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PROPERLY 3-REALIZABLE GROUPS

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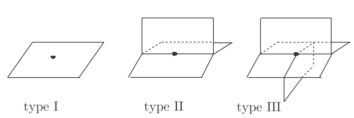
ABSTRACT. A finitely presented group G is said to be properly 3-realizable if there exists a compact 2-polyhedron K with $\pi_1(K) \cong G$ and whose universal cover $\tilde{K}$ has the proper homotopy type of a (p.l.) 3-manifold with boundary. In this paper we show that, after taking wedge with a 2-sphere, this property does not depend on the choice of the compact 2-polyhedron K with $\pi_1(K) \cong G$. We also show that (i) all 0-ended and 2-ended groups are properly 3-realizable, and (ii) the class of properly 3-realizable groups is closed under amalgamated free products (HNN-extensions) over a finite cyclic group (as a step towards proving that ∞-ended groups are properly 3-realizable, assuming 1-ended groups are).

1. INTRODUCTION

The following question was considered in [8] for an arbitrary finitely presented group G: *does there exist a compact* 2*-polyhedron* K *with* $\pi_1(K) \cong G$ *and whose universal cover* $\tilde{K}$ *is proper homotopy equivalent to a* 3*-manifold* M *(with boundary)?* If so, the group G is said to be *properly* 3*-realizable*. According to [2], the proper homotopy type of any 2-dimensional locally finite CW-complex can be represented by a closed subpolyhedron in $\mathbf{R}^4$, whence one can easily check that any finitely presented group would then be properly 4-realizable, by taking regular neighborhoods in $\mathbf{R}^4$. The question of whether or not every finitely presented group G is properly 3-realizable still remains open. In the case of a positive answer, this property would allow us to use duality arguments in the study of certain low-dimensional ((co)homological) proper invariants of the group G (see [8]).

There are several results in the literature regarding this property when we restrict to a particular class of 2-polyhedra. In this respect, we recall that a locally compact 2-polyhedron X is called a (closed) *fake surface* if each point in X has a neighborhood of one of the following types:



It is worth noting that the homotopy type of any compact 2-polyhedron can be represented by a fake surface [14]. In [8] it is shown that in the absence of points

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of type III, the corresponding group $G \cong \pi_1(X)$ is properly 3-realizable, and we denote by $\mathcal{C}$ the class of those finitely presented groups obtained in this way. See also [9] for an extension of this result. On the other hand, in [3] it is observed that the class of groups $\mathcal{C}$ is closed under free product, but not under amalgamated free products in general. It is also shown in [3] that there are properly 3-realizable (non-3-manifold) groups that are not in $\mathcal{C}$. The main results of this paper are:

Theorem 1.1. *Let $H \leq G$ be finitely presented groups with $[G : H] < \infty$. Then G is properly 3-realizable if and only if H is properly 3-realizable.*

For a finitely presented group G one can define the *number of ends of G* as the number of ends of the universal cover $\tilde{X}$ of any compact 2-polyhedron X with $\pi_1(X) \cong G$, and this number equals $1 + rank(H^1(G; \mathbf{Z}G)) = 0, 1, 2$ *or* ∞, this cohomology group being free abelian [5, 6]. The 0-ended groups are the finite groups and the 2-ended groups are those having an infinite cyclic subgroup of finite index. Moreover, Stallings' Structure Theorem characterizes those groups G with more than one end as those that split as an amalgamated free product (or an HNN-extension) over a finite group (see [11, 6]). Dunwoody [4] showed that this process of factorizing G must be finite, i.e., every finitely presented group is the fundamental group of a finite graph of groups in which each edge group is finite and each vertex group has at most one end.

Corollary 1.2. *If G has a free subgroup of finite index, then G is properly 3-realizable. In particular, this applies to all 0-ended and 2-ended groups, as well as to certain ∞-ended groups.*

Proposition 1.3. *Let G be a properly 3-realizable group and X be any compact 2-polyhedron with $\pi_1(X) \cong G$. Then, the universal cover of $X \vee S^2$ has the proper homotopy type of a 3-manifold. Moreover, if G is as in Corollary 1.2, then we may disregard the 2-sphere S^2.*

Finally, assuming one could show all 1-ended groups are properly 3-realizable, one step towards proving that ∞-ended groups are also properly 3-realizable, in view of Stallings' and Dunwoody's results, would be the following (cf. [3], Thm. 1.2).

