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SCALING EXPONENTS OF SELF-SIMILAR FUNCTIONS AND WAVELET ANALYSIS

KOICHI SAKA

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ABSTRACT. In this paper we give estimations of the pointwise scaling exponents of self-similar functions on the n-dimensional Euclidean space $\mathbb{R}^n$. These estimations are derived by using a technique based on wavelet analysis. Examples of such self-similar functions include indefinite integrals of self-similar measures on $\mathbb{R}$, and they also include widely oscillatory functions (e.g. the Takagi function, the Weierstrass function and Lévy's function). Pointwise scaling exponents provide an objective description of an irregularity of a function at a point. Our results are applied to compute the scaling exponents of several oscillatory functions.

1. Introduction

Recently fractal sets and extremely irregular functions are playing an important role in physics, in image or signal processing and in mathematics (e.g. see [6]).

A pointwise scaling exponent of an irregular function $f(x)$ at a point x_0 is aimed at providing an objective description of the irregularity of $f(x)$ at x_0. There are several non-equivalent definitions of scaling exponents. In particular two scalings of the Hölder scaling exponent $\alpha(f, x_0)$ and the weak scaling exponent $\beta(f, x_0)$ have been investigated in Y. Meyer [7] whose works are based on the relation between scaling exponents and estimating the size of wavelet transforms. The weak scaling exponent $\beta(f, x_0)$ is more sensitive to the oscillations of $f(x)$ at x_0. If $\alpha(f, x_0) = \beta(f, x_0)$, then x_0 is called a cusp singularity for a function $f(x)$. If $\alpha(f, x_0) \neq \beta(f, x_0)$, then x_0 is called an oscillating singularity for $f(x)$. Oscillating behavior of a function $f(x)$ at a point x_0 is relevant to the two scaling exponents of $f(x)$ at x_0. The pointwise Hölder scaling exponent of self-similar functions has been studied in relation to multifractal formalism in some particular cases in [1], [2], [4] and [5].

In this paper we give the estimations of the two pointwise scaling exponents of $\alpha(F, x_0)$ and $\beta(F, x_0)$ for self-similar functions $F(x)$ on $\mathbb{R}^n$ in a more general setting. Examples of such self-similar functions include indefinite integrals of self-similar measures on $\mathbb{R}$, and they also include widely oscillatory functions (e.g. the Takagi function, the Weierstrass function and Lévy's function). Properties of self-similar functions are closely related to those of self-similar measures on $\mathbb{R}^n$. Basic properties

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of finite positive self-similar measures on $\mathbb{R}^n$ have been investigated systematically by S. Strichartz [8], [9] and [10] based on Fourier analysis.

In this paper, our results can be applied to compute the pointwise scaling exponents of oscillating functions in the examples.

The plan of the next sections in our paper is as follows:

In the second section the several scaling exponents of a function are defined and we derive their basic properties.

In the third section we give the definition of self-similar functions and we prove the main theorem for scaling exponents of self-similar functions. In the proof of the theorem, a method of wavelet analysis is used.

In the fourth section we give examples of self-similar functions and we compute the scaling exponents of these examples by applying the theorem.

We use C to denote a positive constant different in each occasion. But it will depend on the parameter appearing in each problem. The same notations C are not necessarily the same on any two occurrences.

2. Scaling exponents

If $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space, then $\mathcal{S}_0(\mathbb{R}^n)$ is the closed subspace of $\mathcal{S}(\mathbb{R}^n)$ defined by

$$\int y^\alpha \varphi(y)dy = 0, \quad \forall \alpha \in \mathbb{Z}_+^n,$$

where $\mathbb{Z}_+$ is the set of all nonnegative integers. Let F be a tempered distribution and s a nonnegative real number. We write $F \in \Gamma^s(x)$ if for every φ in $\mathcal{S}_0(\mathbb{R}^n)$, there exists a constant C such that

$$|t^{-n} \int F(y)\varphi(\frac{y-x}{t})dy| \leq Ct^s, \quad 0 < t \leq 1.$$

Lemma 1 (cf. [7, Theorem 3.4])**.** *Let us consider an integer k larger than a nonnegative number s and a function $\varphi \in C^\infty(\mathbb{R}^n)$ with compact support such that*

$$\int y^\alpha \varphi(y)dy = 0, \quad \forall |\alpha| \leq k.$$

Then if $F \in \Gamma^s(x_0)$, there exist positive constants $C,\ \delta_0$ for any given $C_0 > 0$ such that

$$|t^{-n} \int \varphi(\frac{R(y-x)}{t})F(y)dy| \leq Ct^s$$

for any isometry R whenever $|x - x_0| \leq C_0 t$ and $0 < t \leq \delta_0 < 1$.

