PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 4, Pages 1247–1253
S 0002-9939(04)07836-0
Article electronically published on November 19, 2004

VISCOSITY CONVEX FUNCTIONS ON CARNOT GROUPS

CHANGYOU WANG

(Communicated by David S. Tartakoff)

ABSTRACT. We prove that any upper semicontinuous v-convex function in any Carnot group is h-convex.

§1. INTRODUCTION

Convex functions have played very important roles in PDEs, especially fully nonlinear elliptic PDEs in Euclidean spaces (see Caffarelli-Cabré [CC] and Crandall-Ishii-Lions [CIL]). Motivated by this fact and the aim to develop an intrinsic theory of subelliptic fully nonlinear PDEs on Carnot groups, there have been works towards the theory of convex functions on Heisenberg groups by Lu-Manfredi-Stroffolini [LMS], and on general Carnot groups by Danielli-Garofalo-Nhieu [DGN].

Lu-Manfredi-Stroffolini [LMS] have extended the concept of *convex in the viscosity sense* (or v-convex) from the Euclidean space to the sub-Riemannian setting of Heisenberg groups. Using the uniqueness theorem on viscosity solutions of the subelliptic ∞-Laplacian by Bieske [B], they showed that any upper semicontinuous v-convex function on any Heisenberg group is locally Lipschitz continuous.

A geometric approach of convexity on Carnot groups is given by Danielli-Garofalo-Nhieu [DGN], where they have introduced the notation of *horizontally convex* (or h-convex) functions. One of the theorems of [DGN] implies that any locally bounded h-convex function is locally Lipschitz continuous.

It is known by [LMS] that any upper semicontinuous h-convex function u on $\mathbf{G}$ is v-convex, and the converse is also true if, in addition, $u \in \Gamma^2(\mathbf{G})$ (the horizontal C^2 space). Here we are interested in whether the converse remains true under minimal regularity assumptions.

In order to state our theorem, we first recall the basic properties of Carnot groups. A simply connected Lie group $\mathbf{G}$ is called a Carnot group of step $r \geq 1$ if its Lie algebra g admits a vector space decomposition in r layers $g = V_1 + V_2 + \cdots + V_r$ such that (i) g is stratified, i.e., $[V_1, V_j] = V_{j+1}, j = 1, \cdots, r-1$, and (ii) g is r-nilpotent, i.e. $[V_j, V_r] = 0, j = 1, \cdots, r$. We call V_1 the *horizontal* layer and $V_j, j = 2, \cdots, r$, the *vertical* layers. We choose an inner product $\langle \cdot, \cdot \rangle$ on g such that the V_j's are mutually orthogonal for $1 \leq j \leq r$. Let $\{X_{j,1}, \cdots, X_{j,m_j}\}$ denote an orthonormal basis of V_j for $1 \leq j \leq r$, where $m_j = \dim(V_j)$ is the dimension of V_j. Also denote $m = \dim(V_1)$ as the dimension of the horizontal layer and set $X_i = X_{1,i}$ for $1 \leq i \leq m$. It is well known (see [FS]) that the exponential map $\exp : g \equiv R^n \to$

Received by the editors August 21, 2003.
2000 *Mathematics Subject Classification.* Primary 58J05.

G is a global diffeomorphism so that there exists an exponential coordinate system on $\mathbf{G}$ with $n = m + \sum_{i=2}^{r} m_i$ as its topological dimension. More precisely, any $p \in \mathbf{G}$ has a coordinate $((p_1, \cdots, p_m), (p_{2,1}, \cdots, p_{2,m_2}), \cdots, (p_{r,1}, \cdots, p_{r,m_r}))$ such that

(1.1)
$$p = \exp(\xi_1(p) + \cdots + \xi_r(p)), \text{ with } \xi_1(p) = \sum_{l=1}^{m} p_l X_l, \ \xi_i(p) = \sum_{j=1}^{m_i} p_{i,j} X_{i,j}, \ 2 \le i \le r.$$

