod_{j=1}^{k}\,[\hat{0},G_j]\ \xrightarrow{\cong}\ [\hat{0},X] \tag{2.1}$$

with $\varphi_X(\hat{0},\ldots,G_j,\ldots,\hat{0})=G_j$ for $j=1,\ldots,k$. We call $F_{\mathcal{G}}(X):=\max\mathcal{G}_{\leq X}$ the *set of factors* of X in $\mathcal{G}$.

As a simple example we can take the full semilattice $\mathcal{L}_{>\hat{0}}$ as a building set. Besides this maximal building set, there is a minimal building set consisting of all elements X in $\mathcal{L}_{>\hat{0}}$ which do not allow for a product decomposition of the lower interval $[\hat{0},X]$, the so-called *irreducible elements* in $\mathcal{L}$.

Any choice of a building set $\mathcal{G}$ in $\mathcal{L}$ gives rise to a family of so-called *nested sets*. These are, roughly speaking, subsets of $\mathcal{G}$ whose antichains are sets of factors with respect to the building set $\mathcal{G}$. Nested sets form an abstract simplicial complex on the vertex set $\mathcal{G}$ – the *nested set complex*, which is the main character of this article.

Definition 2.2. Let $\mathcal{L}$ be a finite meet-semilattice and $\mathcal{G}$ a building set in $\mathcal{L}$. A subset $\mathcal{S}$ in $\mathcal{G}$ is called *nested* (or $\mathcal{G}$-*nested* if specification is needed) if, for any set of incomparable elements $X_1, \dots, X_t$ in $\mathcal{S}$ of cardinality at least two, the join $X_1 \vee \cdots \vee X_t$ exists and does not belong to $\mathcal{G}$. The $\mathcal{G}$-nested sets form an abstract simplicial complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$, the *nested set complex* with respect to $\mathcal{L}$ and $\mathcal{G}$.

For the maximal building set $\mathcal{L}_{>\hat{0}}$ in $\mathcal{L}$, the nested sets are the chains in $\mathcal{L}_{>\hat{0}}$; in particular, the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{L}_{>\hat{0}})$ coincides with the order complex $\Delta(\mathcal{L}_{>\hat{0}})$.

We also recall here a construction on semilattices that was proposed in [3], the combinatorial blowup of a semilattice $\mathcal{L}$ in an element X of $\mathcal{L}$.

Definition 2.3. For a semilattice $\mathcal{L}$ and an element X in $\mathcal{L}_{>\hat{0}}$ we define a poset $(\mathrm{Bl}_X\mathcal{L}, \prec)$ on the set of elements

$$\mathrm{Bl}_X\mathcal{L} \;=\; \{\, Y \,|\, Y \in \mathcal{L}, Y \not\geq X \,\} \;\cup\; \{\, \hat{Y} \,|\, Y \in \mathcal{L}, Y \not\geq X, \text{ and } Y \vee X \text{ exists in } \mathcal{L} \,\}\,.$$

The order relation $<$ in $\mathcal{L}$ determines the order relation $\prec$ within the two parts of $\mathrm{Bl}_X\mathcal{L}$ described above:

$$\begin{aligned} Y \prec Z\,, &\quad \text{for} \quad Y < Z \ \text{ in } \mathcal{L}\,, \\ \hat{Y} \prec \hat{Z}\,, &\quad \text{for} \quad Y < Z \ \text{ in } \mathcal{L}\,, \end{aligned}$$

and additional order relations between elements of these two parts are defined by

$$Y \prec \hat{Z}\,, \quad \text{for} \quad Y \leq Z \ \text{ in } \mathcal{L}\,,$$

where in all three cases it is assumed that $Y, Z \not\geq X$ in $\mathcal{L}$. We call $\mathrm{Bl}_X\mathcal{L}$ the *combinatorial blowup* of $\mathcal{L}$ in X.