Theorem 1.4. *Let G_0, G_1 be properly 3-realizable groups and F be a finite cyclic group. Then, any amalgamated free product $G_0 *_F G_1$ (HNN-extension $G_k *_F$) is also properly 3-realizable.*

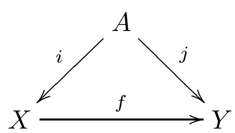
This result generalizes to show that the fundamental group of a finite graph of groups with properly 3-realizable vertex groups and finite cyclic edge groups is properly 3-realizable, since such a group can be expressed as a combination of amalgamated products and HNN-extensions of the vertex groups over the edge groups.

2. Some basic preliminaries

We start with some results from (proper) homotopy theory, which will be used later. Recall that a map $i : A \longrightarrow X$ between topological spaces is said to be a *cofibration* if it satisfies the Homotopy Extension Property, i.e., for any space Y and any two maps $g : X \longrightarrow Y$, $H : A \times I \longrightarrow Y$ with $H|_{A\times\{0\}} = g \circ i$, there is a map $G : X \times I \longrightarrow Y$ with $G|_{X\times\{0\}} = g$ and $H = G \circ (i \times \{1\})$. A proper cofibration is a cofibration in the proper category, i.e., the category of locally compact Hausdorff spaces and proper maps.

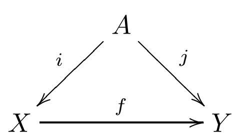
Examples. It is well known that every homeomorphism is a cofibration, and composition of cofibrations is also a cofibration. Moreover, if X is a locally compact polyhedron, then for every closed subpolyhedron $A \subset X$, the inclusion map $i : A \longrightarrow X$ is a proper cofibration.

A space *under* A is a map $i : A \longrightarrow X$, and a map of spaces under A is a commutative diagram



A homotopy H between maps f, f' under A is a homotopy that is a map under A at each time $t \in I$, and we have $H(i(a), t) = j(a)$, for all $a \in A, t \in I$. This leads to a notion of homotopy equivalence under A.

Proposition 2.1. *Consider the following commutative diagram in the proper category:*



where f is a proper homotopy equivalence and i, j are proper cofibrations. Then, f is a proper homotopy equivalence under A.

See ([1], Prop. 4.16) for a proof of this proposition (compare with ([10], Chap. 6, §5)).

We now turn to homology of infinite CW-complexes. Let R be a ring, and let X be an oriented locally finite CW-complex. Let $R(e)$ be the free left R-module generated by the cell e in X, and let $C_n^\infty(X;R) = \prod_{dim(e)=n} R(e)$. Elements in $C_n^\infty(X;R)$ will be denoted by infinite sums and will be called *infinite cellular n-chains with coefficients in R*. Note that the R-module of ordinary cellular n-chains in X, $C_n(X;R)$, is a submodule of $C_n^\infty(X;R)$. Since X is locally finite, the ordinary boundary homomorphism $\partial : C_n(X;R) \longrightarrow C_{n-1}(X;R)$ extends to a boundary homomorphism $\partial : C_n^\infty(X;R) \longrightarrow C_{n-1}^\infty(X;R)$. This way we have a chain complex $(C_*^\infty(X;R), \partial)$ whose homology modules $H_*^\infty(X;R)$ are called the *cellular homology modules of X based on infinite chains* [6]. The short exact sequence of chain complexes

$$0 \longrightarrow C_*(X;R) \longrightarrow C_*^\infty(X;R) \longrightarrow C_*^\infty(X;R)/C_*(X;R) \longrightarrow 0$$

induces a long exact homology sequence

$$\cdots \longrightarrow H_{q+1}(X;R) \longrightarrow H_{q+1}^\infty(X;R) \longrightarrow H_q^e(X;R) \longrightarrow H_q(X;R) \longrightarrow \cdots$$

where $H_q^e(X;R) = H_{q+1}\left(C_*^\infty(X;R)/C_*(X;R)\right)$ is known as the *q^{th}-homology module of ends of X*.