The exponential map induces a homogeneous pseudo-norm $N_{\mathbf{G}}$ on $\mathbf{G}$ (see [FS]):

$$N_{\mathbf{G}}(p) := (\sum_{i=1}^{r} |\xi_i(p)|^{\frac{2r!}{i}})^{\frac{1}{2r!}}, \text{ for } p = \exp(\xi_1(p) + \cdots + \xi_r(p)), \tag{1.2}$$

where $|\xi_1(p)| = (\sum_{l=1}^{m} p_l^2)^{\frac{1}{2}}$ and $|\xi_i(p)| = (\sum_{j=1}^{m_i} p_{i,j}^2)^{\frac{1}{2}}$ $(2 \le i \le r)$. Moreover, $N_{\mathbf{G}}$ yields a pseudo-distance on $\mathbf{G}$ as follows:

$$d_{\mathbf{G}}(p, q) := N_{\mathbf{G}}(p^{-1} \cdot q), \ \forall p, \ q \in \mathbf{G}, \tag{1.3}$$

where $\cdot$ is the group multiplication of $\mathbf{G}$ and p^{-1} is the inverse of p. It is easy to see that $d_{\mathbf{G}}$ satisfies the invariance property

$$d_{\mathbf{G}}(z \cdot x, z \cdot y) = d_{\mathbf{G}}(x, y), \quad \forall x, \ y, \ z \in G, \tag{1.4}$$

and is homogeneous of degree one, i.e.

$$d_{\mathbf{G}}(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d_{\mathbf{G}}(p, q), \ \forall \lambda > 0, \ \forall p, \ q \in \mathbf{G}, \tag{1.5}$$

where $\delta_\lambda(p) = \lambda\xi_1(p) + \sum_{i=2}^{r} \lambda^i \xi_i(p)$ is the nonisotropic dilation on $\mathbf{G}$.

We need some notation: for $u : \mathbf{G} \to R$, let $\nabla_h u := (X_1 u, \cdots, X_m u)$ to denote the horizontal gradient of u, $\nabla_h^2 u := (\frac{X_i X_j + X_j X_i}{2} u)_{1 \le i,j \le m}$ denote the horizontal hessian of u, and use ∇u and $\nabla^2 u$ to denote the full gradient and hessian of u respectively. For any $p \in \mathbf{G}$, let $H_p(\mathbf{G}) = \text{span}\{X_1(p), \cdots, X_m(p)\}$ denote the horizontal tangent plane of $\mathbf{G}$ at p.

We now recall the definition of *horizontal convexity* introduced by [DGN] (§5, Definition 5.5); see also [LMS] (§4, Definition 4.1).

Definition 1.1. For a domain $\Omega \subset \mathbf{G}$, a function $u : \Omega \to R$ is *horizontally convex* (h-convex) if for any $p \in \Omega$ and $q \in H_p(\mathbf{G}) \cap \Omega$, $u|_{[p,q]}$ is convex, where $[p, q]$ denotes the line segment joining p and q.

The v-convexity has been introduced by [LMS] (§3, Definition 3.1); see [CIL] for the general theory of viscosity solutions. More precisely,

Definition 1.2. For a domain $\Omega \subset \mathbf{G}$, an upper semicontinuous function $u : \Omega \to R$ is convex in the viscosity sense (v-convex) if for any vector $\xi \in R^m$, $\xi^T \nabla_h^2 u \xi \ge 0$ in the sense of viscosity, i.e. for any $p \in \Omega$ and any $\phi \in C^2(\Omega)$ touching u from above at p, it follows that

$$\xi^T \nabla_h^2 \phi(p) \xi = \sum_{i,j=1}^{m} \xi_i \xi_j X_i X_j \phi(p) \ge 0. \tag{1.6}$$

We are ready to state

Theorem A. *Any upper semicontinuous v-convex functions on a Carnot group* $\mathbf{G}$ *are h-convex.*

Remark 1.3. Balogh-Rickly [BR] has recently proved Theorem A for Heisenberg groups through a completely different method. While preparing this paper, J. Manfredi has also informed the author that Juutinen-Lu-Manfredi-Stroffolini [JLMS] are able to prove Theorem A by a different method.

Our idea for proving Theorem A is based on the sup-convolution construction (see §2 below) on Carnot groups, which was developed in an earlier paper by the author [W] and was employed to prove the uniqueness for continuous viscosity solutions to the subelliptic ∞-Laplacian equations on any Carnot group $\mathbf{G}$. Roughly speaking, the sup-convolution of a v-convex function is not only v-convex but also semiconvex in the Euclidean sense. The reader can consult Jensen-Lions-Souganidis [JLS] for the sup-convolution in the Euclidean space.

As a byproduct of the proof of Theorem A, we also obtain the following characterization of continuous v-convex functions, analogous to that of convex functions on the Euclidean space in the viscosity sense (cf. [LMS], §2).