Let us remark here that $\mathrm{Bl}_X\mathcal{L}$ is again a meet-semilattice. The combinatorial blowup of a semilattice was used in [3] to analyze the incidence change of strata in the construction process for De Concini-Procesi arrangement models. In the present paper we will need combinatorial blowups to describe the incidence change in polyhedral fans under stellar subdivision following an observation in [3, Prop. 4.9]:

Proposition 2.4. *Let Σ be a polyhedral fan with face poset $\mathcal{F}(\Sigma)$. For a cone σ in Σ, the face poset of the fan obtained by stellar subdivision of Σ in σ, $\mathcal{F}(\mathrm{st}(\Sigma, \sigma))$, can be described as the combinatorial blowup of $\mathcal{F}(\Sigma)$ in σ:*

$$\mathcal{F}(\mathrm{st}(\Sigma, \sigma)) \;=\; \mathrm{Bl}_\sigma(\mathcal{F}(\Sigma))\,.$$

3. The homotopy type of nested set complexes

In this section, we will show that for a given meet-semilattice $\mathcal{L}$ and a building set $\mathcal{G}$ in $\mathcal{L}$ the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$ is homotopy equivalent to the order complex of $\mathcal{L}_{>\hat{0}}$. We will use the following two lemmata on the homotopy type of partially ordered sets going back to Quillen [6].

Lemma 3.1 (Quillen's fiber lemma)**.** *Let $f\colon P \to Q$ be a map of posets such that the order complex of $f^{-1}(Q_{\leq X})$ is contractible for all $X \in Q$. Then f induces a homotopy equivalence between the order complexes of P and Q.*

Lemma 3.2. *Let P be a poset, and assume that there is an element X_0 in P such that the join $X_0 \vee X$ exists for all $X \in P$. Then the order complex of P is contractible. A poset with the property described above is called* join-contractible *via X_0.*

Proposition 3.3. *Let $\mathcal{L}$ be a finite meet-semilattice, and $\mathcal{G}$ a building set in $\mathcal{L}$. Then the nested set complex $\mathcal{N}(\mathcal{L},\mathcal{G})$ is homotopy equivalent to the order complex of $\mathcal{L}_{>\hat{0}}$,*

$$\mathcal{N}(\mathcal{L},\mathcal{G}) \simeq \Delta(\mathcal{L}_{>\hat{0}}).$$

Proof. We denote by $\mathcal{F}(\mathcal{N})$ the poset of non-empty faces of the nested set complex $\mathcal{N}(\mathcal{L},\mathcal{G})$. Consider the following map of posets:

$$\begin{array}{rccl} \phi : & \mathcal{F}(\mathcal{N}) & \longrightarrow & \mathcal{L}_{>\hat{0}} \\ & \mathcal{S} & \longmapsto & \bigvee \mathcal{S} = \bigvee_{X\in\mathcal{S}} X . \end{array}$$

We claim that the order complex of $\mathcal{F}(\mathcal{N})_{\leq X} := \phi^{-1}((\mathcal{L}_{>\hat{0}})_{\leq X})$ is contractible for any $X \in \mathcal{L}_{>\hat{0}}$. An application of the Quillen fiber lemma 3.1 will then prove the statement of the proposition.

Case 1: $X \in \mathcal{G}$. We show that $\mathcal{F}(\mathcal{N})_{\leq X}$ is join-contractible via X and, with an application of Lemma 3.2, thus prove our claim. Let $\mathcal{S}$ be an element in $\mathcal{F}(\mathcal{N})_{\leq X}$, i.e., a nested set with $\bigvee \mathcal{S} \leq X$. We have to show that $\mathcal{S} \cup \{X\}$ is nested with $\bigvee \mathcal{S} \cup \{X\} \leq X$, hence $\mathcal{S} \cup \{X\} \in \mathcal{F}(\mathcal{N})_{\leq X}$. Either $\bigvee \mathcal{S} = X$, in which case $X \in \mathcal{S}$, and our claim is obvious; or $\bigvee \mathcal{S} < X$, in which case we can add X to $\mathcal{S}$, obtaining a nested set, with $\bigvee \mathcal{S} \cup \{X\} = X$, hence $\mathcal{S} \cup \{X\} \in \mathcal{F}(\mathcal{N})_{\leq X}$.

