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PURELY PERIODIC β-EXPANSIONS WITH PISOT UNIT BASE

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ABSTRACT. Let $\beta > 1$ be a Pisot unit. A family of sets $\{X_i\}_{1\le i\le q}$ defined by a β-numeration system has been extensively studied as an atomic surface or Rauzy fractal. For the purpose of constructing a Markov partition, a domain $\hat{X} = \bigcup_{i=1}^{q} \hat{X}_i$ constructed by an atomic surface has appeared in several papers. In this paper we show that the domain $\hat{X}$ completely characterizes the set of purely periodic β-expansions.

1. INTRODUCTION

1.1. **Purely periodic β-expansion.** Let $\beta > 1$ be a real number. The β-expansion of a real number $x \in [0,1]$ is defined as the sequence $(x_i)_{i\ge 1}$ with values in $\{0,1,\dots,[\beta]\}$ produced by the β-transformation $T_\beta : x \mapsto \beta x \pmod 1$ as follows:

$$\forall i \ge 1,\ x_i = \lfloor \beta T_\beta^{(i-1)} x \rfloor, \text{ and thus } x = \sum_{i=1}^{\infty} x_i \beta^{-i} = 0.x_1 \dots x_n \dots .$$

An expansion is *finite* if $(x_i)_{i\ge1}$ is eventually 0. A β-expansion is *periodic* if there exists $p \ge 1$ and $M \ge 1$ such that $x_k = x_{k+p}$ holds for all $k \ge M$; if $x_k = x_{k+p}$ holds for all $k \ge 1$, then it is *purely periodic*. We denote by $\mathrm{Per}(\beta)$ the numbers in $[0,1)$ with periodic β-expansions and by $\mathrm{Pur}(\beta)$ the numbers in $[0,1)$ with purely periodic β-expansions.

Let $\mathbb{Q}(\beta)$ be the smallest fields containing $\mathbb{Q}$ and β. An easy argument shows that $\mathrm{Per}(\beta) \subseteq \mathbb{Q}(\beta) \cap [0,1)$ for every real number $\beta > 1$. K. Schmidt [17] showed that if β is a *Pisot number* (an algebraic integer whose conjugates have modulus < 1), then $\mathrm{Per}(\beta) = \mathbb{Q}(\beta) \cap [0,1)$. The purely periodic β-expansions are also discussed in [17].

Property 1 (Schmidt [17]). *Suppose that β satisfies $\beta^2 = n\beta + 1$ for some integer $n \ge 1$. Then every $x \in \mathbb{Q} \cap [0,1)$ has purely periodic β-expansion.*

A number is called a *Pisot unit* if it is a Pisot number as well as an algebraic unit. The purpose of this paper is to characterize the set $\mathrm{Pur}(\beta)$ when β is a Pisot unit. To state our result, we need some preparations.

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1.2. **Automaton of the β-numeration system.** Let $d_\beta(1) = (b_i)_{i\geq 1}$ denote the β-expansion of 1. Let $d^*_\beta(1) = d_\beta(1)$ if $d_\beta(1)$ is infinite, and let $d^*_\beta(1) = (t_1 \dots t_{m-1}(t_m - 1))^\infty$ if $d_\beta(1) = t_1 \dots t_{m-1} t_m$ is finite (with $t_m \neq 0$). A sequence is *admissible* if and only if starting from any place in the sequence, the right side truncation is lexicographically strictly less than $d^*_\beta(1)$. If the right side truncations are less than or equal to $d^*_\beta(1)$, then the sequence is called *weakly admissible*.

When $d^*_\beta(1)$ is eventually periodic (this holds for any Pisot number), the set of weakly admissible β-representations can be recognized by the following automaton M_β. Assume that

$$d^*_\beta(1) = (b_i)_{i\geq 1} = b_1 \dots b_{q-p} \overline{b_{q-p+1} \dots b_q},$$

so $b_{k+p} = b_k$ for all $k > q - p$. Let $r_k = b_k b_{k+1} \cdots$, $k \geq 1$. (We denote by $\bar{r}_k = 0.b_k b_{k+1} \dots$ the corresponding real number.) Then $\{r_k\}_{k\geq 1}$ is a finite set and actually it is $\{r_1, \cdots, r_q\}$. Let the states of M_β be the integers $1, \cdots, q$ together with a fail state F. The initial state is 1. From a state $j < q$, b_j leads to state $j+1$, while the arrows with labels less than b_j lead to state 1, and all arrows whose labels are greater than b_j fail. From state q, b_q leads to state $q - p + 1$, while all arrows with lower labels lead back to 1 and all arrows with higher labels fail (cf. for example [13, 14, 20]).