Corollary B. *For any bounded domain $\Omega \subset \mathbf{G}$ and $u \in C(\Omega)$, the following statements are equivalent:*

(a) *u is v-convex on Ω.*

(b) *For any subdomain $\tilde{\Omega} \subset\subset \Omega$, there exist a family of v-convex functions $\{u_k\} \subset \Gamma^2(\tilde{\Omega})$ such that $u_k \to u$ uniformly on $\tilde{\Omega}$.*

(c) *For any subdomain $\tilde{\Omega} \subset\subset \Omega$, there exist a family of h-convex functions $\{u_k\} \subset \Gamma^2(\tilde{\Omega})$ such that $u_k \to u$ uniformly on $\tilde{\Omega}$.*

The paper is written as follows. In §2, we outline the sup-convolution construction. In §3, we prove Theorem A.

§2. The construction of sup-convolutions on $\mathbf{G}$

In this section, we outline the construction of sup-convolutions on $\mathbf{G}$, developed by Wang [W] on Carnot groups and by [JLS] on Euclidean spaces, to show the v-convexity of the sup-convolution of any v-convex function.

Let $\Omega \subset \mathbf{G}$ be a bounded domain and $d_{\mathbf{G}}(\cdot,\cdot)$ be the smooth gauge pseudo-distance defined by (1.3). For any $\epsilon > 0$, define

$$\Omega_\epsilon = \{x \in \Omega : \inf_{y \in \mathbf{G}\setminus\Omega} d_{\mathbf{G}}(x^{-1}, y^{-1})^{2r!} \geq \epsilon\}.$$

Definition 2.1. For $\epsilon > 0$ and $u : \bar{\Omega} \to R$ an upper semicontinous and bounded from below function, the sup-convolution u_ϵ of u is defined by

$$u^\epsilon(x) = \sup_{y \in \bar{\Omega}} (u(y) - \frac{1}{2\epsilon} d_{\mathbf{G}}(x^{-1}, y^{-1})^{2r!}),\ \forall x \in \Omega. \tag{2.1}$$

For $p \in \mathbf{G}$, denote by $\|p\|_E := (\sum_{i=1}^m |\xi_i(p)|^2)^{\frac{1}{2}}$ the Euclidean norm of p. We recall

Definition 2.2. An upper semicontinuous function $u : \bar{\Omega} \to R$ is called semiconvex if there is a constant $C > 0$ such that $u(p) + C\|p\|_E^2 : \bar{\Omega} \to R$ is convex in the Euclidean sense, e.g. if $u \in C^2(\bar{\Omega})$ and $\nabla^2 u(p) + CI_n$ is positive semidefinite for $p \in \Omega$, then u is semiconvex.

Now we have the following generalized version of [JLS], which can be found in [W] (§3, Proposition 3.3).

Proposition 2.3. *Suppose that $u : \bar{\Omega} \to R$ is upper semicontinuous, and let $R_0 = \|u\|_{L^\infty(\Omega)} > 0$. Then $u_\epsilon > 0$ satisfies:*

(1) *u^ϵ is locally Lipschitz continuous in Ω with respect to $d_{\mathbf{G}}$ and is semiconvex;*
(2) *$\{u^\epsilon\}$ is monotonically nondecreasing w.r.t. ϵ and converges to u in Ω;*
(3) *if u is a v-convex function in Ω, then u^ϵ is a v-convex function in $\Omega_{(2R_0+1)\epsilon}$;*
(4) *if, in addition, $u \in C(\bar{\Omega})$, then $u_\epsilon \to u$ uniformly on Ω.*

Proof. For $\Omega \subset \mathbf{G}$ bounded, the formula (1.3) for $d_{\mathbf{G}}$ implies that

$$C(\Omega, d_{\mathbf{G}}) \equiv \|\nabla_x^2 (d_{\mathbf{G}}(x^{-1}, y^{-1})^{2r!})\|_{L^\infty(\bar{\Omega}\times\bar{\Omega})} < \infty.$$

Therefore, for any $y \in \bar{\Omega}$, the full hessian of

$$\tilde{u^\epsilon}(x,y) := u(y) - \frac{1}{2\epsilon} d_{\mathbf{G}}(x^{-1}, y^{-1})^{2r!} + \frac{C(\Omega, d_{\mathbf{G}})}{2\epsilon}\|x\|_E^2, \ \forall x \in \Omega,$$

is positive semidefinite so that $\tilde{u^\epsilon}$ is convex. Note that the supremum for a family of convex functions is convex. Hence we have that

$$\sup_{y\in\bar{\Omega}} \tilde{u^\epsilon}(x,y) = u^\epsilon(x) + \frac{C(\Omega, d_{\mathbf{G}})}{2\epsilon}\|x\|_E^2, \quad x \in \Omega,$$

is convex. Hence u^ϵ is semiconvex in Ω. It is well known that semiconvex functions are locally Lipschitz continuous with respect to the Euclidean metric. In particular, u^ϵ is locally Lipschitz continuous with respect to $d_{\mathbf{G}}$. This gives (1).