Case 2: $X \notin \mathcal{G}$. We show that $\mathcal{F}(\mathcal{N})_{\leq X}$ is join-contractible via the set of factors of X, $F_{\mathcal{G}}(X)$. Again, let $\mathcal{S}$ be a nested set with $\bigvee \mathcal{S} \leq X$; we have to show that $\mathcal{S} \cup F_{\mathcal{G}}(X)$ is nested with join less than or equal to X, hence $\mathcal{S} \cup F_{\mathcal{G}}(X) \in \mathcal{F}(\mathcal{N})_{\leq X}$.

If $\bigvee \mathcal{S} = X$, then $X = \bigvee \max \mathcal{S}$ and $F_{\mathcal{G}}(X) = \max \mathcal{S} \subseteq \mathcal{S}$ by [3, Prop. 2.8(2)], which makes our claim obvious.

For $\bigvee \mathcal{S} < X$, assume that $A \subseteq \mathcal{S} \cup F_{\mathcal{G}}(X)$ is an antichain with at least two elements, and $\bigvee A \in \mathcal{G}$. Since the $\mathcal{G}$-factors of X, $F_{\mathcal{G}}(X) = \{G_1, \ldots, G_t\}$, give a partition of $\mathcal{G}_{\leq X}$ into subsets $\mathcal{G}_{\leq G_i}$, $i = 1, \ldots, t$ [3, Prop. 2.5(1)], we find that $\bigvee A \leq G$ for some $G \in F_{\mathcal{G}}(X)$. If A contains any elements of $F_{\mathcal{G}}(X)$, then it must contain G, which contradicts A being an antichain with more than one element. We conclude that A does not contain any factors of X. In particular, it is a subset of the nested set $\mathcal{S}$, thus should have a join outside $\mathcal{G}$, and we again reach a contradiction. We conclude that $\mathcal{S} \cup F_{\mathcal{G}}(X)$ is nested with join X, hence belongs to $\mathcal{F}(\mathcal{N})_{\leq X}$. □

Remark 3.4. The homotopy equivalence in 3.3 can be viewed as a generalization of the classical crosscut theorem [1, Thm. 10.8] applied to a particular class of posets and crosscuts. Let P be a simplicial poset, i.e., P contains a minimal element $\hat{0}$, and each interval $[\hat{0}, X]$, $X \in P$, is isomorphic to a boolean lattice. Observe that P is a meet-semilattice, and the set of atoms $\mathfrak{A}$ is a building set in P. The crosscut complex of P with respect to $\mathfrak{A}$,

$$\Gamma(P,\mathfrak{A}) = \{A \subseteq \mathfrak{A} \mid A \text{ is bounded in } P\},$$

coincides with the nested set complex $\mathcal{N}(P,\mathfrak{A})$.

4. Simplicial fans realizing nested set complexes

We recall the definition of the simplicial fan $\Sigma(\mathcal{L},\mathcal{G})$ for a given atomic meet-semilattice $\mathcal{L}$ and a building set $\mathcal{G}$ in $\mathcal{L}$. For details see [4, Section 5].

Given a finite meet-semilattice $\mathcal{L}$ with set of atoms $\mathfrak{A}(\mathcal{L}) = \{A_1, \ldots, A_n\}$, we will frequently use the following notation: For $X \in \mathcal{L}$, define $\lfloor X \rfloor := \{A \in \mathfrak{A}(\mathcal{L}) \mid X \geq A\}$, the set of atoms below a specific element X in $\mathcal{L}$. We define characteristic vectors v_X in $\mathbb{R}^n$ for lattice elements $X \in \mathcal{L}$ by

$$(v_X)_i \;:=\; \begin{cases} 1 & \text{if } A_i \in \lfloor X \rfloor, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } i = 1, \ldots, n.$$

These characteristic vectors will appear as spanning vectors of simplicial cones in $\mathbb{R}^n$. For a subset $\mathcal{S} \subseteq \mathcal{L}$, we agree to denote by $V(\mathcal{S})$ the cone spanned by the vectors v_X for $X \in \mathcal{S}$.