A pointwise weak scaling exponent $\beta(f, x)$ for $x \in \mathbb{R}^n$ is defined as

$$\beta(f,x) = \sup\{s \geq 0 : f \in \Gamma^s(x)\}. \tag{1}$$

If $k < s < k + 1$ for a nonnegative integer k, then for $x \in \mathbb{R}^n$, a function $f \in C^s(x)$ means that there exists a polynomial P_x of degree less than or equal to k such that

$$|f(y) - P_x f(x-y)| \leq C|x-y|^s \tag{2}$$

on a neighborhood of x. For an open set Ω in $\mathbb{R}^n$, $f \in C^s(\Omega)$ means that f is bounded on Ω and (2) holds for all $x \in \Omega$ with a uniform constant C on Ω. When s is a nonnegative integer, we need some modification for the definition above. See

[7, p. 6] for details. The pointwise Hölder scaling exponent of a function f at a point x is defined as

$$\alpha(f,x) = \sup\{s \geq 0 : f \in C^s(x)\}. \tag{3}$$

We also define

$$\alpha(f,\Omega) = \sup\{s \geq 0 : f \in C^s(\Omega)\}. \tag{4}$$

In particular when $\Omega = \mathbb{R}^n$, we write

$$\alpha(f) = \alpha(f,\mathbb{R}^n). \tag{5}$$

Let $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ be the Besov space and the Triebel-Lizorkin space respectively for $0 < p, q \leq \infty$. Then we define

$$\alpha_{pq}(f) = \sup\{s \geq 0 : f \in B^s_{pq}(\mathbb{R}^n)\},$$
$$\alpha_p(f) = \alpha_{p\infty}(f).$$

By the definition it follows that $\alpha(f) = \alpha_\infty(f)$ where $\alpha(f)$ is given in (5). By the embedding theorem of function space theory (e.g. see [11]), we have the following proposition:

Proposition 1. (a) *$\alpha_{pq}(f) = \sup\{s \geq 0 : f \in F^s_{pq}(\mathbb{R}^n)\}$, for $0 < p < \infty, \quad 0 < q \leq \infty$.*

(b) *$\alpha_p(f) = \alpha_{p\eta}(f)$ for $0 < p, \eta \leq \infty$.*

(c) *$\alpha(f) \geq \alpha_p(f) - \frac{n}{p} \geq \alpha_q(f) - \frac{n}{q}$ for $0 < q \leq p < \infty$.*

Let for $x \in \mathbb{R}^n, \quad t > 0$,

$$\mathrm{osc}^k_p f(x,t) = \inf_{\deg P \leq k} \Big(\frac{1}{|B(x,t)|}\int_{B(x,t)} |f(y) - P(y)|^p dy\Big)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\mathrm{osc}^k_\infty f(x,t) = \inf_{\deg P \leq k} \sup_{|x-y|<t} |f(y) - P(y)|,$$

where the infimum is taken over all polynomials P of degree less than or equal to a nonnegative integer k and $|B(x,t)|$ means the volume of the ball $B(x,t)$.

We write $f \in T^s_{pq}(x)$ defined by (2) being replaced with

$$\int_0^1 (t^{-s}\mathrm{osc}^k_p f(x,t))^q \frac{dt}{t} < \infty \quad (0 < q < \infty, \quad s < k+1),$$
$$\sup_{0<t\leq 1} t^{-s}\mathrm{osc}^k_p f(x,t) < \infty \quad (q = \infty),$$

where $1 \leq p \leq \infty$. We define

$$\alpha_{pq}(f,x) = \sup\{s \geq 0 : f \in T^s_{pq}(x)\}$$

and

$$\alpha_p(f,x) = \alpha_{p\infty}(f,x).$$

By the definition it follows that $\alpha(f,x) = \alpha_\infty(f,x)$ where $\alpha(f,x)$ is given in (3). In the easy routine of function space theory, we can see the following proposition:

Proposition 2 (cf. [7, p. 3])**.** (a) *$\alpha_{p\xi}(f,x) = \alpha_{p\eta}(f,x)$ for $0 < \xi, \eta \leq \infty$ and $1 \leq p \leq \infty$.*