For any $\epsilon_1 < \epsilon_2$, it is easy to see that $u^{\epsilon_1}(x) \le u^{\epsilon_2}(x)$ for any $x \in \Omega$ so that $\{u^\epsilon\}$ is monotonically nondecreasing with respect to ϵ. For any $x \in \Omega$, there exists an $x_\epsilon \in \bar{\Omega}$ such that

$$u(x_\epsilon) - \frac{1}{2\epsilon} d_{\mathbf{G}}(x^{-1}, x_\epsilon^{-1})^{2r!} = u^\epsilon(x). \tag{2.2}$$

This implies that

$$\frac{1}{2\epsilon} d_{\mathbf{G}}(x^{-1}, x_\epsilon^{-1})^{2r!} \le u(x_\epsilon) - u^\epsilon(x). \tag{2.3}$$

Since

$$u(x) \le u^\epsilon(x) \le \sup_{y\in\bar{\Omega}} u(y), \ \forall x \in \Omega, \tag{2.4}$$

we have

$$\frac{1}{2\epsilon} d_{\mathbf{G}}(x^{-1}, x_\epsilon^{-1})^{2r!} \le 2R_0, \forall x \in \Omega. \tag{2.5}$$

Hence $x_\epsilon \to x$. Moreover, since

$$u_{\frac{\epsilon}{2}}(x) \ge u(x_\epsilon) - \frac{1}{\epsilon} d_{\mathbf{G}}(x^{-1}, x_\epsilon^{-1})^{2r!} = u_\epsilon(x) - \frac{1}{2\epsilon} d_{\mathbf{G}}(x^{-1}, x_\epsilon^{-1})^{2r!},$$

we have

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} d_{\mathbf{G}}(x^{-1}, x_\epsilon^{-1})^{2r!} = 0.$$

Therefore, the upper semicontinuity of u implies that $\lim_{\epsilon\downarrow 0} u^\epsilon(x) = u(x)$ for any $x \in \Omega$. This gives (2).

For (3), we first observe that for any $x^0 \in \Omega_{(1+2R_0)\epsilon}$ there exists an $x_\epsilon^0 \in \Omega$ such that

$$u_\epsilon(x^0) = u(x_\epsilon^0) - \frac{1}{2\epsilon} d_{\mathbf{G}}((x^0)^{-1}, (x_\epsilon^0)^{-1})^{2r!}.$$

Let $\phi \in C^2(\Omega)$ be such that

$$u_\epsilon(x^0) - \phi(x^0) \geq u_\epsilon(x) - \phi(x), \quad \forall x \in \Omega.$$

Then we have

$$u(x^0_\epsilon) - \frac{1}{2\epsilon} d_{\mathbf{G}}((x^0)^{-1}, (x^0_\epsilon)^{-1})^{2r!} - \phi(x^0) \geq u(y) - \frac{1}{2\epsilon} d_{\mathbf{G}}(x^{-1}, y^{-1})^{2r!} - \phi(x), \forall x, y \in \Omega.$$

For any y close to x^0_ϵ, note that $x = x^0 \cdot (x^0_\epsilon)^{-1} \cdot y \in \Omega$. By substituting this into the above inequality, we obtain

$$u(x^0_\epsilon) - \phi(x^0) \geq u(y) - \phi(x^0 \cdot (x^0_\epsilon)^{-1} \cdot y).$$

Set $\bar{\phi}(y) = \phi(x_0 \cdot (x^0_\epsilon)^{-1} \cdot y)$, for $y \in \Omega$ near y_0. Then $\bar{\phi}$ touches u at $y = x^0_\epsilon$ from above so that v-convexity of u implies

$$\nabla^2_h \bar{\phi}(x^0_\epsilon) \geq 0.$$

On the other hand, the left-invariance of $\{X_i\}_{i=1}^m$ implies

$$\nabla^2_h \bar{\phi}(y) = (\nabla^2_h \phi)(x^0 \cdot (x^0_\epsilon)^{-1} \cdot y).$$

Therefore we have

$$\nabla^2_h \phi(x^0) \geq 0.$$

This implies that u^ϵ is v-convex in $\Omega_{(2R_0+1)\epsilon}$.