Definition 4.1. Let $\mathcal{L}$ be a finite atomic meet-semilattice and $\mathcal{G}$ a building set in $\mathcal{L}$. We define a rational, polyhedral fan $\Sigma(\mathcal{L}, \mathcal{G})$ in $\mathbb{R}^n$ as the collection of cones $V(\mathcal{S})$ for all nested sets $\mathcal{S}$ in $\mathcal{L}$,

$$\Sigma(\mathcal{L}, \mathcal{G}) \;:=\; \{\, V(\mathcal{S}) \,|\, \mathcal{S} \in \mathcal{N}(\mathcal{L}, \mathcal{G}) \,\}\,. \tag{4.1}$$

By definition, rays in $\Sigma(\mathcal{L}, \mathcal{G})$ are in 1-1 correspondence with elements in $\mathcal{G}$. In fact, the face poset of $\Sigma(\mathcal{L}, \mathcal{G})$ coincides with the face poset of $\mathcal{N}(\mathcal{L}, \mathcal{G})$.

If there is no risk of confusion we will denote the fan in (4.1) by $\Sigma(\mathcal{G})$.

Theorem 4.2. *Let $\mathcal{L}$ be a finite atomic meet-semilattice, and $\mathcal{G}, \mathcal{H}$ building sets in $\mathcal{L}$ with $\mathcal{G} \supseteq \mathcal{H}$. Then, the fan $\Sigma(\mathcal{G})$ is obtained from $\Sigma(\mathcal{H})$ by a sequence of stellar subdivisions. In particular, the supports of the fans $\Sigma(\mathcal{G})$ and $\Sigma(\mathcal{H})$ coincide.*

Proof. For building sets $\mathcal{G} \supseteq \mathcal{H}$ in $\mathcal{L}$ and G minimal in $\mathcal{G} \setminus \mathcal{H}$, set $\overline{\mathcal{G}} := \mathcal{G} \setminus \{G\}$. Obviously, $\max \overline{\mathcal{G}}_{\leq G} = F_{\mathcal{H}}(G)$, and for any $X \in \mathcal{L}$ we find that

$$\max \overline{\mathcal{G}}_{\leq X} \;=\; \begin{cases} F_{\mathcal{G}}(X) & \text{if } G \notin F_{\mathcal{G}}(X)\,, \\ (F_{\mathcal{G}}(X) \setminus \{G\}) \cup F_{\mathcal{H}}(G) & \text{if } G \in F_{\mathcal{G}}(X)\,. \end{cases}$$

Isomorphisms of posets required for the building set property of $\overline{\mathcal{G}}$ expand accordingly in the second case, and we find that $\overline{\mathcal{G}}$ is again a building set for $\mathcal{L}$.

We thus conclude that, for any two building sets $\mathcal{G}$, $\mathcal{H}$ with $\mathcal{G} \supseteq \mathcal{H}$, there is a sequence of building sets

$$\mathcal{G} \;=\; \mathcal{G}_1 \;\supseteq\; \mathcal{G}_2 \;\supseteq\; \ldots \;\supseteq\; \mathcal{G}_t \;=\; \mathcal{H}\,,$$

such that $\mathcal{G}_i$ and $\mathcal{G}_{i+1}$ differ by exactly one element G_i, and G_i is minimal in $\mathcal{G}_i \setminus \mathcal{H}$ for $i = 1, \ldots, t{-}1$.

We can thus assume that $\mathcal{H} = \mathcal{G} \setminus \{G\}$, and it suffices to show that $\Sigma(\mathcal{G})$ is obtained from $\Sigma(\mathcal{H})$ by a sequence of stellar subdivisions.

In fact, we claim that $\Sigma(\mathcal{G})$ is obtained by a single stellar subdivision of $\Sigma(\mathcal{H})$ in $V(F_{\mathcal{H}}(G))$, introducing a new ray that is generated by the characteristic vector v_G, i.e.,

$$\Sigma(\mathcal{G}) \;=\; \mathrm{st}\,(\Sigma(\mathcal{H}), V(F_{\mathcal{H}}(G)), v_G)\,. \tag{4.2}$$

Observe that the two fans in (4.2) share the same set of generating vectors for rays, so all we have to show is that they have the same combinatorial structure, i.e., their face posets coincide.