(b) *$\alpha(f) \leq \alpha(f,x) \leq \alpha_p(f,x) \leq \alpha_q(f,x) \leq \beta(f,x)$ for $1 \leq q \leq p < \infty$.*

3. Self-similar functions

Definition. A function F on $\mathbb{R}^n$ is said to be self-similar relative to a function g on $\mathbb{R}^n$ if

$$F(x) = \sum_{j=1}^{d} \lambda_j F(S_j^{-1}x) + g(x),\ x \in \mathbb{R}^n, \tag{6}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_d$ are real or complex numbers with $0 < |\lambda_j| < 1, j = 1, 2, \ldots, d$, and $S_1, S_2, \ldots, S_d$ are contractive similarities with ratios $\mu_1, \mu_2, \ldots, \mu_d$ satisfying $0 < \mu_j < 1, j = 1, 2, \ldots, d$. We remark that (6) implies $\alpha(F) \leq \alpha(g)$.

From now on we will use the notation $S_J = S_{j_1} S_{j_2} \cdots S_{j_l},\ \lambda_J = \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_l}$, $\mu_J = \mu_{j_1}\mu_{j_2}\cdots\mu_{j_l},\ |J| = l$ for a multi-index $J = (j_1, j_2, \ldots, j_l) \in \{1, 2, \ldots, d\}^l$, and $S_J = \text{Identity}$, $\lambda_J = 1,\ \mu_J = 1,\ |J| = 0$ for $J = 0$. If a function F is self-similar relative to g, then we have for any N,

$$F(x) = \sum_{|J|<N} \lambda_J g(S_J^{-1}x) + \sum_{|J|=N} \lambda_J F(S_J^{-1}x),\ x \in \mathbb{R}^n. \tag{7}$$

We say that the open set condition holds if there exists a bounded open set Ω in $\mathbb{R}^n$ such that

$$S_i(\Omega) \subset \Omega, \quad i = 1, 2, \ldots, d,$$

and

$$S_i(\Omega) \cap S_j(\Omega) = \emptyset \quad (i \neq j), \quad i, j = 1, 2, \ldots, d \quad (d \geq 2).$$

We denote $\Omega_0 = \Omega,\ \Omega_i = S_i(\Omega)$ and $\Omega_J = S_J(\Omega)$ for $J = (j_1, j_2, \ldots, j_l)$. $K = \bigcap_{l=0}^{\infty} \bigcup_{|J|=l} \bar{\Omega}_J$ is called the invariant set with respect to similarities $\{S_j\}_{j=1}^d$. Assume that the functions F and g are bounded and zero outside Ω. When the open set condition holds, then (6) has a unique solution which is given by the series

$$F(x) = \sum_J \lambda_J g(S_J^{-1}x), \quad x \in \mathbb{R}^n.$$

Then we can see easily that $\alpha(F, x) \geq \alpha(g)$ for each $x \notin K$.

Let

$$a(x) = \liminf_{N\to\infty} \inf_{K_N(x)\ni J} \frac{\log|\lambda_J|}{\log \mu_J},\ x \in K, \tag{8}$$

where $K_N(x) = \{J : B(x, \mu_J) \cap \Omega_J \neq \emptyset,\ |J| = N\}$ and $B(x, r)$ is a ball centered at x with a radius r.

Lemma 2. *Let μ be any real number with $0 < \mu < 1$ and $x \in K$. We put*

$$K_N^{\mu}(x) = \{J : B(x, \mu^N) \cap \Omega_J \neq \emptyset,\ \mu^{N+1} \leq \mu_J < \mu^N\}.$$

Then we have

$$a(x) = \liminf_{N\to\infty} \inf_{K_N^{\mu}(x)\ni J} \frac{\log|\lambda_J|}{\log \mu_J}.$$

Proof. In an easy routine we can prove the lemma. We will omit the details (cf. [5]). ∎

When $x \in \bigcap_{l=0}^{\infty} \bigcup_{|J|=l} \Omega_J$, there exists a unique sequence $J(x) = (j_1, j_2, \ldots)$ such that $x \in \Omega_{J_l(x)}$ where $J_l(x) = (j_1, j_2, \ldots, j_l)$ for $l = 1, 2, \ldots$ and $J_0(x) = 0$. Let

$$b(x) = \liminf_{N\to\infty} \frac{\log|\lambda_{J_N(x)}|}{\log \Delta_N(x)}, \tag{9}$$

for $x \in \bigcap_{l=0}^{\infty} \bigcup_{|J|=l} \Omega_J$ where $\Delta_N(x) = \operatorname{dist}(x, \partial\Omega_{J_N(x)})$ is the distance from x to the boundary of $\Omega_{J_N(x)}$.