For (4), since $u \in C(\bar{\Omega})$, we can see easily from (2.3)-(2.4) that

$$\begin{aligned} |u^\epsilon(x) - u(x)| &\leq |u(x_\epsilon) - u(x)| + \frac{1}{2\epsilon} d_{\mathbf{G}}(x^{-1}, x_\epsilon^{-1})^{2r!} \\ &\leq 2|u(x_\epsilon) - u(x)| \leq 2\omega(d_{\mathbf{G}}(x, x_\epsilon)), \end{aligned} \tag{2.6}$$

where ω is the modulus of continuity of u. On the other hand, (2.5) implies

$$d_{\mathbf{G}}(x_\epsilon, x) \leq (4\epsilon \|u\|_{C(\bar{\Omega})})^{\frac{1}{2r!}}. \tag{2.7}$$

Therefore, we have

$$\max_{x \in \Omega} |u^\epsilon(x) - u(x)| \leq 2\omega((4\epsilon \|u\|_{C(\bar{\Omega})})^{\frac{1}{2r!}}) \to 0, \quad \text{as } \epsilon \to 0. \tag{2.8}$$

The proof is complete. □

§3. Proof of Theorem A and Corollary B

Proof of Theorem A. It suffices to prove Theorem A for any bounded domain $\Omega \subset \mathbf{G}$. Moreover we assume that u is locally bounded from below since $u_c = \sup\{u, c\}$ is v-convex and locally bounded for any finite $c \in \mathbf{R}$. Therefore, if we can show that u_c is h-convex for any finite $c \in \mathbf{R}$, then it is easy to see that $u = \sup_{c \in \mathbf{R}} u_c$ is also h-convex.

For any $\epsilon > 0$, let $u^\epsilon : \Omega \to \mathbf{R}$ be the sup-convolution of u by Proposition 2.3. Then (1) and (3) of Proposition 2.3 imply that u^ϵ is both semiconvex in the Euclidean sense and v-convex in $\Omega_{(2R_0+1)\epsilon}$. It follows from the well-known theorem on convex functions in the Euclidean space (cf. [EG]) that u^ϵ is twice differentiable in the Euclidean sense for a.e. $x \in \Omega_{(2R_0+1)\epsilon}$ and $\nabla^2 u^\epsilon \in L^1(\Omega_{(2R_0+1)\epsilon})$. In particular, the horizontal hessian $\nabla^2_h u^\epsilon(x)$ exists for a.e. $x \in \Omega_{(2R_0+1)\epsilon}$ and $\nabla^2_h u^\epsilon \in L^1(\Omega_{(2R_0+1)\epsilon})$. This, combined with the v-convexity of u^ϵ on $\Omega_{(2R_0+1)\epsilon}$ and the standard theory on viscosity solutions (see [CIL]), implies that $\nabla_h^2 u^\epsilon(x) \geq 0$ is positive semidefinite for a.e. $x \in \Omega_{(2R_0+1)\epsilon}$, i.e.

$$\sum_{i,j=1}^{m} \eta_i \eta_j X_i X_j u^\epsilon(x) \geq 0, \ \forall \eta \in R^m. \tag{3.1}$$

Let $\phi \in C_0^\infty(\mathbf{G})$ be nonnegative such that $\mathrm{supp}(\phi) \subset B_1(0)$ and $\int_{\mathbf{G}} \phi(p)\, dp = 1$. For any small $\delta > 0$, consider the mollification $u^\epsilon_\delta = \phi_\delta * u^\epsilon$ of u^ϵ defined by

$$u^\epsilon_\delta = \phi_\delta * u^\epsilon(q) = \int_{\mathbf{G}} u^\epsilon(p^{-1} \cdot q)\phi_\delta(p)\, dp, \ \forall q \in \Omega_{(2R_0+1)\epsilon+\delta},$$

where $\phi_t(p) = t^{-Q}\phi(\delta_t(p))$ for $t > 0$ and Q is the homogeneous dimension of $\mathbf{G}$. Since

$$\nabla_h^2 u^\epsilon_\delta(p) = \int_{\mathbf{G}} (\nabla_h^2 u^\epsilon)(q^{-1} \cdot p)\phi_\delta(q)\, dq, \quad \forall p \in \Omega_{(2R_0+1)\epsilon+\delta},$$

