

HILBERT-SAMUEL COEFFICIENTS AND POSTULATION NUMBERS OF GRADED COMPONENTS OF CERTAIN LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian homogeneous ring with one-dimensional local base ring (R_0, \mathfrak{m}_0) . Let $\mathfrak{q}_0 \subseteq R_0$ be an \mathfrak{m}_0 -primary ideal, let M be a finitely generated graded R -module and let $i \in \mathbb{N}_0$. Let $H_{R_+}^i(M)$ denote the i -th local cohomology module of M with respect to the irrelevant ideal $R_+ := \bigoplus_{n > 0} R_n$ of R . We show that the first Hilbert-Samuel coefficient $e_1(\mathfrak{q}_0, H_{R_+}^i(M)_n)$ of the n -th graded component of $H_{R_+}^i(M)$ with respect to \mathfrak{q}_0 is antipolynomial of degree $< i$ in n . In addition, we prove that the postulation numbers of the components $H_{R_+}^i(M)_n$ with respect to \mathfrak{q}_0 have a common upper bound.

1. INTRODUCTION

Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian homogeneous ring, so that R is \mathbb{N}_0 -graded with Noetherian base ring R_0 and of the form $R = R_0[\ell_0, \dots, \ell_r]$ with finitely many elements $\ell_0, \dots, \ell_r \in R_1$. Let $R_+ := \bigoplus_{n > 0} R_n$ denote the irrelevant ideal of R . Moreover let M denote a finitely generated graded R -module. For $i \in \mathbb{N}_0$ let $H_{R_+}^i(M) = \bigoplus_{n \in \mathbb{Z}} H_{R_+}^i(M)_n$ denote the i -th local cohomology module of M with respect to R_+ . Keep in mind that the n -th graded component $H_{R_+}^i(M)_n$ of $H_{R_+}^i(M)$ is a finitely generated R_0 -module for all $n \in \mathbb{Z}$ and vanishes for all $n \gg 0$.

Motivated by the “tameness-problem” for coherent sheaves over projective schemes (cf. [B-H]) one is led to ask for the “asymptotic behaviour” of the graded components $H_{R_+}^i(M)_n$ for $n \rightarrow -\infty$. Namely, the mentioned tameness-problem is equivalent to the question of whether for each $i \in \mathbb{N}_0$ either $H_{R_+}^i(M)_n = 0$ for all $n \ll 0$ or $H_{R_+}^i(M)_n \neq 0$ for all $n \ll 0$. We do not know any example in which this tameness property is not satisfied. On the other hand, tameness has been established until now in fairly special cases only (cf. [L], [B-F-L]).

A more specific question is, whether for each $i \in \mathbb{N}_0$, the set $\text{Ass}_{R_0}(H_{R_+}^i(M)_n)$ of associated primes of the R_0 -module $H_{R_+}^i(M)_n$ is asymptotically stable for $n \rightarrow -\infty$. This is true, for example, if R_0 is semilocal and either of dimension one or of dimension two and essentially of finite type over a field (cf. [B-F-L]). On the other

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hand this “asymptotic stability of associated primes” need not hold in general, as can be deduced from examples of Singh [S] (cf. [B-K-S]) or Katzman [K] (cf. [B-F-T]). It may fail, for example, if R_0 is regular local and of dimension four.

Finally, to be even more specific, one could ask whether certain numerical invariants of the R_0 -modules $H_{R_+}^i(M)_n$ “behave well” if i is fixed and n goes to $-\infty$. To be more precise we say that a numerical function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is *antipolynomial (of degree $< i$)* if there is some polynomial $p \in \mathbb{Q}[x]$ (of degree $< i$) such that $f(n) = p(n)$ for all $n \ll 0$. Moreover, if (R_0, \mathfrak{m}_0) is local and $\mathfrak{q}_0 \subseteq R_0$ is an \mathfrak{m}_0 -primary ideal, let $e_0(\mathfrak{q}_0, T)$ denote the *Hilbert-Samuel multiplicity* of the finitely generated R_0 -module T with respect to \mathfrak{q}_0 . Then, in this terminology we can say (cf. [B-F-T]):