Theorem. *Let F be a self-similar bounded function relative to a bounded function g on $\mathbb{R}^n$, for which the open set condition holds. Assume that $F(x) = g(x) = 0$ for all $x \notin \Omega$.*

(a) *Then we have*

$$\alpha(F, x) \geq \min(\alpha(g), a(x)), \;\; x \in K,$$

where $\alpha(F, x)$, $\alpha(g)$ and $a(x)$ are given in (3), (5) *and* (8) *respectively.*

(b) *Suppose that $g \in C^\infty(\Omega_i), i = 1, 2, \ldots, d$ (i.e. piecewise smooth), and that $\inf_{x\in\Omega} \beta(F, x) < \infty$. Then we have*

$$a(x_0) \geq \beta(F, x_0), \;\; x_0 \in K,$$

where $\beta(F, x_0)$ is given in (1).

(c) *Suppose that $x \in \bigcap_{l=0}^{\infty} \bigcup_{|J|=l} \Omega_J$ with $\sup_N \dfrac{\Delta_N(x)}{\Delta_{N+1}(x)} < \infty$. Then we have*

$$\alpha(F, x) \geq \min_i(\alpha(g, \Omega_i), b(x))$$

where $\alpha(g, \Omega_i)$ and $b(x)$ are given in (4) *and* (9) *respectively.*

Proof. (a) We fix any element $x \in K$. We may assume $\min(\alpha(g), a(x)) > 0$. We choose a positive number s such that

$$\min(\alpha(g), a(x)) > s > 0.$$

We claim that $F \in C^s(x)$. We may assume that there is a nonnegative integer k such that $k + 1 > s > k$ and we can choose a positive number s' such that $k + 1 > s' > s$ and $g \in C^{s'}(\mathbb{R}^n)$.

From Lemma 2 there is a positive integer N_0 such that

$$\forall N \geq N_0, \; |\lambda_J| \leq \mu_J^s, \; J \in K_N^\mu(x),$$

where μ is any fixed number with $0 < \mu < 1$. Let $T_x g(x - y)$ be the k-th Taylor polynomial of g at x and let $PF(x - y)$ be a polynomial such that

$$PF(x - y) = \sum_J \lambda_J T_{S_J^{-1}x} g(S_J^{-1}x - S_J^{-1}y).$$

Let N be an integer such that $N \geq N_0$ and consider any $y \in \mathbb{R}^n$ with

$$\mu^{N+1} \leq |x - y| < \mu^N.$$

Then we have

$$
\begin{aligned}
&F(y) - PF(x-y) \\
&\quad= \sum_{l=0}^{N-1} \sum_{J \in B_l} \lambda_J (g(S_J^{-1}y) - T_{S_J^{-1}x} g(S_J^{-1}x - S_J^{-1}y)) + \sum_{l=N}^{\infty} \sum_{J \in B_l} \lambda_J g(S_J^{-1}y) \\
&\qquad - \sum_{l=N}^{\infty} \sum_{J \in B_l} \lambda_J T_{S_J^{-1}x} g(S_J^{-1}x - S_J^{-1}y) = I + II + III,
\end{aligned}
$$

where $B_l = \{J : \mu^{l+1} \le \mu_J < \mu^l\}$.

The first sum is split into two parts:

$$
I = \sum_{l=0}^{N_0-1} + \sum_{l=N_0}^{N-1} = I_0 + I_1.
$$

From the fact that $g \in C^{s'}(\mathbb{R}^n)$ the sum I_0 is estimated in

$$
\begin{aligned}
|I_0| &\le C \sum_{l=0}^{N_0-1} \sum_{J \in B_l} |\lambda_J| |S_J^{-1}x - S_J^{-1}y|^{s'} \le C \sum_{l=0}^{N_0-1} \sum_{J \in B_l} |\lambda_J| \mu_J^{-s'} |x-y|^{s'} \\
&\le C|x-y|^{s'} \le C|x-y|^s.
\end{aligned}
$$