1.3. **Canonical embedding.** We denote by $\mathbb{Z}(\beta)$ the smallest ring containing $\mathbb{Z}$ and β. From now on we assume that $\beta > 1$ is a Pisot unit with minimal polynomial $x^d = a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$, where $a_0 = \pm 1$. Let

$$A_\beta = \begin{pmatrix} 0 & \cdots & 0 & a_0 \\ 1 & & 0 & a_1 \\ & \ddots & & \vdots \\ 0 & & 1 & a_{d-1} \end{pmatrix}$$

be the *companion matrix* of β. Since $\{1, \beta, \cdots, \beta^{d-1}\}$ is a basis of the field $\mathbb{Q}(\beta)$, we define a mapping $\phi: \mathbb{Q}(\beta) \mapsto \mathbb{Q}^d$ by

$$\phi(z_1 + z_2\beta + \cdots + z_d\beta^{d-1}) = (z_1, z_2, \cdots, z_d)^T,$$

where v^T is the transpose of a vector v. Clearly for $x \in \mathbb{Q}(\beta)$, it follows that

$$\phi(\beta x) = A_\beta \phi(x). \tag{1.1}$$

We consider A_β as a linear transformation on $\mathbb{R}^d$. Each real root of the minimal polynomial of β is an eigenvalue for this transformation, with a 1-dimensional eigenspace, and for each pair of complex conjugate roots there is a 2-dimensional invariant subspace. Denote by V the eigenspace of β. Then V is the expanding eigenspace and the other eigenspaces are contractive. Let P be the direct sum of all these contractive eigenspaces. Then $\mathbb{R}^d = V \oplus P$, $\dim P = d - 1$ and $\dim V = 1$. According to this direct sum we define two natural projections $\pi: \mathbb{R}^d \mapsto P$ and $\pi': \mathbb{R}^d \mapsto V$. It is easy to see that

$$A_\beta \circ \pi = \pi \circ A_\beta, \quad A_\beta \circ \pi' = \pi' \circ A_\beta. \tag{1.2}$$

1.4. **Construction of Markov partition.** Let $w_n \dots w_1 w_0$ be an admissible sequence. We say $w_n \dots w_1 w_0$ stops at state i if there is a path with label $w_n \dots w_1 w_0$ which starts from state 1 and ends at state i. Let

$$\begin{aligned} Y &= \{\sum_{i=0}^{n} w_i \beta^i : \ w_n \dots w_1 w_0 \text{ is admissible}, \ n \geq 0\}, \\ Y_i &= \{\sum_{i=0}^{n} w_i \beta^i : \ w_n \dots w_1 w_0 \text{ is admissible and stops at state } i, \ n \geq 0\}. \end{aligned}$$

Clearly $\{Y_1, \dots, Y_q\}$ is a partition of Y. Define

$$X = \overline{\pi \circ \phi(Y)} \text{ and } X_i = \overline{\pi \circ \phi(Y_i)}, \ 1 \leq i \leq q.$$

Then $X = \bigcup_{i=1}^{q} X_i$ is a subset of the contractive subspace P and is called the *atomic surface* by [11, 8]. (It is called a *Rauzy fractal* by [6, 7, 19] and is called a central tile by [20, 3, 4].) Denote by $e_1, \dots, e_d$ the canonical basis of $\mathbb{R}^d$, and let

$$I_i = \{\theta \pi'(e_1) : \ 0 \leq \theta \leq r_i\}, \quad 1 \leq i \leq q,$$

be subsets of V. Then

$$\hat{X}_i = -X_i + I_i := \{-x + y : \ x \in X_i, y \in I_i\}, \ \ 1 \leq i \leq q,$$

are d-dimensional tubes. Taking their union, we get

$$\hat{X} := \bigcup_{i=1}^{q} \hat{X}_i.$$

(See Figure 1.1.) This construction has appeared in several papers (cf. [15, 18, 11, 8]), where its purpose is to construct Markov partitions for the group automorphism A_β. However, in this paper, we will show that $\hat{X}$ completely characterizes the set of purely periodic β-expansions.

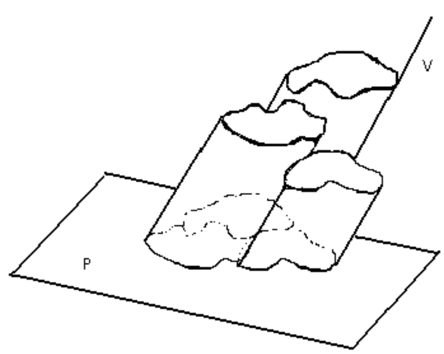

FIGURE 1.1.

1.5. **Main result.** Our purpose is to study Pur(β), the set of purely periodic β-representations which are admissible. However, if we consider Pur$'(\beta)$, the set of purely periodic β-representations which are *weakly* admissible, it is slightly different from Pur(β). A purely periodic sequence is weakly admissible but not admissible if and only if it is a shift of the sequence $d_\beta^*(1)$ and $d_\beta^*(1)$ is purely periodic. Notice that $d_\beta^*(1)$ is purely periodic if and only if $d_\beta(1)$ is finite. So

$$\mathrm{Pur}'(\beta) = \begin{cases} \mathrm{Pur}(\beta), & \text{when } d_\beta(1) \text{ is not finite,} \\ \mathrm{Pur}(\beta) \cup \{r_1, \cdots, r_q\}, & \text{when } d_\beta(1) \text{ is finite.} \end{cases}$$

Since Pur(β) and Pur$'(\beta)$ have very small differences, so in the following we will study Pur$'(\beta)$ instead of Pur(β).