Theorem 2.2. *Let X be as above and $\{K_i\}_{i\geq 1}$ be an exhaustive sequence of X consisting of finite subcomplexes $K_i \subset X$. Then, we have the following short exact sequences ($q \geq 0$):*

$$0 \longrightarrow \varprojlim{}^1 H_{q+1}(X, X - K_i; R) \longrightarrow H_q^{\infty}(X; R) \longrightarrow \varprojlim H_q(X, X - K_i; R) \longrightarrow 0,$$

$$0 \longrightarrow \varprojlim{}^1 H_{q+1}(X - K_i; R) \longrightarrow H_q^{e}(X; R) \longrightarrow \varprojlim H_q(X - K_i; R) \longrightarrow 0.$$

See [6] or ([7], 4.11) for a proof of this result.

3. Properly 3-realizable groups

The purpose of this section is to prove the main results of this paper. For this we will need some previous results.

Proposition 3.1. *Let M be a polyhedral n-manifold of the same proper homotopy type of a locally compact polyhedron K with $dim(K) < dim(M)$. Then, any Freudenthal end $\epsilon \in \mathcal{F}(M)$ can be represented by a sequence of points in ∂M.*

Proof. Let $\epsilon \in \mathcal{F}(M)$ and $\{M_i\}_{i\geq 1}$ be an exhaustive sequence of M consisting of compact submanifolds. Suppose $\epsilon \in \mathcal{F}(M)$ cannot be represented by a sequence of points in ∂M. Then, there exist an integer $k_0 \geq 1$ and a component $C_{i(k_0)} \subseteq cl(M - M_{k_0})$ containing ϵ such that $C_{i(k_0)} \cap \partial M = \emptyset$. Therefore, for all $k \geq k_0$, the set of components $\mathcal{C}_k = \{C \subseteq C_{i(k_0)} \cap cl(M - M_k)\}$ misses ∂M. Observe that each boundary ∂C is compact and $\partial(C_{i(k_0)} \cap cl(M - M_k)) = \bigsqcup_{C\in\mathcal{C}_k} \partial C$. Thus, one can exhibit a non-trivial element $(x_k)_{k\geq k_0} \in \varprojlim H_{n-1}(cl(M - M_k); \mathbf{Z}_2)$ (and hence a non-trivial element in the $(n-1)^{st}$-homology group of ends $H^e_{n-1}(M; \mathbf{Z}_2)$; see §2) by setting $x_k = \sum_{C\in\mathcal{C}_k} [z_C]$, where z_C is the $(n-1)$-cycle in $C_{i(k_0)} \cap cl(M - M_k)$ represented by ∂C, for all $C \in \mathcal{C}_k$. Indeed, for each $C \in \mathcal{C}_k$, we have the equation

$$[z_C] = \sum_{C'\subseteq C, C'\in\mathcal{C}_{k+1}} [z_{C'}] \in H_{n-1}(cl(M - M_k); \mathbf{Z}_2)$$

determined by the difference $cl(C - (M - M_{k+1}))$.

On the other hand, the proper homotopy equivalence between M and K yields an isomorphism $0 \neq H^e_{n-1}(M; \mathbf{Z}_2) \cong H^e_{n-1}(K; \mathbf{Z}_2)$, which is a contradiction, since $dim(K) \leq n - 1$ (see §2). □

Lemma 3.2. *Let K, L be two compact 2-polyhedra with the property that the universal cover of each of them is proper homotopy equivalent to a 3-manifold. Then, the universal cover of the wedge $K \vee L$ has the same property.*

Proof. Suppose K and L are compact 2-polyhedra so that their universal covers $\tilde{K}$, $\tilde{L}$ are now proper homotopy equivalent to 3-manifolds M and N, respectively. Let $I = [0, 1] \subset \mathbf{R}$, and let P be the 2-polyhedron obtained as a quotient from the disjoint union $K \sqcup L \sqcup I$ by identifying $0 \in I$ with the base point $x_0 \in K$ and $1 \in I$ with the base point $y_0 \in L$. Note that P has the same homotopy type as the wedge $K \vee L$, and hence the universal cover $\tilde{P}$ of P has the same proper homotopy type as the universal cover of $K \vee L$ (see [6]).