We write $B_l(y) = \{J : \mu^{l+1} \le \mu_J < \mu^l,\ y \in \Omega_J\}$. We can see easily that the cardinality of $B_l(y)$ is bounded independently of y and l. Then the sum I_1 is bounded by

$$
\begin{aligned}
|I_1| &\le C \sum_{l=N_0}^{N-1} \sum_{J \in B_l(x) \cup B_l(y)} |\lambda_J| |S_J^{-1}x - S_J^{-1}y|^{s'} \\
&\le C \sum_{l=N_0}^{N-1} \sum_{J \in B_l(x) \cup B_l(y)} |\lambda_J| \mu_J^{-s'} |x-y|^{s'} \\
&\le C \sum_{l=N_0}^{N-1} \sum_{J \in B_l(x) \cup B_l(y)} \mu_J^{-(s'-s)} |x-y|^{s'} \\
&\le C \sum_{l=N_0}^{N-1} \sum_{J \in B_l(x) \cup B_l(y)} \mu^{-(l+1)(s'-s)} |x-y|^{s'} \\
&\le C\mu^{-N(s'-s)} |x-y|^{s'} \le C|x-y|^s.
\end{aligned}
$$

We estimate the sum II by

$$
\begin{aligned}
|II| &\le \sum_{l=N}^{\infty} \sum_{J \in B_l(y)} |\lambda_J| |g(S_J^{-1}y)| \le C \sum_{l=N}^{\infty} \sum_{B_l(y) \ni J} |\lambda_J| \\
&\le C \sum_{l=N}^{\infty} \sum_{J \in B_l(y)} \mu_J^s \le C \sum_{l=N}^{\infty} \mu^{ls} \le C\mu^{Ns} \le C|x-y|^s.
\end{aligned}
$$

The sum III is bounded in

$$\begin{aligned}|III| &\leq C\sum_{l=N}^{\infty}\sum_{J\in B_l(x)}\sum_{|\alpha|\leq k}\mu_J^{-|\alpha|}|x-y|^{|\alpha|}|\lambda_J| \\ &\leq C\sum_{|\alpha|\leq k}\sum_{l=N}^{\infty}\sum_{J\in B_l(x)}\mu_J^{s-|\alpha|}|x-y|^{|\alpha|} \\ &\leq C\sum_{|\alpha|\leq k}\mu^{N(s-|\alpha|)}|x-y|^{|\alpha|}\leq C\mu^{Ns}\leq C|x-y|^s.\end{aligned}$$

Hence we obtain

$$|F(y)-PF(x-y)|\leq |I_0|+|I_1|+|II|+|III|\leq C|x-y|^s.$$

This completes part (a) of the theorem.

(b) We start from $F\in\Gamma^s(x_0)$ and we will claim that $s\leq a(x_0)$. Let us choose k as a large enough integer that will be determined later with $k>s$ and let us consider a function φ satisfying the conditions in Lemma 1 and $\int_0^\infty|\hat{\varphi}(t\xi)|^2\frac{dt}{t}\neq 0$ for $\xi\neq 0$.

Then we have from Lemma 1 for $C_0 = 1 + \text{diam}\,\Omega$,

$$|t^{-n}\int\varphi(\frac{R(y-x)}{t})F(y)dy|\leq Ct^s$$

for any orthogonal R whenever $0<t\leq\delta_0$ and $|x-x_0|<C_0t$. We may assume that $0\in\Omega$ and $F\notin\Gamma^k(0)$. Hence $v_J=S_J0\in\Omega_J$ for any J. We can choose a large integer N such that $0<\mu_J\leq\delta_0,\ |J|=N$. For this integer N, we fix any element $J^0=(j_1^0,j_2^0,\ldots,j_N^0)\in K_N(x_0)$. Since $|x_0-v_J|<C_0\mu_J$ for all $J\in K_N(x_0)$ and we can write $S_{J^0}^{-1}y=\dfrac{R_{J^0}^{-1}(y-v_{J^0})}{\mu_{J^0}}$ for some orthogonal R_{J^0}, it follows that

$$|\mu_{J^0}^{-n}\int\varphi(S_{J^0}^{-1}y)F(y)dy|\leq C\mu_{J^0}^s. \tag{10}$$

Hence we have from (7),

$$\begin{aligned}\mu_{J^0}^{-n}\int\varphi(S_{J^0}^{-1}y)F(y)dy &= \int\varphi(y)F(S_{J^0}y)dy \\ &= \sum_{|J|<N}\lambda_J\int\varphi(y)g(S_J^{-1}S_{J^0}y)dy+\sum_{|J|=N}\lambda_J\int\varphi(y)F(S_J^{-1}S_{J^0}y)dy \\ &= \sum_{l=0}^{N-1}\lambda_{J_l^0}\int\varphi(y)g(S_{J_l^0}^{-1}S_{J^0}y)dy+\lambda_{J^0}\int\varphi(y)F(y)dy \qquad (11)\end{aligned}$$