The face poset of the subdivided fan can be described as the combinatorial blowup of the face poset $\mathcal{F}(\Sigma(\mathcal{H})) = \mathcal{F}(\mathcal{N}(\mathcal{H}))$ in the $\mathcal{H}$-nested set $F_{\mathcal{H}}(G)$ (cf. [3, Sect. 4.2]); hence we are left to show that

$$\mathcal{F}(\mathcal{N}(\mathcal{G})) \;=\; \mathrm{Bl}_{F_{\mathcal{H}}(G)}(\mathcal{F}(\mathcal{N}(\mathcal{H})))\,. \tag{4.3}$$

Let us abbreviate notation and denote the poset on the right-hand side by $\mathrm{Bl}\,\mathcal{F}$.

We first show the left-to-right inclusion in (4.3). Let $\mathcal{S}$ be a $\mathcal{G}$-nested set in $\mathcal{L}$. We need to show that $\mathcal{S}$ is an element in $\mathrm{Bl}\,\mathcal{F}$. For the matter of this proof, we agree to freely switch between sets of atoms and their joins in the respective semilattices.

For $G \notin \mathcal{S}$, we note that $\mathcal{S}$ is $\mathcal{H}$-nested. Moreover, $\mathcal{S}$ does not contain $F_{\mathcal{H}}(G)$, since the latter is certainly not $\mathcal{G}$-nested. We conclude that $\mathcal{S}$ is an element in $\mathrm{Bl}\,\mathcal{F}$.

For $G \in \mathcal{S}$, we need to show that $(S \setminus \{G\}) \cup F_{\mathcal{H}}(G)$ is $\mathcal{H}$-nested. Let A be an antichain with at least two elements that is contained in $(S \setminus \{G\}) \cup F_{\mathcal{H}}(G)$; we need to see that $\bigvee A \notin \mathcal{H}$. If $A \subseteq F_{\mathcal{H}}(G)$, then clearly $\bigvee A$ either equals G or lives between G and its $\mathcal{H}$-factors, hence in any case is not contained in $\mathcal{H}$. If A does not contain any $\mathcal{H}$-factor of G, then $A \subset \mathcal{S}$ is $\mathcal{G}$-nested; in particular, $\bigvee A \notin \mathcal{H}$.

We can thus assume that the antichain A is of the form $A = \{S_1, \dots, S_t, F_1, \dots, F_k\}$, where $S_i \in \mathcal{S} \setminus (\{G\} \cup F_{\mathcal{H}}(G))$ for $i = 1, \dots, t$, and $F_j \in F_{\mathcal{H}}(G)$ for $j = 1, \dots, k$, and both types of elements occur in A.

Let us assume that $\bigvee A \in \mathcal{H}$. We have

$$\bigvee A \;\leq\; \bigvee_{i=1}^{t} S_i \;\vee\; G \;=\; \bigvee_{\substack{i \in \{1,\dots t\},\ S_i \text{ in-} \\ \text{comparable with } G}} S_i \;\vee\; G\,, \tag{4.4}$$

where the last equality holds since any S_j comparable with G has to be smaller than G; otherwise $S_j \geq G > F_1$ gives a contradiction to A being an antichain.

If there are no S_i, $i \in \{1, \dots, t\}$, that are incomparable with G, the right-hand side of (4.4) equals G. Assuming that $\bigvee A \in \mathcal{H}$ we find that $\bigvee A \leq F$ for some $F \in F_{\mathcal{H}}(G)$ since the $\mathcal{H}$-factors of G partition the elements of $\mathcal{H}$ below G [3, Prop. 2.5.(1)]. We assumed that A contains some of the $\mathcal{H}$-factors of G, and thus conclude that it must contain F. This however contradicts A being an antichain with at least two elements.