and $\nabla_h^2 u^\epsilon \in L^1(\Omega_{(2R_0+1)\epsilon})$ is positive semidefinite, we have $u^\epsilon_\delta \in C^\infty(\Omega_{(2R_0+1)\epsilon+\delta})$ and $\nabla_h^2 u^\epsilon_\delta$ is positive semidefinite everywhere in $\Omega_{(2R_0+1)\epsilon+\delta}$. Therefore, [DGN] (§5, Theorem 5.11) or [LMS] (§4, Proposition 4.1) implies that u^ϵ_δ is h-convex on $\Omega_{(2R_0+1)\epsilon+\delta}$, i.e. for any $0 \leq \lambda \leq 1$,

$$u^\epsilon_\delta(p_\lambda) \leq u^\epsilon_\delta(p) + (1-\lambda)(u^\epsilon_\delta(p') - u^\epsilon_\delta(p)), \ \forall p \in \Omega_{(2R_0+1)\epsilon+\delta}, \ p' \in H_0(\mathbf{G}) \cap \Omega_{(2R_0+1)\epsilon+\delta}. \tag{3.2}$$

Since $u^\epsilon_\delta \to u^\epsilon$ uniformly on $\Omega_{(2R_0+1)\epsilon}$ as $\delta \downarrow 0$, it follows that, by taking δ to zero in (3.2), u^ϵ is h-convex on $\Omega_{(2R_0+1)\epsilon}$. Since (2) of Proposition 2.3 implies $u^\epsilon \to u$ on Ω as $\epsilon \downarrow 0$, u is h-convex on Ω. $\square$

Proof of Corollary B. It is clear that (b) and (c) are equivalent (see, e.g., [DGN], [LMS]). It is also easy to see that (b) implies (a). To see that (a) implies (b), let $\tilde{\Omega} \subset\subset \Omega$ be fixed and u^ϵ be the sup-convolution of u on Ω given by Proposition 2.3, and u^ϵ_δ be the δ-mollifer of u^ϵ on Ω constructed in the proof of Theorem A, where $\epsilon > 0, \delta > 0$ are sufficiently small. Then (4) of Proposition 2.3 implies $u^\epsilon \to u$ uniformly on Ω. Moreover, the proof of Theorem A implies that $u^\epsilon_\delta \in \Gamma^2(\tilde{\Omega})$ is v-convex and $u^\epsilon_\delta \to u^\epsilon$ uniformly on $\tilde{\Omega}$. Therefore, by the Cauchy diagonal process, we may assume that $u^\epsilon_\delta \to u$ uniformly on $\tilde{\Omega}$ for $\epsilon \to 0$ and $\delta = \delta(\epsilon) \to 0$. This finishes the proof. $\square$

References

[B] T. Bieske, *On ∞-harmonic functions on the Heisenberg group.* Comm. Partial Differential Equations 27 (2002), no. 3-4, 727–761. MR1900561 (2003g:35033)

[BR] Z. Balogh, M. Rickly, *Regularity of convex functions on Heisenberg groups.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) 2 (2003), no. 4, 847–868. MR2040646

[CC] X. Cabré, L. Caffarelli, Fully nonlinear elliptic equations. AMS Colloquium Publications 43, AMS, Providence, RI, 1995. MR1351007 (96h:35046)

[CIL] M. Crandall, H. Ishii, P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations.* Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1–67. MR1118699 (92j:35050)

[DGN] D. Danielli, N. Garofalo, D. Nhieu, *Notations of convexity in Carnot groups.* Comm. Anal. Geom. 11 (2003), no. 2, 263–341. MR2014879 (2004m:22014)

[EG] L. Evans, R. Gariepy, Measure theory and fine properties of functions. CRC Press, 1992. MR1158660 (93f:28001)

[FS] G. Folland, E. Stein, Hardy spaces on homogeneous groups. Mathematical Notes, 28. Princeton University Press, Princeton, N.J., 1982. MR0657581 (84h:43027)

[JLS] R. Jensen, P. L. Lions, P. Souganidis, *A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations*. Proc. Amer. Math. Soc. 102 (1988), no. 4, 975–978. MR0934877 (89f:35027)

[JLMS] P. Juutinen, G. Lu, J. Manfredi, B. Stroffolini. In preparation.

[LMS] G. Lu, J. Manfredi, B. Stroffolini, *Convex functions on the Heisenberg group*. Calc. Var., 19 (2004), no. 1, 1-22. MR2027845

[W] C. Y. Wang, *The Aronsson equation for absolute minimizers of L^∞-functionals associated with vector fields satisfying Hörmander's condition*. To appear in Trans. Amer. Math. Soc.

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506