where $J_0^0=0$ and $J_l^0=(j_1^0,\ldots,j_l^0),\ \ 1\leq l<N$. We may assume that $\text{supp}\,\varphi\subset\Omega$. We have from $g\in C^\infty(\Omega_i)$ for $i=1,2,\ldots,d$,

$$\begin{aligned}|\int_\Omega\varphi(y)g(S_{J_l^0}^{-1}S_{J^0}y)dy| &= |\int_\Omega\varphi(y)(g(S_{J_l^0}^{-1}S_{J^0}y)-Tg(S_{J_l^0}^{-1}S_{J^0}y-S_{J_l^0}^{-1}S_{J^0}0))dy| \\ &\leq C\int|\varphi(y)||S_{J_l^0}^{-1}S_{J^0}y-S_{J_l^0}^{-1}S_{J^0}0|^kdy\leq C(\mu_{j_{l+1}^0}\ldots\mu_{j_N^0})^k\int|\varphi(y)||y|^kdy,\end{aligned}$$

for $0 \leq l < N$ where Tg is the $(k-1)$-th Taylor polynomial of g at $S_{J_l^0}^{-1}S_{J^0}0$. From above, we have

$$\begin{aligned} |\sum_{l=0}^{N-1} \lambda_{J_l^0} \int \varphi(y) g(S_{J_l^0}^{-1} S_{J^0} y) dy| &\leq C \int |\varphi(y)||y|^k dy \sum_{l=0}^{N-1} |\lambda_{J_l^0}| (\mu_{j_{l+1}^0} \dots \mu_{j_N^0})^k \\ &\leq C \mu_{J^0}^k \int |\varphi(y)||y|^k dy \frac{|\lambda_{J^0}|}{(\min_i |\lambda_i|)^N} \frac{(\max_i \mu_i)^{Nk}}{\mu_{J^0}^k} \sum_{l=0}^{N-1} \frac{(\min_i |\lambda_i|)^l}{(\max_i \mu_i)^{kl}} \\ &\leq C |\lambda_{J^0}| \int |\varphi(y)||y|^k dy \end{aligned}$$

for an integer k such that $\min_i |\lambda_i| > \max_i \mu_i^k$.

From [7, Theorem 3.5] we may assume that $C \int |\varphi(y)||y|^k dy \leq \frac{1}{2}$ in the right-hand side of the above and $|\int \varphi(y) F(y) dy| = 1$ because of the fact that $F \notin \Gamma^k(0)$. Thus we obtain from (11),

$$\begin{aligned} & |\mu_{J^0}^{-n} \int \varphi(S_{J^0}^{-1} y) F(y) dy| \\ \geq \quad & |\lambda_{J^0}| |\int \varphi(y) F(y) dy| - |\sum_{l=0}^{N-1} \lambda_{J_l^0} \int \varphi(y) g(S_{J_l^0}^{-1} S_{J^0} y) dy| \\ \geq \quad & |\lambda_{J^0}| - \frac{1}{2} |\lambda_{J_0}| \geq \frac{1}{2} |\lambda_{J^0}|. \end{aligned} \tag{12}$$

(10) and (12) imply that $a(x_0) \geq s$. This yields part (b) of the theorem.

The proof of part (c) of the theorem is the same as the proof of part (a). We use the same notation in the proof of part (a). We fix any element x in $\bigcap_{l=0}^{\infty} \bigcup_{|J|=l} \Omega_J$. Let $J(x) = (j_1, j_2, \dots)$ be a sequence such that $x \in \Omega_{J_l(x)}$ for all $l \geq 0$ where $J_l(x) = (j_1, j_2, \dots, j_l)$ and $J_0(x) = 0$. We may choose a positive number s such that

$$\min_i (\alpha(g, \Omega_i), b(x)) > s > 0.$$

We will claim that $F \in C^s(x)$. We may assume that $k+1 > s > k$ with a nonnegative integer k and we choose a positive number s' such that $k+1 > s' > s$ and $g \in C^{s'}(\Omega_i), i = 1, 2, \dots, d$. From the definition of $b(x)$ in (9), there is a positive integer N_0 such that

$$|\lambda_{J_N(x)}| \leq \Delta_N(x)^s, \ \forall N \geq N_0.$$

We consider any $y \in \mathbb{R}^n$ such that $\Delta_{N+1}(x) \leq |x-y| < \Delta_N(x)$. Hence $y \in \Omega_{J_N(x)}$. We put

$$\begin{aligned} F(y) - PF(x-y) &= \sum_{|J|<N} \lambda_J (g(S_J^{-1} y) - T_{S_J^{-1} x} g(S_J^{-1} x - S_J^{-1} y)) \\ &\quad + \sum_{|J|=N} \lambda_J F(S_J^{-1} y) - \sum_{|J| \geq N} \lambda_J T_{S_J^{-1} x} g(S_J^{-1} x - S_J^{-1} y) \\ &= I + II + III. \end{aligned}$$