(1.1) If (R_0, \mathfrak{m}_0) is local of dimension one, $\mathfrak{q}_0 \subseteq R_0$ is \mathfrak{m}_0 -primary and $i \in \mathbb{N}_0$, the functions given by

$$n \mapsto \text{length}_{R_0} \left(\Gamma_{\mathfrak{m}_0} \left(H_{R_+}^i(M)_n \right) \right), n \mapsto \text{length}_{R_0} \left(0_{H_{R_+}^i(M)_n} :_{H_{R_+}^i(M)_n} \mathfrak{q}_0 \right),$$

$$n \mapsto \text{length}_{R_0} \left(H_{R_+}^i(M)_n / \mathfrak{q}_0 H_{R_+}^i(M)_n \right) \text{ and } n \mapsto e_0 \left(\mathfrak{q}_0, H_{R_+}^i(M)_n \right)$$

are all antipolynomial of degree $< i$.

The first aim of this note is to show that under the above hypotheses a further numerical invariant of the R_0 -modules $H_{R_+}^i(M)_n$ is antipolynomial of degree $< i$. Namely, assume that (R_0, \mathfrak{m}_0) is local of dimension one, let $\mathfrak{q}_0 \subseteq R_0$ be an \mathfrak{m}_0 -primary ideal and fix $i \in \mathbb{N}_0$. We then have “asymptotic stability of associated primes” and so $\dim_{R_0} \left(H_{R_+}^i(M)_n \right)$ takes a constant value ≤ 1 if $n \rightarrow -\infty$. Assume that this constant value is 1. Then, for all $n \ll 0$ the *Hilbert-Samuel polynomial of $H_{R_+}^i(M)_n$ with respect to \mathfrak{q}_0* can be written as

$$(1.2) \quad P_{H_{R_+}^i(M)_n, \mathfrak{q}_0}(\mathbf{x}) = e_0 \left(\mathfrak{q}_0, H_{R_+}^i(M)_n \right) (\mathbf{x} + 1) - e_1 \left(\mathfrak{q}_0, H_{R_+}^i(M)_n \right),$$

where $e_1 \left(\mathfrak{q}_0, H_{R_+}^i(M)_n \right) = -P_{H_{R_+}^i(M)_n, \mathfrak{q}_0}(-1) \in \mathbb{Z}$ is the so-called *first Hilbert-Samuel coefficient of $H_{R_+}^i(M)_n$ with respect to \mathfrak{q}_0* .

We shall prove that the function given by $n \mapsto e_1 \left(\mathfrak{q}_0, H_{R_+}^i(M)_n \right)$ is antipolynomial of degree $< i$ (cf. Theorem 3.1).

In addition we show that the *postulation numbers* of the R_0 -modules $H_{R_+}^i(M)_n$ with respect to \mathfrak{q}_0 , hence the numbers

$$(1.3) \quad \mu \left(\mathfrak{q}_0, H_{R_+}^i(M)_n \right) := \inf \{ r \in \mathbb{N}_0 \mid \text{length}_{R_0} \left(H_{R_+}^i(M)_n / \mathfrak{q}_0^{t+1} H_{R_+}^i(M)_n \right) = P_{H_{R_+}^i(M)_n, \mathfrak{q}_0}(t), \forall t \geq r \}$$

have a common upper bound for all $n \in \mathbb{Z}$ (cf. Theorem 3.3).

As for the unexplained terminology we refer to [B-S] and to [E].

2. HILBERT-SAMUEL COEFFICIENTS IN DIMENSION 1

Throughout this section let (A, \mathfrak{m}) be a local Noetherian ring of dimension one, let $\mathfrak{q} \subseteq A$ be an \mathfrak{m} -primary ideal and let T be a finitely generated A -module of dimension one.