Fix proper homotopy equivalences $g : \tilde{K} \longrightarrow M$ and $h : \tilde{L} \longrightarrow N$. We show that one can obtain new proper homotopy equivalences $\tilde{K} \longrightarrow M, \tilde{L} \longrightarrow N$ from g and h

so as to produce a proper homotopy equivalence between the universal cover $\tilde{P}$ of P and a 3-manifold constructed from copies of M and N. Next, we roughly describe what the universal cover $\tilde{P}$ of P looks like (see [11], §3). See also [6] for a combinatorial description of the universal cover of a CW-complex. Fix points $\tilde{x}_0, \tilde{y}_0 \in \tilde{P}$ in the fibre of $x_0, y_0 \in P$ respectively, and let $\Gamma = \pi_1(K, x_0), \overline{\Gamma} = \pi_1(L, y_0)$. The universal cover $\tilde{P}$ of P comes together with the following data:

(a) a disjoint union $\bigsqcup_{p\in\mathbf{N}} \tilde{K}_p$ of copies of $\tilde{K}$ together with a locally finite sequence of points $\{\gamma^p_q \cdot \tilde{x}_0\}_{q\in\mathbf{N}} \subset \tilde{K}_p$ ($\gamma^p_q \in \Gamma * \overline{\Gamma}$), for each $p \in \mathbf{N}$;

(b) a disjoint union $\bigsqcup_{r\in\mathbf{N}} \tilde{L}_r$ of copies of $\tilde{L}$ together with a locally finite sequence of points $\{\overline{\gamma}^r_s \cdot \tilde{y}_0\}_{s\in\mathbf{N}} \subset \tilde{L}_r$ ($\overline{\gamma}^r_s \in \Gamma * \overline{\Gamma}$), for each $r \in \mathbf{N}$;

(c) a disjoint union $\bigsqcup_{p,q\in\mathbf{N}} I_{p,q}$ of copies of the unit interval I; and

(d) a bijective function $\varphi : \mathbf{N}\times\mathbf{N} \longrightarrow \mathbf{N}\times\mathbf{N}, (p,q) \mapsto (r,s)$, given by the free group action of $\Gamma * \overline{\Gamma}$ on $\tilde{P}$, in such a way that $0 \in I_{p,q}$ is being identified with $\gamma^p_q \cdot \tilde{x}_0 \in \tilde{K}_p$ and $1 \in I_{p,q}$ is being identified with $\overline{\gamma}^r_s \cdot \tilde{y}_0 \in \tilde{L}_r$, for each $p, q \in \mathbf{N}$. Notice that for each $p \in \mathbf{N}$, the composition with the first projection $q \mapsto \pi_1 \circ \varphi(p,q) \in \mathbf{N}$ is injective, for $\tilde{P}$ is simply connected.

Given this data, set $A = \mathbf{N}\times\mathbf{N}$ and consider maps $i : A \longrightarrow \bigsqcup_{p\in\mathbf{N}} \tilde{K}_p \subset \tilde{P}, i' : A \longrightarrow \bigsqcup_{r\in\mathbf{N}} \tilde{L}_r \subset \tilde{P}$ given by $i(p,q) = \gamma^p_q \cdot \tilde{x}_0$ and $i'(p,q) = \overline{\gamma}^r_s \cdot \tilde{y}_0$, where $(r,s) = \varphi(p,q)$. It is easy to check that i and i' are proper cofibrations; see §2. Next, we take exhaustive sequences $\{C^p_m\}_{m\in\mathbf{N}}$ and $\{D^r_n\}_{n\in\mathbf{N}}$ of each copy M_p and N_r of the 3-manifolds M and N respectively by compact submanifolds $C^p_m \subset M_p$, $D^r_n \subset N_r$, and define proper cofibrations $j : A \longrightarrow \bigsqcup_{p\in\mathbf{N}} M_p$, $j' : A \longrightarrow \bigsqcup_{r\in\mathbf{N}} N_r$ as follows. Given $(p,q) \in A$ and the proper homotopy equivalences $g_p = g : \tilde{K}_p \longrightarrow M_p$, $h_r = h : \tilde{L}_r \longrightarrow N_r$ (with $(r,s) = \varphi(p,q)$), we take $m(q), n(s) \in \mathbf{N}$ to be the least natural numbers such that $g_p \circ i(p,q) \notin C^p_{m(q)} \subset M_p$ and $h_r \circ i'(p,q) \notin D^r_{n(s)} \subset N_r$. Then, we make use of Proposition 3.1 and define $j(p,q)$ and $j'(p,q)$ to be points $j(p,q) = a_{p,q} \in \partial M_p - C^p_{m(q)}$ and $j'(p.q) = b_{r,s} \in \partial N_r - D^r_{n(s)}$ so that (i) j, j' are one-to-one maps (note that g, h need not be one-to-one); and (ii) $a_{p,q}$ and $g_p \circ i(p,q)$ (resp. $b_{r,s}$ and $h_r \circ i'(p,q)$) are in the same path component of $M_p - C^p_{m(q)}$ (resp. $N_r - D^r_{n(s)}$). Notice that j and j' are proper maps by construction. Consider now maps