We split the sum I into two parts:

$$I = \sum_{|J|<N_0} + \sum_{N_0 \leq |J| < N} = I_0 + I_1.$$

Since $\Delta_l(x) \le (\text{diam } \Omega)\mu_{J_l(x)}$ for $l > 0$, these sums are bounded in

$$
\begin{aligned}
|I_0| &\le C \sum_{l=0}^{N_0-1} |\lambda_{J_l(x)}| \mu_{J_l(x)}^{-s'} |x-y|^{s'} \le C|x-y|^{s'} \le C|x-y|^s, \\
|I_1| &\le C \sum_{l=N_0}^{N-1} |\lambda_{J_l(x)}| \mu_{J_l(x)}^{-s'} |x-y|^{s'} \le C \sum_{l=N_0}^{N-1} \Delta_l(x)^s \mu_{J_l(x)}^{-s'} |x-y|^{s'} \\
&\le C \sum_{l=0}^{N-1} \mu_{J_l(x)}^{-(s'-s)} |x-y|^{s'} \le C \mu_{J_N(x)}^{-(s'-s)} |x-y|^{s'} \\
&\le C\Delta_N(x)^{-(s'-s)} |x-y|^{s'} \le C|x-y|^s, \\
|II| &\le |\lambda_{J_N(x)}| |F(S_{J_N(x)}^{-1} y)| \le C\Delta_N(x)^s \le C\Delta_{N+1}(x)^s \le C|x-y|^s, \\
|III| &\le C \sum_{l=N}^{\infty} |\lambda_{J_l(x)}| \sum_{|\alpha| \le k} \mu_{J_l(x)}^{-|\alpha|} |x-y|^{|\alpha|} \\
&\le C \sum_{|\alpha| \le k} \sum_{l=N}^{\infty} |\lambda_{J_l(x)}| \Delta_l(x)^{-|\alpha|} |x-y|^{|\alpha|} \\
&\le C \sum_{|\alpha| \le k} \sum_{l=N}^{\infty} |\lambda_{J_l(x)}|^{(1-\frac{|\alpha|}{s})} |x-y|^{|\alpha|} \le C \sum_{|\alpha| \le k} |\lambda_{J_N(x)}|^{(1-\frac{|\alpha|}{s})} |x-y|^{|\alpha|} \\
&\le C \sum_{|\alpha| \le k} \Delta_N(x)^{s-|\alpha|} |x-y|^{|\alpha|} \le C|x-y|^s.
\end{aligned}
$$

These estimations yield the proof of part (c) of the theorem. □

4. Examples

We consider the similarities $S_j x = \dfrac{x+j-1}{2}$, $j = 1, 2$, on $\mathbb{R}$. Then the open set condition holds for the open interval $\Omega = (0,1)$. In the case when x is a nondyadic point in $\Omega = (0,1)$, (*i.e.* $x \in \bigcap_{l=0}^{\infty} \bigcup_{|J|=l} \Omega_J$), we have that

$$
a(x) = \liminf_{N \to \infty} \frac{\log |\lambda_{J_N(x)}|}{\log 2^{-N}}
$$

where $J_N(x)$ is given in part (c) of the proof in the theorem, and for a nondyadic point x in $\Omega = (0,1)$ with $\sup_N \dfrac{\Delta_N(x)}{\Delta_{N+1}(x)} < \infty$, we have $a(x) = b(x)$.