2.1. *Notation and Remark.* A) Let $P_{T,\mathfrak{q}}(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}]$ denote the *Hilbert-Samuel polynomial* of T with respect to \mathfrak{q} , so that

$$P_{T,\mathfrak{q}}(n) = \text{length}_A(T/\mathfrak{q}^{n+1}T) \text{ for all } n \gg 0.$$

As $\dim(T) = 1$, $P_{T,\mathfrak{q}}(\mathbf{x})$ is of degree 1 and we may write

$$P_{T,\mathfrak{q}}(\mathbf{x}) = e_0(\mathfrak{q}, T)(\mathbf{x} + 1) - e_1(\mathfrak{q}, T)$$

where $e_0(\mathfrak{q}, T) \in \mathbb{N}$ is the (*Hilbert-Samuel multiplicity*) of T with respect to \mathfrak{q} and $e_1(\mathfrak{q}, T) \in \mathbb{Z}$ is the *first Hilbert-Samuel coefficient* of T with respect to \mathfrak{q} .

B) We set

$$\overline{T} := T/\Gamma_{\mathfrak{m}}(T),$$

where $\Gamma_{\mathfrak{m}}(T) = \bigcup_{n \in \mathbb{N}} (0 \underset{T}{:} \mathfrak{m}^n)$ denotes the \mathfrak{m} -torsion of T . Keep in mind that $\text{Ass}_A(\overline{T}) = \text{Ass}_A(T) \setminus \{\mathfrak{m}\}$, so that $\dim(\overline{T}) = \text{depth}_A(\overline{T}) = 1$.

C) Keep the above hypotheses and notation. Then, for each $n \in \mathbb{N}_0$, there is a short exact sequence

$$0 \rightarrow \Gamma_{\mathfrak{m}}(T)/(\Gamma_{\mathfrak{m}}(T) \cap \mathfrak{q}^n T) \rightarrow T/\mathfrak{q}^n T \rightarrow \overline{T}/\mathfrak{q}^n \overline{T} \rightarrow 0.$$

In particular,

$$\text{length}_A(T/\mathfrak{q}^n T) = \text{length}_A(\overline{T}/\mathfrak{q}^n \overline{T}) + \text{length}_A(\Gamma_{\mathfrak{m}}(T)) - \text{length}_A(\Gamma_{\mathfrak{m}}(T) \cap \mathfrak{q}^n T).$$

By Artin-Rees there is some $n_0 \in \mathbb{N}$ such that $\Gamma_{\mathfrak{m}}(T) \cap \mathfrak{q}^n T = 0$ for all $n \geq n_0$. This gives the relations

$$\begin{aligned} \text{length}_A(T/\mathfrak{q}^n T) &= \text{length}_A(\overline{T}/\mathfrak{q}^n \overline{T}) + \text{length}_A(\Gamma_{\mathfrak{m}}(T)) \text{ for all } n \geq n_0; \\ P_{T,\mathfrak{q}}(\mathbf{x}) &= P_{\overline{T},\mathfrak{q}}(\mathbf{x}) + \text{length}_A(\Gamma_{\mathfrak{m}}(T)); \\ e_0(\mathfrak{q}, T) &= e_0(\mathfrak{q}, \overline{T}); \\ e_1(\mathfrak{q}, T) &= e_1(\mathfrak{q}, \overline{T}) - \text{length}_A(\Gamma_{\mathfrak{m}}(T)). \end{aligned}$$

2.2. *Notation.* A) Let

$$\mathcal{R}(\mathfrak{q}) := \bigoplus_{n \geq 0} \mathfrak{q}^n, \quad \mathcal{R}(\mathfrak{q}, T) := \bigoplus_{n \geq 0} \mathfrak{q}^n T$$

denote the *Rees ring of \mathfrak{q}* and the *Rees module of T with respect to \mathfrak{q}* , respectively.