We are left with the case of the join on the right-hand side of (4.4) being taken over more than one element. Since $\mathcal{S}_0 = \{S_i \in A \mid S_i \text{ incomparable with } G\} \cup \{G\} \subseteq \mathcal{S}$ is a $\mathcal{G}$-nested antichain, we conclude that $\bigvee \mathcal{S}_0$ is not contained in $\mathcal{G}$ and $\mathcal{S}_0$ is its set of factors. Since these factors partition $\mathcal{G}$-elements below $\bigvee \mathcal{S}_0$ we find that either $\bigvee A \leq S_i$, for some $S_i \in \mathcal{S}_0$, which is a contradiction to A being an antichain, or $\bigvee A \leq G$, which again places $\bigvee A$ below one of the $\mathcal{H}$-factors F of G, and, as argued above, leads to a contradiction. We conclude that $(S \setminus \{G\}) \cup F_{\mathcal{H}}(G)$ is $\mathcal{H}$-nested; thus any $\mathcal{G}$-nested set $\mathcal{S}$ is an element of $\mathrm{Bl}\,\mathcal{F}$ as claimed.

Let us now turn to the right-to-left inclusion in (4.3). Let $\mathcal{S} \in \mathrm{Bl}_{F_{\mathcal{H}}(G)}(\mathcal{F}(\mathcal{N}(\mathcal{H})))$. We have to show that $\mathcal{S}$, respectively the set of atoms below $\mathcal{S}$ in $\mathrm{Bl}\,\mathcal{F}$, is nested with respect to $\mathcal{G}$.

Let us first consider the case when $\mathcal{S}$ is $\mathcal{H}$-nested and does not contain $F_{\mathcal{H}}(G)$; i.e., $\mathcal{S}$ is one of the elements of the face poset $\mathcal{F}(\mathcal{N}(\mathcal{H}))$ that remain after the blowup. Assume that $\mathcal{S}$ is not $\mathcal{G}$-nested; hence there exists an antichain A in $\mathcal{S}$ with $\bigvee A \in \mathcal{G} \setminus \mathcal{H}$, i.e., $\bigvee A = G$. We conclude that A coincides with the set of $\mathcal{H}$-factors of G (cf. [3, Prop. 2.8.(2)]), which contradicts our assumption about $\mathcal{S}$ not containing $F_{\mathcal{H}}(G)$.

Let us now consider the remaining case, i.e., $\mathcal{S} = \mathcal{S}' \cup \{G\}$, where $\mathcal{S}'$ is $\mathcal{H}$-nested, $\mathcal{S}' \not\supseteq F_{\mathcal{H}}(G)$, and $\mathcal{S}' \cup F_{\mathcal{H}}(G)$ is $\mathcal{H}$-nested. We have to show that $\mathcal{S}$ is $\mathcal{G}$-nested.

Let A be an antichain contained in $\mathcal{S}$. If $G \notin A$, then $A \subseteq \mathcal{S}'$ and $\bigvee A \in \mathcal{G} \setminus \mathcal{H}$ implies as above that $A = F_{\mathcal{H}}(G)$, contradicting our assumptions.

If $G \in A$, then $A = A' \cup \{G\}$ where A' is an antichain in $\mathcal{S}'$. If $\bigvee A = G$, then A would not be an antichain; hence it suffices to show that $\bigvee A \notin \mathcal{H}$. Consider

$$\bigvee A \;=\; \bigvee A' \;\vee\; G \;=\; \bigvee A' \;\vee\; \bigvee F_{\mathcal{H}}(G) \;=\; \bigvee A' \;\vee \bigvee_{\substack{F \in F_{\mathcal{H}}(G),\ F \text{ incom-} \\ \text{parable to elements in } A'}} F\,,$$

where the last equality holds since any $\mathcal{H}$-factor F of G comparable with an element a in the antichain A' must be smaller than a; otherwise $F \geq a$ implies $G > a$, which contradicts A being an antichain.