Let g be a bounded function on $\mathbb{R}$ such that $g \in C^\infty(\Omega_j), j = 1, 2$, and $g = 0$ outside Ω. Consider a self-similar function F given by

$$
F(x) = \sum_{j=1}^{2} \lambda_j F(S_j^{-1} x) + g(x), \ x \in \mathbb{R}, \tag{13}
$$

with $0 < |\lambda_j| < 1$, $j = 1, 2$, and $F(x) = 0$ outside $\Omega = (0,1)$. From the theorem, if $\inf_{x \in \Omega} \beta(F, x) < \infty$ we have

$$
a(x) \ge \beta(F, x) \ge \alpha(F, x) \ge \min(\alpha(g), a(x)), \ x \in K = [0,1], \tag{14}
$$

and for a nondyadic point x in $\Omega = (0,1)$ with $\sup_N \dfrac{\Delta_N(x)}{\Delta_{N+1}(x)} < \infty$,

$$\alpha(F,x) = \beta(F,x) = a(x) = b(x). \tag{15}$$

(a) Let μ be the Bernoulli measure which is a probability measure supported on [0,1] such that $\mu(I_1) = \lambda_1\mu(I)$ and $\mu(I_2) = \lambda_2\mu(I)$ when I is a dyadic interval, I_1 is the left half of I and I_2 is the right half of I with $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 \neq \lambda_2$ and $\lambda_1 + \lambda_2 = 1$. Let $F_0(x) = \mu[0,x)$ be the continuous function whose distributional derivative is μ with $F_0(x) = 0\ \ (x \leq 0)$ and $F_0(x) = 1\ \ (x \geq 1)$. Then we have

$$\beta(F_0,x) = \alpha(F_0,x) = a(x), \quad x \in K = [0,1]$$

(see [7, Proposition 1.9 and Proposition 1.10]). Let $F(x) = F_0(x) - x\ (0 < x < 1)$, and $F(x) = 0\ $ (otherwise). Then F is a self-similar function such that

$$F(x) = \lambda_1 F(S_1^{-1}x) + \lambda_2 F(S_2^{-1}x) + g(x),\ \forall x \in \mathbb{R},$$

where $g(x) = (2\lambda_1 - 1)x\ (0 < x \leq \frac{1}{2}),\ g(x) = (2\lambda_1 - 1)(1-x)\ (\frac{1}{2} \leq x < 1),\ g(x) = 0$ (otherwise). Then from (14) we can see that

$$\alpha(F,x) = \beta(F,x), \quad x \in K = [0,1].$$

In particular we have $a(x) = \alpha(F,x) = \beta(F,x)$ for each $x \in \Omega = (0,1)$.

(b) We consider the Takagi function such that

$$F(x) = \sum_J \lambda^{|J|} g(S_J^{-1}x),\ \forall x \in \mathbb{R}, \tag{16}$$

where $0 < \lambda < 1$ and g is a bounded function such that $g(x) = x\ (0 < x \leq \frac{1}{2}),\ g(x) = 1 - x\ (\frac{1}{2} \leq x < 1),\ g(x) = 0$ (otherwise). Then F is a self-similar function such that

$$F(x) = \sum_{j=1}^{2} \lambda F(S_j^{-1}x) + g(x), \quad \forall x \in \mathbb{R}.$$

Let $a = \dfrac{\log\lambda}{\log 2^{-1}}$. Then from (14), if $a \leq 1$, $a = \alpha(F,x) = \beta(F,x)$ for each $x \in K$.

(c) We consider the Weierstrass function $F(x) = \sum_{l=0}^{\infty} \lambda^l g(2^l x)$ with $0 < \lambda < 1$ and $g(x) = \sin 2\pi x\ \ (x \in \mathbb{R})$. Then F is a self-similar function such that

$$F(x) = \sum_{j=1}^{2} (2^{-1}\lambda) F(S_j^{-1}x) + g(x), \quad \forall x \in \mathbb{R}.$$

The proof of the theorem can also be applied to this self-similar function case. Then we have

$$a = \alpha(F,x) = \beta(F,x), \quad \forall x \in \mathbb{R},$$

where the constant $a = \dfrac{\log\lambda}{\log 2^{-1}}$ is given in part (b) above.

(d) We consider Lévy's function $F(x)$ in (16) being replaced by $g(x) = x - \frac{1}{2}\ \ (0 < x < 1),\ \ g(x) = 0\ $ (otherwise) and $\lambda = 2^{-1}$. In [3, Proposition 4] it follows that $\alpha(F,x) = b(x)$ for each nondyadic point x in $\Omega = (0,1)$. Moreover we can see by (15) that $1 = b(x) = \alpha(F,x) = \beta(F,x)$ for a nondyadic point x in $\Omega = (0,1)$ with $\sup_N \dfrac{\Delta_N(x)}{\Delta_{N+1}(x)} < \infty$.

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Department of Mathematics, Akita University, Akita, 010-8502 Japan
E-mail address: saka@math.akita-u.ac.jp