B) In addition, let

$$\begin{aligned} \text{Gr}(\mathfrak{q}) &:= \mathcal{R}(\mathfrak{q})/\mathfrak{q}\mathcal{R}(\mathfrak{q}) = \bigoplus_{n \geq 0} \mathfrak{q}^n/\mathfrak{q}^{n+1}, \\ \text{Gr}(\mathfrak{q}, T) &:= \mathcal{R}(\mathfrak{q}, T)/\mathfrak{q}\mathcal{R}(\mathfrak{q}, T) = \bigoplus_{n \geq 0} \mathfrak{q}^n T/\mathfrak{q}^{n+1} T \end{aligned}$$

denote the *associated graded ring of \mathfrak{q}* and the *associated graded module of T with respect to \mathfrak{q}* , respectively.

2.3. *Notation.* For a finitely generated graded module M over the Noetherian homogeneous ring $R = \bigoplus_{n \geq 0} R_n$ let $\text{reg}(M)$ denote the (*Castelnuovo-Mumford*) *regularity* of M , thus

$$\text{reg}(M) := \inf\{r \in \mathbb{Z} \mid H_{R_+}^i(M)_{n-i+1} = 0, \forall i \in \mathbb{N}_0, \forall n \geq r\}.$$

2.4. **Lemma.** *Let $\text{depth}_A(T) \neq 0$. Then $\text{reg}(\text{Gr}(\mathfrak{q}, T)) \leq \text{reg}(\text{Gr}(\mathfrak{q}))$.*

Proof. Let \mathbf{x} be an indeterminate. Then $B := A[\mathbf{x}]_{\mathfrak{m}_A[\mathbf{x}]}$ is a Noetherian local flat one-dimensional extension ring of A with maximal ideal $\mathfrak{n} := \mathfrak{m}B$ and infinite residue field B/\mathfrak{n} . Moreover $\mathfrak{q}B \subseteq B$ is an \mathfrak{n} -primary ideal and $T \otimes_A B$ is a finitely generated B -module of dimension 1 and depth $\neq 0$.

In addition, there is an isomorphism of graded rings $\text{Gr}(\mathfrak{q}B) \cong \text{Gr}(\mathfrak{q}) \otimes_A B$ and an isomorphism of graded $\text{Gr}(\mathfrak{q}B)$ -modules $\text{Gr}(\mathfrak{q}B, T \otimes_A B) \cong \text{Gr}(\mathfrak{q}, T) \otimes_A B$. In view of the graded flat base-change property of local cohomology (cf. [B-S]), these latter isomorphisms induce

$$\text{reg}(\text{Gr}(\mathfrak{q}B)) = \text{reg}(\text{Gr}(\mathfrak{q}))$$

and

$$\text{reg}(\text{Gr}(\mathfrak{q}B, T \otimes_A B)) = \text{reg}(\text{Gr}(\mathfrak{q}, T)).$$

Altogether, we thus may replace $(A, \mathfrak{m}), \mathfrak{q}$ and T respectively by $(B, \mathfrak{n}), \mathfrak{q}B$ and $T \otimes_A B$ and hence assume that A/\mathfrak{m} is infinite.

As \mathfrak{q} is \mathfrak{m} -primary, its analytic spread equals $\dim(A) = 1$. So there is some $x \in \mathfrak{q}$ such that Ax is a minimal reduction of \mathfrak{q} . As A/\mathfrak{m} is infinite the reduction number of \mathfrak{q} with respect to Ax is $\leq \text{reg}(\text{Gr}(\mathfrak{q}))$ (cf. [T, Proposition 3.2] or [B-S, 18.3.12]) so that $xq^n = \mathfrak{q}^{n+1}$ for all $n \geq \text{reg}(\text{Gr}(\mathfrak{q}))$. Let $x^* = (0, x, 0, \dots) \in \mathcal{R}(\mathfrak{q})_1$ be the element $x \in \mathfrak{q}$ considered as a one-form in $\mathcal{R}(\mathfrak{q})$. It follows that $\sqrt{\mathcal{R}(\mathfrak{q})_+} = \sqrt{x^* \mathcal{R}(\mathfrak{q})}$ so that $H^i_{\mathcal{R}(\mathfrak{q})_+}(\mathcal{R}(\mathfrak{q}, T)) = H^i_{(x^*)}(\mathcal{R}(\mathfrak{q}, T))$ for all $i \geq 0$. Therefore $H^i_{\mathcal{R}(\mathfrak{q})_+}(\mathcal{R}(\mathfrak{q}, T)) = 0$ for all $i > 1$. As $x \in A$ is a parameter, $\text{depth}_A(T) = 1 = \dim(A)$ implies that x is T -regular. So x^* is $\mathcal{R}(\mathfrak{q}, T)$ -regular and hence $H^0_{\mathcal{R}(\mathfrak{q})_+}(\mathcal{R}(\mathfrak{q}, T)) = 0$. Moreover we have the following short exact sequence of graded $\mathcal{R}(\mathfrak{q})$ -modules (cf. [B-S, 2.2.17]):