Main Theorem. *Let $\beta > 1$ be a Pisot unit. Then $x \in Pur'(\beta)$ if and only if $x \in \mathbb{Q}(\beta) \cap [0,1)$ and $\phi(x) \in \hat{X}$.*

This theorem has been proved when β is of degree 2 by Hara and Ito [10], and proved by Sano and Ito [12] when β is a Pisot unit with the minimal polynomial $x^d = a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ where $a_{d-1} \geq a_{d-2} \geq \cdots \geq a_1 \geq a_0 = 1$. In the present paper, we use a simpler argument and obtain a complete result.

2. Algebraic version of the Main Theorem

A two-sided sequence $\{a_i\}_{i\in\mathbb{Z}}$ is (weakly) admissible provided each left truncation $\{a_i\}_{i\geq M}$ is (weakly) admissible. Let (Ω, σ) denote the two-sided symbolic dynamical system associated with β, where Ω is the set of admissible two-sided sequences and σ the shift operator. We write a two-sided sequence in the form (w, u), where $w = \ldots w_2 w_1 w_0$ is a backward sequence and $u = u_1 u_2 \ldots$ is a forward sequence.

2.1. **First realization of (Ω, σ).** Let β be a Pisot unit of degree d. Denote by

$$\beta = \beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(d)} \tag{2.1}$$

the algebraic conjugates of β. We arrange the sequence $\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(d)}$ in such an order: the real roots are ahead of the complex roots, and for a complex root $\beta^{(j)}$ we put its complex conjugate $\overline{\beta^{(j)}}$ next to it to make a pair. Suppose $\beta^{(1)}, \ldots, \beta^{(r)}$ are real roots and the others are complex roots. For a backward sequence w, we define

$$\rho(w) := (-\sum_{i=0}^{\infty} w_i (\beta^{(2)})^i, \ldots, -\sum_{i=0}^{\infty} w_i (\beta^{(d)})^i).$$

Abusing the notation a little, we define a projection $\rho : \Omega \mapsto \mathbb{R}^r \times \mathbb{C}^{d-r}$ as

$$\rho(w, u) := (\sum_{i=1}^{\infty} u_i \beta^{-i}, \rho(w)). \tag{2.2}$$

Let $K := \{\rho(w, u) : \ (w, u) \in \Omega\}$ be the projection of Ω. Clearly K is a bounded subset of $[0,1) \times (\mathbb{R}^{r-1} \times \mathbb{C}^{d-r})$. We write a point of K as (x, y), where $x \in [0,1)$ and $y \in \mathbb{R}^{r-1} \times \mathbb{C}^{d-r}$. The last ingredient we need is a mapping $S : \ K \mapsto K$ defined by

$$S \begin{pmatrix} x \\ y \end{pmatrix} = I_\beta \begin{pmatrix} x \\ y \end{pmatrix} - \lfloor \beta x \rfloor \cdot \mathbf{1}, \tag{2.3}$$

where $\mathbf{1}$ denotes the vector $(1,1,\dots,1)^T$ and I_β is the diagonal matrix

$$I_\beta = \begin{pmatrix} \beta^{(1)} & & 0 \\ & \ddots & \\ 0 & & \beta^{(d)} \end{pmatrix}.$$

Proposition 2.1. (K,S) *is a realization of* (Ω,σ)*. Precisely,* (i) $S\circ\rho = \rho\circ\sigma$, (ii) $S(K)=K$*, that is,*

$$\begin{array}{ccc} & \sigma & \\ \Omega & \longrightarrow & \Omega \\ \rho\downarrow & & \downarrow\rho \\ K & \longrightarrow & K \\ & S & \end{array}$$

Proof. (i) By direct calculation.
(ii) $S(K) = S\circ\rho(\Omega) = \rho\circ\sigma(\Omega) = \rho(\Omega) = K$. □

2.2. **Characterization of** $\mathrm{Pur}(\beta)$**.** For $x = c_0 + c_1\beta + \cdots + c_{d-1}\beta^{d-1}$ with $c_j \in \mathbb{Q}$, the j-th conjugate of x in the field $\mathbb{Q}(\beta)$ is defined by

$$x^{(j)} = c_0 + c_1\beta^{(j)} + \cdots + c_{d-1}(\beta^{(j)})^{d-1}.$$

We define x', the (total) conjugate of x, to be $x' = (x^{(2)},\cdots,x^{(d)})$. Particularly, $\beta' = (\beta^{(2)},\cdots,\beta^{(d)})$. Clearly

$$\begin{pmatrix} \beta x \\ (\beta x)' \end{pmatrix} = I_\beta \begin{pmatrix} x \\ x' \end{pmatrix}.$$

Proposition 2.2 illustrates the relation between S and the β-transformation T_β.

