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RELATING EXPONENTIAL GROWTH IN A MANIFOLD AND ITS FUNDAMENTAL GROUP

ANTHONY MANNING

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ABSTRACT. We relate the growth rate of volume in the universal cover of a compact Riemannian manifold to the growth in the fundamental group in terms of word length in a given set of generators and the length of geodesics representing these generators.

Given a group Γ and a finite set S of generators for Γ, we have the *word length* $l_S : \Gamma \to \mathbb{N} \cup \{0\}$ for which $l_S(\gamma)$ is the least n such that there exist $s_1, \ldots, s_n \in S \cup S^{-1}$ with $\gamma = s_1 \ldots s_n$. The exponential growth rate of Γ is defined as $\psi(\Gamma, S) := \lim_{k\to\infty} k^{-1} \log \#\{\gamma \in \Gamma : l_S(\gamma) \leq k\}$. It is easy to check that $\psi(\Gamma, S) > 0$ if and only if $\psi(\Gamma, S') > 0$ for *any* finite set S' of generators of Γ; in this case Γ is said to be of *exponential growth*. Without a way of comparing the length of elements of different sets S of generators there is no single natural value of the exponential growth rate of Γ.

When Γ is the fundamental group $\pi_1(M, *)$ of a compact Riemannian manifold (M, g) we can define $L_g : \Gamma \to \mathbb{R}$ by putting $L_g(\gamma)$ equal to the shortest length of a geodesic from the base point $*$ to itself representing γ. In the Cayley graph (see [3], for example) of (Γ, S) we shall use $L_g|S$ instead of 1 for the lengths of edges and incorporate this length into a definition of the growth rate of Γ for the generating set S. Thus we define $L_{g,S} : \Gamma \to \mathbb{R}$ by

$$L_{g,S}(\gamma) := \inf\{\sum_{j=1}^{n} L_g(s_j) : \gamma = s_1 \ldots s_n, \{s_1, \ldots, s_n\} \subset S \cup S^{-1}, n \in \mathbb{N} \cup \{0\}\}.$$

(Note that this infimum is attained and that $L_{g,S}|S = L_g|S$.) This gives rise to the growth function

$$\beta_{g,S} : \mathbb{R} \to \mathbb{R},\ \beta_{g,S}(t) := \#\{\gamma \in \Gamma : L_{g,S}(\gamma) \leq t\}$$

and the *exponential growth rate*

$$\varphi(g, S) := \lim_{t\to\infty} t^{-1} \log \beta_{g,S}(t).$$

This limit exists because $\beta_{g,S}(t + u) \leq \beta_{g,S}(t)\beta_{g,S}(u)$. Thus $\varphi(g, S)$ differs from $\psi(\Gamma, S)$ by incorporating the g-length of geodesics representing the generators in S.

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The *volume entropy* of (M, g) is defined by

$$h(g) := \lim_{R\to\infty} R^{-1} \log \mathrm{Vol}\, B(*, R),$$

the growth rate of the $\widetilde{g}$-volume of the ball of radius R and centre the base point $*$ in the universal cover $(\widetilde{M}, \widetilde{g})$ of (M, g). See [6] or [5, 8] for this and its connection with the topological entropy of the geodesic flow. By tiling $\widetilde{M}$ with the translates of a fundamental domain by the elements of the covering group Γ we see that

$$h(g) = \lim_{R\to\infty} R^{-1} \log \#\{\gamma \in \Gamma : L_g(\gamma) \le R\}.$$

Our theorem connects the volume entropy with the exponential growth rate of Γ for generating subsets $S \subset \Gamma$.

Theorem 1.

$$h(g) = \sup\{\varphi(g, S) : S \textit{ is a finite subset generating } \Gamma = \pi_1(M, *)\}.$$

Proof. First we fix a finite generating set $S \subset \Gamma$ and argue that $\varphi(g, S) \le h(g)$. For $\gamma \in \Gamma$, $L_{g,S}(\gamma)$ is the infimum of the length of certain piecewise geodesic loops representing γ, and so $L_g(\gamma) \le L_{g,S}(\gamma)$. Thus

$$\#\{\gamma \in \Gamma : L_{g,S}(\gamma) \le R\} \le \#\{\gamma \in \Gamma : L_g(\gamma) \le R\},$$

from which we obtain $\varphi(g, S) \le h(g)$.

Choose a fundamental domain N for $\widetilde{M}$ of diameter A, say, using the metric d on $\widetilde{M}$ arising from the Riemannian metric $\widetilde{g}$. Consider the fibre $\{\alpha * : \alpha \in \Gamma\}$ over the base point $* \in M$. Given $\gamma \in \Gamma$ with $L_g(\gamma) \le kR$, we pick $\alpha_j \in \Gamma$, $1 \le j < k$, such that $d(\alpha_j *, \gamma(jR)) \le A$ and put $\alpha_0 = \mathrm{id}_\Gamma$, $\alpha_k = \gamma$. Then, for $1 \le j < k$,

$$\begin{aligned} L_g(\alpha_j^{-1}\alpha_{j+1}) &= d(\alpha_j *, \alpha_{j+1} *) \\ &\le d(\alpha_j *, \gamma(jR)) + d(\gamma(jR), \gamma((j+1)R)) + d(\gamma((j+1)R), \alpha_{j+1} *) \\ &\le R + 2A. \end{aligned}$$

Put

$$S := \{\alpha \in \Gamma : L_g(\alpha) \le R + 2A\}.$$

Then

$$L_{g,S}(\gamma) = L_{g,S}(\alpha_1(\alpha_1^{-1}\alpha_2)(\alpha_2^{-1}\alpha_3)\dots(\alpha_{k-1}^{-1}\gamma)) \le k(R + 2A).$$

So $\#\{\gamma \in \Gamma : L_g(\gamma) \le kR\} \le \beta_{g,S}(k(R + 2A))$. Thus

$$\begin{aligned} h(g) &= \lim_{k\to\infty} (kR)^{-1} \log \#\{\gamma \in \Gamma : L_g(\gamma) \le kR\} \\ &\le \frac{k(R+2A)}{kR} \lim_{k\to\infty} (k(R+2A))^{-1} \log \beta_{g,S}(k(R+2A)) \\ &= \frac{R+2A}{R} \varphi(g, S). \end{aligned}$$

Thus, for all R there is $S \subset \Gamma$ such that

$$\frac{R}{R+2A} h(g) \le \varphi(g, S) \le h(g),$$

and letting $R \to \infty$ completes the proof. □

Remark 2. We have shown the geometrical relevance of the *supremum* of $\varphi(g, S)$ over generating sets S. By contrast, the *uniform exponential growth rate* is defined as the *infimum* of $\exp\psi(\Gamma, S)$ taken over finite sets S that generate Γ; see [2, 5.11] or [4, 1, 7]. This is relevant to the abstract group Γ rather than to its geometrical properties.

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Mathematics Institute, University of Warwick, Coventry, CV4 7AL, United Kingdom
E-mail address: `akm@maths.warwick.ac.uk`