$$0 \rightarrow \mathcal{R}(\mathfrak{q}, T) \xrightarrow{\eta} \mathcal{R}(\mathfrak{q}, T)_{x^*} \rightarrow H^1_{(x^*)}(\mathcal{R}(\mathfrak{q}, T)) \rightarrow 0.$$

As $x^* \mathcal{R}(\mathfrak{q}, T)_n = xq^n T = \mathfrak{q}^{n+1} T = \mathcal{R}(\mathfrak{q}, T)_{n+1}$ for all $n \geq \text{reg}(\text{Gr}(\mathfrak{q}))$, the natural map η becomes an isomorphism in all degrees $\geq \text{reg}(\text{Gr}(\mathfrak{q}))$ so that

$$H^1_{\mathcal{R}(\mathfrak{q})_+}(\mathcal{R}(\mathfrak{q}, T))_n = H^1_{(x^*)}(\mathcal{R}(\mathfrak{q}, T))_n = 0$$

for all $n \geq \text{reg}(\text{Gr}(\mathfrak{q}))$. Thus, finally we get $\text{reg}(\mathcal{R}(\mathfrak{q}, T)) \leq \text{reg}(\text{Gr}(\mathfrak{q}))$.

Now, in view of the well-known behaviour of regularities in short exact sequences (cf. [B-S, 15.2.15]) and as $\text{reg}(\mathcal{R}(\mathfrak{q}, T)) \geq 0$ (cf. [B-S, 15.3.1]), we get our claim by the graded exact sequences

$$\begin{aligned} 0 \rightarrow \mathfrak{q}\mathcal{R}(\mathfrak{q}, T) \rightarrow \mathcal{R}(\mathfrak{q}, T)(1) \rightarrow [T]_{-1} \rightarrow 0, \\ 0 \rightarrow \mathfrak{q}\mathcal{R}(\mathfrak{q}, T) \rightarrow \mathcal{R}(\mathfrak{q}, T) \rightarrow \text{Gr}(\mathfrak{q}, T) \rightarrow 0. \end{aligned}$$

□

2.5. Lemma. *Let $r \geq \text{reg}(\text{Gr}(\mathfrak{q}))$, let $n \geq r$ and assume that $\text{depth}_A(T) = 1$. Then*

$$\text{length}_A(T/\mathfrak{q}^{n+1}T) = e_0(\mathfrak{q}, T)n - re_0(\mathfrak{q}, T) + \text{length}_A(T/\mathfrak{q}^{r+1}T).$$

Proof. For all $n \in \mathbb{N}_0$ we have

$$\text{length}_{A/\mathfrak{q}}(\text{Gr}(\mathfrak{q}, T)_n) = \text{length}_A(T/\mathfrak{q}^{n+1}T) - \text{length}_A(T/\mathfrak{q}^nT).$$