We find that $A' \cup \{F \in F_{\mathcal{H}}(G) \,|\, F \text{ incomparable to elements in } A'\}$ is an antichain in $\mathcal{S}' \cup F_{\mathcal{H}}(G)$. With the latter being $\mathcal{H}$-nested by assumption, we conclude that $\bigvee A \notin \mathcal{H}$ as required, which completes our proof. □

Corollary 4.3. *Let $\mathcal{L}$ be a finite atomic meet-semilattice, and $\mathcal{G}$ a building set in $\mathcal{L}$. Then the nested set complex $\mathcal{N}(\mathcal{L},\mathcal{G})$ is homeomorphic to the order complex of $\mathcal{L}_{>\hat{0}}$,*

$$\mathcal{N}(\mathcal{L},\mathcal{G}) \;\cong\; \Delta(\mathcal{L}_{>\hat{0}})\,.$$

Proof. By Theorem 4.2 the simplicial fan $\Sigma(\mathcal{L}_{>\hat{0}})$ is a stellar subdivision of $\Sigma(\mathcal{G})$ for any building set $\mathcal{G}$ in $\mathcal{L}$. This in particular implies that the abstract simplicial complexes encoding the face structure of the respective fans are homeomorphic. The observation that the nested set complex for the maximal building set, $\mathcal{N}(\mathcal{L},\mathcal{L}_{>\hat{0}})$, coincides with the order complex of $\mathcal{L}_{>\hat{0}}$ finishes our proof. □

Remark 4.4. In most of the literature on the topology of posets, the order complex of a poset P that has a maximal and a minimal element, $\hat{1}$ and $\hat{0}$, respectively, is understood to be the order complex of the proper part $P \setminus \{\hat{0},\hat{1}\}$. Both our theorems can be used to study the topology of lattices in this sense:

Let $\mathcal{L}$ be a lattice, $\mathcal{G}$ a building set in $\mathcal{L}$. We assume that $\mathcal{G}$ contains $\hat{1}$, observing that we can always add $\hat{1}$ to a given building set. The nested set complex $\mathcal{N}(\mathcal{L},\mathcal{G})$ is a cone with apex $\hat{1}$:

$$\mathcal{N}(\mathcal{L},\mathcal{G}) \;=\; \{\hat{1}\} \;*\; \mathcal{N}(\mathcal{L},\mathcal{G})_{\lceil \mathcal{G}\setminus\{\hat{1}\}}\,.$$

Its base can be interpreted as a nested set complex, namely of the meet-semilattice $\mathcal{L} \setminus \{\hat{1}\}$ with respect to $\mathcal{G} \setminus \{\hat{1}\}$:

$$\mathcal{N}(\mathcal{L},\mathcal{G})_{\lceil \mathcal{G}\setminus\{\hat{1}\}} \;=\; \mathcal{N}(\mathcal{L} \setminus \{\hat{1}\}, \mathcal{G} \setminus \{\hat{1}\})\,.$$

Our theorems state homotopy equivalence, resp. homeomorphism, between the nested set complex $\mathcal{N}(\mathcal{L} \setminus \{\hat{1}\}, \mathcal{G} \setminus \{\hat{1}\})$ and the order complex of the proper part of $\mathcal{L}$, $\Delta(\mathcal{L} \setminus \{\hat{0},\hat{1}\})$.

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References

1. A. Björner: *Topological methods*; in: *Handbook of Combinatorics* (R. Graham, M. Grötschel, L. Lovász, eds); North-Holland, Amsterdam 1995, pp. 1819–1872. MR1373690 (96m:52012)
2. C. De Concini, C. Procesi: *Wonderful models of subspace arrangements*; Selecta Math. (N.S.) 1 (1995), 459–494. MR1366622 (97k:14013)
3. E.M. Feichtner, D.N. Kozlov: *Incidence combinatorics of resolutions*; Selecta Math. (N.S.) 10 (2004), 37–60. MR2061222

4. E.M. Feichtner, S. Yuzvinsky: *Chow rings of toric varieties defined by atomic lattices*; Invent. Math. 155 (2004), 515–536. MR2038195
5. W. Fulton, R. MacPherson: *A compactification of configuration spaces*; Ann. of Math. 139 (1994), 183–225. MR1259368 (95j:14002)
6. D. Quillen: *Homotopy properties of the poset of non-trivial p-subgroups of a group*; Advances in Math. 21 (1978), 101–128. MR0493916 (80k:20049)
7. R. Stanley: *Enumerative combinatorics, Vol.* I, Wadsworth & Brooks/Cole, Monterey, Calif., 1986. MR0847717 (87j:05003)

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