Proposition 2.2. *For any* $x \in \mathbb{Q}(\beta)\cap[0,1)$, $S(x,x') = (T_\beta(x),(T_\beta(x))')$.

Proof. $S\begin{pmatrix} x \\ x' \end{pmatrix} = I_\beta\begin{pmatrix} x \\ x' \end{pmatrix} - \lfloor\beta x\rfloor\cdot\mathbf{1} = \begin{pmatrix} \beta^{(1)}x^{(1)} - \lfloor\beta x\rfloor \\ \vdots \\ \beta^{(d)}x^{(d)} - \lfloor\beta x\rfloor \end{pmatrix} = \begin{pmatrix} T_\beta(x) \\ (T_\beta(x))' \end{pmatrix}.$ □

Theorem 2.1. *Let* $\beta > 1$ *be a Pisot unit. Then* $x \in Pur(\beta)$ *if and only if* $x \in \mathbb{Q}(\beta)\cap[0,1)$ *and* $(x,x')\in K$*.*

Proof. (i) Suppose $x = 0.\overline{a_1\dots a_L} \in \mathrm{Pur}(\beta)$. Then $x\in\mathbb{Q}(\beta)\cap[0,1)$. Let $w = \dots\overline{a_1\dots a_L}$ and $u = \overline{a_1\dots a_L}\dots$. Clearly $(w,u)\in\Omega$. The first coordinate of $\rho(w,u)$ is

$$x = 0.\overline{a_1\dots a_L} = \frac{a_1\beta^{L-1}+\cdots+a_{L-1}\beta+a_L}{\beta^L-1}.$$

The j-th coordinate of $\rho(w,u)$ is $(j\geq 2)$

$$\begin{aligned} & -(a_L + a_{L-1}\beta^{(j)} + \cdots + a_1(\beta^{(j)})^{L-1})(1+(\beta^{(j)})^L+(\beta^{(j)})^{2L}+\cdots) \\ = & \tfrac{a_1(\beta^{(j)})^{L-1}+\cdots+a_{L-1}(\beta^{(j)})+a_L}{(\beta^{(j)})^L-1} = x^{(j)}. \end{aligned}$$

Thus $(x,x') = \rho(w,u)\in K$.

(ii) Now we prove the other direction. Suppose $x \in \mathbb{Q}(\beta) \cap [0,1)$ and $(x,x') \in K$. Let b be the smallest integer such that $bx \in \mathbb{Z}(\beta)$ and set

$$\mathcal{R}_b := \{(x,x') : \ x \in b^{-1}\mathbb{Z}(\beta)\} \cap K.$$

Then $\mathcal{R}_b$ is a finite set because K is bounded.

First we assert that $S(\mathcal{R}_b) \subseteq \mathcal{R}_b$. For $(x,x') \in \mathcal{R}_b$, $S(x,x') = (T_\beta x, (T_\beta x)')$ by Proposition 2.2. Since $T_\beta x = \beta x - \lfloor \beta x \rfloor \in b^{-1}\mathbb{Z}(\beta)$ and $S(K) = K$, we have that $S(\mathcal{R}_b) \subseteq \mathcal{R}_b$.

Secondly we claim that S is surjective on $\mathcal{R}_b$. For $(x,x') \in \mathcal{R}_b$, there exists at least one sequence $(w,u) \in \Omega$ such that $\rho(w,u) = (x,x')$. Let w_0 be the first symbol of w and let $y = \beta^{-1}(x + w_0)$. We argue that $\rho \circ \sigma^{-1}(w,u) = (y,y')$. Write

$$\rho \circ \sigma^{-1}(w,u) = \rho(\dots w_2 w_1 . w_0 u_1 u_2 \dots) = (t_1, t_2, \dots, t_d).$$

Then

$$t_1 = \frac{w_0}{\beta} + \frac{u_1}{\beta^2} + \frac{u_2}{\beta^3} + \cdots = \frac{x + w_0}{\beta} = y,$$

and for $j \geq 2$,

$$\begin{aligned} t_j &= -w_1 - \beta^{(j)} w_2 - (\beta^{(j)})^2 w_3 - \cdots \\ &= (\beta^{(j)})^{-1}((-w_0 - \beta^{(j)} w_1 - (\beta^{(j)})^2 w_2 - \cdots) + w_0) \\ &= (\beta^{(j)})^{-1}(x^{(j)} + w_0) = y^{(j)}. \end{aligned}$$

So $\rho \circ \sigma^{-1}(w,u) = (y,y')$. This implies $(y,y') \in K$ and hence $S(y,y') = (x,x')$. On the other hand, $y \in b^{-1}\mathbb{Z}(\beta)$ because β is an algebraic unit. So (y,y') belongs to $\mathcal{R}_b$ and it is a preimage of (x,x'). This proves that S is surjective on $\mathcal{R}_b$.