For all $n \gg 0$, the right-hand side of this equality takes the value $P_{T, \mathfrak{q}}(n) - P_{T, \mathfrak{q}}(n - 1) = e_0(\mathfrak{q}, T)$, so that the characteristic function of the graded $\text{Gr}(\mathfrak{q})$ -module $\text{Gr}(\mathfrak{q}, T)$ takes the constant value $e_0(\mathfrak{q}, T)$. As $\text{reg}(\text{Gr}(\mathfrak{q}, T)) \leq r$ it follows

that $\text{length}_{A/\mathfrak{q}}(\text{Gr}(\mathfrak{q}, T)_n) = e_0(\mathfrak{q}, T)$ for all $n > r$ and hence $\text{length}_A(T/\mathfrak{q}^{n+1}T) - \text{length}_A(T/\mathfrak{q}^nT) = e_0(\mathfrak{q}, T)$ for all $n \geq r + 1$. So, for all $n \geq r$ we get

$$\begin{aligned} \text{length}_A(T/\mathfrak{q}^{n+1}T) &= \sum_{m=r+1}^n (\text{length}_A(T/\mathfrak{q}^{m+1}T) - \text{length}_A(T/\mathfrak{q}^mT)) \\ &\quad + \text{length}_A(T/\mathfrak{q}^{r+1}T) = e_0(\mathfrak{q}, T)(n - r) + \text{length}_A(T/\mathfrak{q}^{r+1}T) \\ &= e_0(\mathfrak{q}, T)n - re_0(\mathfrak{q}, T) + \text{length}_A(T/\mathfrak{q}^{r+1}T). \end{aligned}$$

□

2.6. Proposition. *Let $r \geq \text{reg}(\text{Gr}(\mathfrak{q}))$. Then*

$$e_1(\mathfrak{q}, T) = (r + 1)e_0(\mathfrak{q}, T) - \text{length}_A(T/\mathfrak{q}^{r+1}T) - \text{length}_A(\mathfrak{q}^{r+1}T \cap \Gamma_{\mathfrak{m}}(T)).$$

Proof. As \overline{T} has dimension and depth 1, Lemma 2.5 gives $e_1(\mathfrak{q}, \overline{T}) = (r+1)e_0(\mathfrak{q}, \overline{T}) - \text{length}_A(\overline{T}/\mathfrak{q}^{r+1}\overline{T})$. As

$$e_0(\mathfrak{q}, T) = e_0(\mathfrak{q}, \overline{T}), \quad e_1(\mathfrak{q}, T) = e_1(\mathfrak{q}, \overline{T}) - \text{length}_A(\Gamma_{\mathfrak{m}}(T))$$

and

$$\begin{aligned} \text{length}_A(\overline{T}/\mathfrak{q}^{r+1}\overline{T}) &= \text{length}_A(T/\mathfrak{q}^{r+1}T) - \text{length}_A(\Gamma_{\mathfrak{m}}(T)) \\ &\quad + \text{length}_A(\Gamma_{\mathfrak{m}}(T) \cap \mathfrak{q}^{r+1}T), \end{aligned}$$

(cf. 2.1 C)), we get our claim. □

3. ANTIPOLYNOMIAL GROWTH OF FIRST HILBERT-SAMUEL COEFFICIENTS AND BOUNDEDNESS OF POSTULATION NUMBERS

Now, we are ready to formulate and to prove our first main result.

3.1. Theorem. *Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian homogeneous ring such that (R_0, \mathfrak{m}_0) is local and of dimension 1. Let $\mathfrak{q}_0 \subseteq R_0$ be an \mathfrak{m}_0 -primary ideal, let M be a finitely generated graded R -module and let $i \in \mathbb{N}_0$. Assume that $\dim_{R_0}(H_{R_+}^i(M)_n) = 1$ for all $n \ll 0$. Then there is a polynomial $S(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}]$ of degree $< i$ such that*

$$e_1(\mathfrak{q}_0, H_{R_+}^i(M)_n) = S(n) \text{ for all } n \ll 0.$$

Proof. By our hypotheses there is some $r \geq \text{reg}(\text{Gr}(\mathfrak{q}_0))$ such that $\dim_{R_0}(H_{R_+}^i(M)_n) = 1$ for all $n \leq -r$. So by Proposition 2.6 we obtain for all $n \leq -r$:

$$\begin{aligned} e_1(\mathfrak{q}_0, H_{R_+}^i(M)_n) &= (r + 1)e_0(\mathfrak{q}_0, H_{R_+}^i(M)_n) \\ &\quad - \text{length}_{R_0}(H_{R_+}^i(M)_n/\mathfrak{q}_0^{r+1}H_{R_+}^i(M)_n) \\ &\quad - \text{length}_{R_0}(\mathfrak{q}_0^{r+1}H_{R_+}^i(M)_n \cap \Gamma_{\mathfrak{m}_0}(H_{R_+}^i(M)_n)). \end{aligned}$$