Hence $S|_{\mathcal{R}_b}$ is a one-to-one mapping, and thus there exists an integer n such that

$$(x,x') = S^n(x,x') = (T_\beta^n x, (T_\beta^n x)').$$

Therefore $x = T_\beta^n x$; namely, the β-expansion of x is purely periodic. $\square$

2.3. **Example 2.1.** We apply Theorem 2.1 to Pisot units of degree two (cf. [10]). the domain K is illustrated by Figure 2.1. We leave the easy calculations to the readers.

Case $\beta^2 = n\beta + 1$, $n \geq 1$**.** In this case $d_\beta^*(1) = \overline{n0}$, and M_β has two states $\bar{r}_1 = 1$ and $\bar{r}_2 = 0.\overline{0n} = \frac{1}{\beta}$. The domain K is illustrated by Figure 2.1.a. Since the line segment from o to $(1,1)$ is contained in K, $(x,x') = (x,x)$ belongs to K for any rational $0 \leq x < 1$. Hence $\mathbb{Q} \cap [0,1) \subseteq \mathrm{Pur}(\beta)$, which is the result of K. Schmidt (cf. [17]).

Case $\beta^2 = n\beta - 1$, $n \geq 3$. In this case $d_\beta^* = (n-1)\overline{(n-2)}$, M_β has two states $\bar{r}_1 = 1$ and $\bar{r}_2 = 0.\overline{(n-1)} = \frac{\beta - 1}{\beta}$. The domain K is illustrated by Figure 2.1.b. Now for any rational number $x \in [0,1)$, $(x,x) \notin K$ and the β-expansion of x is not purely periodic.

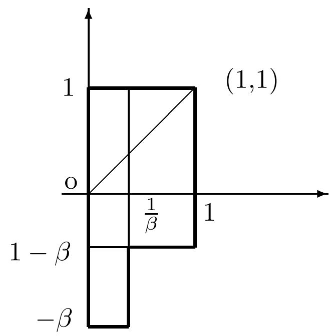

FIGURE 2.1.a

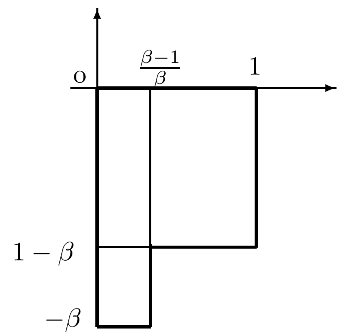

FIGURE 2.1.b

2.4. **Admissible and weakly admissible.** Let $\bar{\Omega}$ be the closure of Ω in the symbolic metric. Then $\bar{\Omega}$ is the set of all weakly admissible sequences. The set $\bar{\Omega}$ is compact and $\bar{K} = \rho(\bar{\Omega})$. Let $\overline{\lfloor \beta x \rfloor}$ denote the largest integer which is *strictly* less than βx. Replacing $\lfloor \beta x \rfloor$ in (2.3) by $\overline{\lfloor \beta x \rfloor}$, we get a new operator $\bar{S}$. Similar to the above discussion, we have

Theorem 2.1′. *Let $\beta > 1$ be a Pisot unit. Then $x \in Pur'(\beta)$ if and only if $x \in \mathbb{Q}(\beta) \cap [0, 1]$ and $(x, x') \in \bar{K}$.*

3. Proof of the Main Theorem

3.1. **Second realization of (Ω, σ).** Recall that a sequence $w_n \dots w_1 w_0$ stops at state i if there is a path form state 1 to state i with label $w_n \dots w_1 w_0$. It is equivalent to $w_n \dots w_1 w_0 \cdot r_i$ being weakly admissible, but for any $r' \succ r$ (in lexicographical order), $w_n \dots w_1 w_0 \cdot r'$ is not weakly admissible. We generalize this definition to an infinite backward sequence. We say $w = (\dots w_n \dots w_1 w_0)$ stops at state i if $w \cdot r_i$ is weakly admissible but $w \cdot r'$ is not weakly admissible for any $r' \succ r$. We claim that

$$X_i = \{\sum_{k=0}^{\infty} w_k A_\beta^k \pi(e_1) : \ (\dots w_n \dots w_1 w_0) \text{ stops at state } i\}. \tag{3.1}$$

By the definition of X_i, we have

$$X_i = \overline{\pi \circ \phi\{\sum_{k=0}^{n} w_k \beta^k : \ w_n \dots w_1 w_0 \text{ stops at state } i\}}.$$

Since π and ϕ are linear, by (1.1) and (1.2) we see that

$$\begin{aligned} \pi \circ \phi(\sum_{k=0}^{n} w_k \beta^k) &= \pi(\sum_{k=0}^{n} w_k A_\beta^k \phi(1)) \\ &= \sum_{k=0}^{n} w_k A_\beta^k \pi(e_1). \end{aligned}$$

Hence the right side of (3.1) is a subset of X_i. On the other hand, for any $x \in X_i$, there is a sequence of finite words $\{w^{[k]}\}_{k \ge 1}$ such that $w^{[k]}$ stops at state i and the $\pi \circ \phi(w^{[k]})$ converge to x. Since $\bar{\Omega}$ is compact, there is at least one limit point w.