According to (1.1) there are polynomials $P, Q \in \mathbb{Q}[\mathbf{x}]$ of degree $< i$ such that

$$e_0(\mathfrak{q}_0, H_{R_+}^i(M)_n) = Q(n), \quad \text{length}_{R_0}(H_{R_+}^i(M)_n/\mathfrak{q}_0^{r+1}H_{R_+}^i(M)_n) = P(n)$$

for all $n \ll 0$.

By [B-F-T, Theorem 2.5], the graded R -module $\Gamma_{\mathfrak{m}_0 R} \left(H_{R_+}^i(M) \right)$ is Artinian, and hence so is its graded submodule

$$U := \mathfrak{q}_0^{r+1} H_{R_+}^i(M) \cap \Gamma_{\mathfrak{m}_0 R} \left(H_{R_+}^i(M) \right).$$

So, there is a polynomial $F \in \mathbb{Q}[\mathbf{x}]$ such that $\text{length}_{R_0}(U_n) = F(n)$ for all $n \ll 0$ (cf. [Ki]). According to (1.1) there is a polynomial $\overline{Q} \in \mathbb{Q}[\mathbf{x}]$ of degree $< i$ such that $\text{length}_{R_0} \left(\Gamma_{\mathfrak{m}_0} \left(H_{R_+}^i(M)_n \right) \right) = \overline{Q}(n)$ for all $n \ll 0$. As $U_n \subseteq \Gamma_{\mathfrak{m}_0} \left(H_{R_+}^i(M)_n \right)$ for all n , it follows that $\deg(F) \leq \deg(\overline{Q}) < i$.

As $U_n = \mathfrak{q}_0^{r+1} H_{R_+}^i(M)_n \cap \Gamma_{\mathfrak{m}_0} \left(H_{R_+}^i(M)_n \right)$, the equality given at the beginning of this proof yields

$$e_1 \left(\mathfrak{q}_0, H_{R_+}^i(M)_n \right) = (r + 1)Q(n) - P(n) - F(n) \quad \text{for all } n \ll 0.$$

This proves our claim. □

Now we prove the announced boundedness of postulation numbers (cf. (1.3)). We begin with a preliminary remark.

3.2. Remark. Let $R = \bigoplus_{n \in \mathbb{Z}} R_n$ be a graded ring, let $W = \bigoplus_{n \in \mathbb{Z}} W_n$ be a graded and Artinian R -module and let $(U^{(m)})_{m \in \mathbb{N}_0}$ be a descending sequence of graded submodules of W . Assume that for each $n \in \mathbb{Z}$ there is some $m_n \in \mathbb{N}_0$ such that $U_n^{(m_n)} = 0$.

As W is Artinian, there is some $t \in \mathbb{N}_0$ such that $U^{(m)} = U^{(t)}$ for all $m \geq t$. So, for each $n \in \mathbb{Z}$ and for each $m \geq \max\{t, m_n\}$, we obtain $U_n^{(t)} = U_n^{(m)} \subseteq U_n^{(m_n)} = 0$. Therefore $U^{(t)} = 0$.

3.3. Theorem. Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian homogeneous ring such that (R_0, \mathfrak{m}_0) is local and of dimension ≤ 1 . Let $\mathfrak{q}_0 \subseteq R_0$ be an \mathfrak{m}_0 -primary ideal, let M be a finitely generated graded R -module and let $i \in \mathbb{N}_0$. Then, there is some $c \in \mathbb{N}_0$ such that $\mu(\mathfrak{q}_0, H_{R_+}^i(M)_n) \leq c$ for all $n \in \mathbb{Z}$.