One can show that w stops at state i and $\sum_{k=0}^{\infty} w_k A_\beta^k \pi(e_1) = x$. This proves the other direction of (3.1).

From

$$[0, \bar{r}_i] = \{\sum_{k=1}^{\infty} u_k \beta^k : \ u = (u_1 u_2 \dots) \text{ weakly admissible and } u \preceq r_i\}$$

we get

(3.2)

$$I_i = \{\sum_{k=1}^{\infty} u_k A_\beta^{-k} \pi'(e_1) : \ u = (u_1 u_2 \dots) \text{ weakly admissible and } u \preceq r_i\}.$$

Motivated by (3.1) and (3.2), we define $\psi : \ \bar{\Omega} \mapsto \hat{X}$ as

$$\psi(w, u) := -\sum_{k=0}^{\infty} w_k A_\beta^k \pi(e_1) + \sum_{k=1}^{\infty} u_k A_\beta^{-k} \pi'(e_1).$$

Proposition 3.1. *$\psi(\bar{\Omega}) = \hat{X}$.*

Proof. Pick any $(w, u) \in \bar{\Omega}$. There is a state i such that w stops at state i and $u \preceq r_i$. Therefore $\sum_{k=0}^{\infty} w_k A_\beta^k \pi(e_1) \in X_i$ and $\sum_{k=1}^{\infty} u_k A_\beta^{-k} \pi'(e_1) \in I_i$. So $\psi(w, u) \in X_i + I_i = \hat{X}_i \subset \hat{X}$. The other direction can be proved similarly. □

We will show in §3.4 that $(\hat{X}, \hat{T}_\beta)$ is another realization of (Ω, σ).

3.2. **Proof of the Main Theorem.** Let $v = v_1, v_2, \dots, v_d$ be eigenvectors of A_β corresponding to eigenvalues $\beta, \beta^{(2)}, \dots, \beta^{(d)}$ respectively. The order is arranged by (2.1). Every vector of $\mathbb{R}^d$ can be written in a unique way as a linear combination of $v_1, v_2, \dots, v_d$. If we assume that

$$e_1 = v_1 + v_2 + \cdots + v_d, \tag{3.3}$$

then $v_1, v_2, \dots, v_d$ are uniquely determined. Let $U = (v_1, v_2, \dots, v_d)$ be a $d \times d$ matrix with entries in $\mathbb{C}$. Regarding U as a linear transformation, we have

Proposition 3.2. *$U(\bar{K}) = \hat{X}$.*

Proof. Equation (3.3) implies that $\pi'(e_1) = v_1$ and $\pi(e_1) = v_2 + \cdots + v_d$. Hence

$$A_\beta^k \pi(e_1) = (\beta^{(2)})^k v_2 + \cdots + (\beta^{(d)})^k v_d.$$

Therefore,

$$\begin{aligned}
& U(\bar{K}) = U \circ \rho(\bar{\Omega}) \\
= \ & \{U(\textstyle\sum_{k=1}^{\infty} u_k \beta^{-k}, -\sum_{k=0}^{\infty} w_k (\beta^{(2)})^k, \cdots, -\sum_{k=0}^{\infty} w_k (\beta^{(d)})^k) : \ (w, u) \in \bar{\Omega}\} \\
= \ & \{\textstyle\sum_{k=1}^{\infty} u_k \beta^{-k} v_1 - \sum_{k=0}^{\infty} w_k ((\beta^{(2)})^k v_2 + \cdots + (\beta^{(d)})^k v_d) : \ (w, u) \in \bar{\Omega}\} \\
= \ & \{\textstyle\sum_{k=1}^{\infty} u_k A_\beta^{-k} \pi'(e_1) - \sum_{k=0}^{\infty} w_k A_\beta^k \pi(e_1) : \ (w, u) \in \bar{\Omega}\} \\
= \ & \psi(\bar{\Omega}) = \hat{X}.
\end{aligned}$$

□

Lemma 3.3. *For $x \in \mathbb{Q}(\beta)$,*

$$U(x, x') = \phi(x). \tag{3.4}$$

Proof. We claim that if (3.4) is true for x, then it is also true for βx. For

$$\begin{aligned} U\begin{pmatrix}\beta x\\(\beta x)'\end{pmatrix} &= UI_\beta\begin{pmatrix}x\\x'\end{pmatrix} = (\beta^{(1)}v_1,\ldots,\beta^{(d)}v_d)\begin{pmatrix}x\\x'\end{pmatrix}\\ &= A_\beta(v_1,\ldots,v_d)\begin{pmatrix}x\\x'\end{pmatrix} = A_\beta U\begin{pmatrix}x\\x'\end{pmatrix} = A_\beta\phi(x) = \phi(\beta x).\end{aligned}$$

By (3.3), we see that (3.4) is true for $x = 1$. Hence (3.4) holds for $x = \beta^k$, $0 \le k \le d-1$. Notice that both ϕ and $x \mapsto x'$ are linear mappings; therefore, (3.4) hold for all $x \in \mathbb{Q}(\beta)$. □

Proof of the Main Theorem. Notice that the matrix U is non-singular; hence,

$$(x,x') \in \bar{K} \;\Leftrightarrow\; U(x,x') \in U\bar{K} \;\Leftrightarrow\; \phi(x) \in \hat{X}.$$

Now our Main Theorem follows from Theorem 2.1′. □

3.3. **Example 3.1.** Again we consider a Pisot unit of degree two.

Case $\beta^2 = n\beta + 1$. The domain $\hat{X}$ is illustrated by Figure 3.1.a. Since the line segment from o to e_1 is contained in $\hat{X}$, so $\phi(x) \in \hat{X}$ holds for any rational x in $[0,1)$. Hence $\mathbb{Q} \cap [0,1) \subseteq \mathrm{Pur}'(\beta)$.

Case $\beta^2 = n\beta - 1$. The domain $\hat{X}$ is illustrated by Figure 3.1.b.

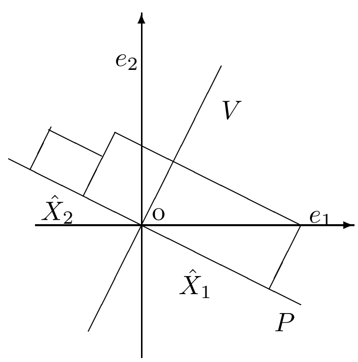

FIGURE 3.1.a

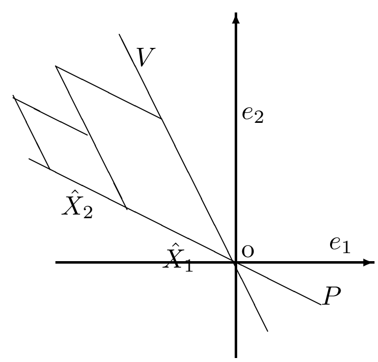

FIGURE 3.1.b

3.4. **Transformation** $\hat{T}_\beta$**.** Thurston [20] described the following dynamics: Let $t \in \mathbb{R}^d$ be a point in the tube between P and the hyperplane $P + e_1$. Multiply t by A_β and subtract the largest multiple of e_1 which keeps t on the same side of P. (Remember that $e_1 = \phi(1)$.) Let us denote this transformation by $\hat{T}_\beta$. If $t = \phi(x)$ with $x \in \mathbb{Q}(\beta)$ (we say t is algebraic), then it follows that

$$\hat{T}_\beta(t) = \phi(T_\beta x).$$

Observe that $\hat{T}_\beta^k(t)$ always remains in a bounded region near the origin as $\hat{T}_\beta$ is iterated. It can never escape very far from P since we always guide it back, and it can never escape very far from V since A_β squeezes every point toward V.

If t starts out as an algebraic integer in $\mathbb{Q}(\beta)$, then it always remains an algebraic integer. The set of all algebraic integers forms a lattice in $\mathbb{R}^d$ by mapping ϕ, so $\hat{T}_\beta^k(t)$ can only take a finite number of values. Therefore, its orbit eventually arrives back at a previous point, and from then on it repeats. If t is in $\mathbb{Q}(\beta)$, then there is some integer b such that bt is an algebraic integer. Then we apply the above argument. This proves that every number in $\mathbb{Q}(\beta)$ has eventually a periodic β-expansion.

From the discussion of Section 2 and Section 3, we can easily confirm the following commutative diagram, and thus $(\hat{X}, \hat{T}_\beta)$ is a realization of $(\bar{\Omega}, \sigma)$.

$$\begin{array}{ccccc} \hat{X} & \xleftarrow{\ \psi\ } & \bar{\Omega} & \xrightarrow{\ \rho\ } & \bar{K} \\ {\scriptstyle \hat{T}_\beta}\Big\downarrow & & {\scriptstyle \sigma}\Big\downarrow & & \Big\downarrow{\scriptstyle \bar{S}} \\ \hat{X} & \xleftarrow{\ \psi\ } & \bar{\Omega} & \xrightarrow{\ \rho\ } & \bar{K} \end{array} \tag{3.5}$$

We obtain a more exact picture about the behavior of $\hat{T}_\beta$: starting from an algebraic point t, sooner or later it will fall into $\hat{X}$. Then it moves inside $\hat{X}$, and comes back to the entering point after a finite number of steps, and from then on it repeats.