Proof. By [B-F-T, Theorem 3.5 e)] there is some $\delta \in \{0, \pm 1\}$ such that

$$\dim_{R_0}(H_{R_+}^i(M)_n) = \delta \quad \text{for all } n \ll 0.$$

If $\delta = -1$ we have $H_{R_+}^i(M)_n = 0$ for all but finitely many values of n and our claim is clear.

If $\delta \geq 0$ we shall apply Remark 3.2 to the graded Artinian R -module

$$W := \Gamma_{\mathfrak{m}_0 R}(H_{R_+}^i(M))$$

(cf. [B-F-T, Theorem 2.5 b)]). First, let $\delta = 0$, so that $W_n = H_{R_+}^i(M)_n$ for all $n \ll 0$. If we apply Remark 3.2 with $U^{(m)} := \mathfrak{q}_0^m W$, we find some $t \in \mathbb{N}_0$ with $\mathfrak{q}_0^t W = 0$ so that $\mathfrak{q}_0^t H_{R_+}^i(M)_n = 0$ for all $n \ll 0$. As $H_{R_+}^i(M)_n = 0$ for all $n \gg 0$ we get our claim if $\delta = 0$.

So, let $\delta = 1$. If we apply Remark 3.2 with $U^{(m)} := W \cap \mathfrak{q}_0^m H_{R_+}^i(M)$ and use the lemma of Artin-Rees we find some $t \in \mathbb{N}$ with $\Gamma_{\mathfrak{m}_0 R}(H_{R_+}^i(M)) \cap \mathfrak{q}_0^t H_{R_+}^i(M) = 0$, so that $\Gamma_{\mathfrak{m}_0}(H_{R_+}^i(M)_n) \cap \mathfrak{q}_0^t H_{R_+}^i(M)_n = 0$ for all $n \in \mathbb{Z}$. Now, let n_0 be such that $\dim_{R_0}(H_{R_+}^i(M)_n) = 1$ for all $n \leq n_0$. Then, for each $r \geq \max\{\text{reg}(\text{Gr}(\mathfrak{q}_0)), t - 1\}$

and each $n \leq n_0$, Proposition 2.6 — applied with $A = R_0, \mathfrak{q} = \mathfrak{q}_0$ and $T = H_{R_+}^i(M)_n$ — gives

$$\begin{aligned} & \text{length}_{R_0} \left(H_{R_+}^i(M)_n / \mathfrak{q}_0^{r+1} H_{R_+}^i(M)_n \right) \\ &= (r+1)e_0 \left(\mathfrak{q}_0, H_{R_+}^i(M)_n \right) - e_1 \left(\mathfrak{q}_0, H_{R_+}^i(M)_n \right) = P_{H_{R_+}^i(M)_n, \mathfrak{q}_0}(r). \end{aligned}$$

As $H_{R_+}^i(M)_n = 0$ for all $n \gg 0$, this proves our claim. \square

3.4. Corollary. *Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian homogeneous ring such that (R_0, \mathfrak{m}_0) is local and of dimension ≤ 1 . Let $\mathfrak{q}_0 \subseteq R_0$ be an \mathfrak{m}_0 -primary ideal, let M be a finitely generated graded R -module and let $i \in \mathbb{N}_0$. Then, there are integers $n_0 \in \mathbb{Z}, c \in \mathbb{N}_0$ and polynomials $S, Q \in \mathbb{Q}[\mathbf{x}]$ of degree $< i$ such that for each $n \leq n_0$ and each $r \geq c$ we have*

$$\text{length}_{R_0} \left(H_{R_+}^i(M)_n / \mathfrak{q}_0^{r+1} H_{R_+}^i(M)_n \right) = Q(n)(r+1) - S(n).$$

Proof. Immediate by Theorem 3.1, Theorem 3.3 and [B-F-T, Theorem 3.5]. \square

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