4. Some remarks

4.1. **(F)-property.** We denote by $\mathrm{Fin}(\beta)$ the set of all $\alpha \in \mathbb{Z}(\beta)_{\geq 0}$ that have finite β-expansion. We say β has the *(F)-property* (finite expansion property) provided $\mathbb{Z}(\beta)_{\geq 0} \subseteq \mathrm{Fin}(\beta)$. The (F)-property of algebraic numbers is introduced by C. Frougny and B. Solomyak [9]. Akiyama found the connection between the (F)-property and tiling.

Lemma 4.1 (Akiyama [3])**.** *If $\beta > 1$ is a Pisot unit with the (F)-property, then $0 \in P$ is an interior point of the atomic surface $X = \bigcup_{i=1}^{q} X_i$.*

The following result of Akiyama can be derived from the Main Theorem. (Akiyama's original proof is an algebraic argument.)

Theorem A (Akiyama [2])**.** *If $\beta > 1$ is a Pisot unit with the (F)-property, then there exists a constant $c = c(\beta) > 0$ such that $\mathbb{Q} \cap [0, c) \subseteq Pur(\beta)$.*

Proof. Let Δ be the line passing through 0 and e_1. Since $0 \in P$ is an interior point of X, $\hat{X}$ contains a small half-ball with center 0 and sitting on the positive side of P (the side containing e_1). Hence a small part of the line Δ near 0 is contained in $\hat{X}$. We start from the origin and walk along the line Δ. Let ce_1 be the first point we meet on the boundary of $\hat{X} \cap \Delta$. For a rational number $x \in [0, c)$, $\phi(x) = x\phi(1) = xe_1$ belongs to the line segment from 0 to $c\phi(1)$ and thus is contained in $\hat{X}$. So by the Main Theorem, $x \in \mathrm{Pur}'(\beta)$. Since x is not an algebraic integer, it cannot be one of $r_1, \cdots, r_q$. Therefore $x \in \mathrm{Pur}(\beta)$. Moreover, this number c is the best bound. □

Let $\beta > 1$ satisfy $\beta^2 = n\beta + 1$. Then β has the (F)-property, and 0 is an interior point of X as we see in Figure 3.1.a. In this case $c = 1$. If $\beta > 1$ satisfies $\beta^2 = n\beta - 1$, then β does not have the (F)-property. From Figure 3.1.b we see that 0 sits on the boundary of X, and Theorem A is false. When $d \geq 3$, it is not so easy to get the best bound c in Theorem A because of the fractal nature of atomic surfaces.

4.2. **Atomic surfaces and Markov partitions.** The family $\{X_i\}_{1 \leq i \leq q}$ has been studied extensively in many papers. It is proved that $\{X_i\}_{1 \leq i \leq q}$ are the invariant sets of a graph iterated function system (graph-IFS). The graph-IFS satisfies the open set condition, and the X_i have non-empty interiors. So $\{X_i\}_{1 \leq i \leq q}$ forms a self-similar tiling system (cf. [16, 6, 2, 3, 4, 7, 19, 11, 8]). For detailed discussions of the tiling and dynamical properties of $\{X_i\}_{1 \leq i \leq q}$, we refer to [11, 8].

Akiyama [4] introduced the *weakly finite property*: β is said to satisfy the weakly finite property if for any $x \in \mathbb{Z}(\beta)_{\geq 0}$ and any $\epsilon > 0$, there exist $y, z \in \mathrm{Fin}(\beta)$ such that $x = y - z$ and $z < \epsilon$. Notice that the (F)-property implies the weakly finite property since we can always choose $z = 0$. It is proved that many tiling and dynamical properties are rooted in the weakly finite property of β (cf. [4, 11, 8]).

Notice that the domains in Figure 3.1 are essentially the Markov partitions of A_β constructed by Adler and Weiss [1]. It is easy to check that in Example 3.1, $\hat{X} + \mathbb{Z}^2$ is a lattice tiling of $\mathbb{R}^2$ and thus $\hat{X}$ is a two-dimensional torus. Moreover, if we regard A_β as a group automorphism acting on $\hat{X}$, then it coincides with $\hat{T}_\beta$, and $\{\hat{X}_1, \hat{X}_2\}$ is a Markov partition for A_β. In general, we have the following result.

Theorem B (Ei, Ito and Rao [8])**.** *Let β be a Pisot unit. Then the following statements are equivalent:*

(1) *β satisfies the weakly finite property;*

(2) *$\hat{X} + \mathbb{Z}^d$ is a tiling of $\mathbb{R}^d$, and consequently $\{\hat{X}_1, \ldots, \hat{X}_d\}$ is a Markov partition of the group automorphism A_β.*

Akiyama [4] conjectured that any Pisot unit satisfies the weakly finite property. Some partial results are obtained in [5]. If the conjecture is true, then $\{\hat{X}_1, \ldots, \hat{X}_q\}$ always gives a Markov partition of A_β and it is a generalization of the construction of Adler and Weiss [1]. It is interesting that the Markov partition of A_β characterizes the set of purely periodic β-expansions.

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