$$G : \left(\bigsqcup_{p\in\mathbf{N}} \tilde{K}_p\right) \times \{0\} \cup (i(A)\times I) \longrightarrow \bigsqcup_{p\in\mathbf{N}} M_p,$$

$$H : \left(\bigsqcup_{r\in\mathbf{N}} \tilde{L}_r\right) \times \{0\} \cup (i'(A)\times I) \longrightarrow \bigsqcup_{r\in\mathbf{N}} N_r$$

with $G|_{\tilde{K}_p \times \{0\}} = g_p = g$ and $H|_{\tilde{L}_r \times \{0\}} = h_r = h$ $(p, r \in \mathbf{N})$, and so that $\alpha_{p,q} = G|_{i(p,q)\times I}$ (resp. $\beta_{r,s} = H|_{i'(p,q)\times I}$) is a path in $M_p - C^p_{m(q)}$ from $g_p \circ i(p,q)$ to $a_{p,q}$ (resp. a path in $N_r - D^r_{n(s)}$ from $h_r \circ i'(p,q)$ to $b_{r,s}$). Observe that G and H are proper maps, since g, h, j and j' are proper. By the Homotopy Extension Property, the maps G, H extend to proper maps

$$\widehat{G} : \left(\bigsqcup_{p\in\mathbf{N}} \tilde{K}_p \right) \times I \longrightarrow \bigsqcup_{p\in\mathbf{N}} M_p \ , \ \widehat{H} : \left(\bigsqcup_{r\in\mathbf{N}} \tilde{L}_r \right) \times I \longrightarrow \bigsqcup_{r\in\mathbf{N}} N_r$$

which yield commutative diagrams

$$\begin{array}{ccc} & A & \\ {\scriptstyle i}\swarrow & & \searrow{\scriptstyle j} \\ \bigsqcup_{p\in\mathbf{N}} \tilde{K}_p & \xrightarrow{\ \hat{g}\ } & \bigsqcup_{p\in\mathbf{N}} M_p \end{array} \qquad \begin{array}{ccc} & A & \\ {\scriptstyle i'}\swarrow & & \searrow{\scriptstyle j'} \\ \bigsqcup_{r\in\mathbf{N}} \tilde{L}_r & \xrightarrow{\ \hat{h}\ } & \bigsqcup_{r\in\mathbf{N}} N_r \end{array}$$

where $\hat{g} = \widehat{G}|_{(\bigsqcup_{p\in\mathbf{N}} \tilde{K}_p)\times\{1\}}$ and $\hat{h} = \widehat{H}|_{(\bigsqcup_{r\in\mathbf{N}} \tilde{L}_r)\times\{1\}}$ are proper homotopy equivalences. Moreover, $\hat{g}$ and $\hat{h}$ are proper homotopy equivalences under A, by Proposition 2.1. Hence, they induce a proper homotopy equivalence between the universal cover $\tilde{P}$ of P and the quotient space obtained from the following data:

(a) the disjoint unions $\bigsqcup_{p\in\mathbf{N}} M_p$ and $\bigsqcup_{r\in\mathbf{N}} N_r$ of copies of the 3-manifolds M and N respectively;

(b) a disjoint union $\bigsqcup_{p,q\in\mathbf{N}} I_{p,q}$ of copies of the unit interval; and

(c) the bijective function $\varphi : A \longrightarrow A, (p,q) \mapsto (r,s)$, so that $0 \in I_{p,q}$ is being identified with $a_{p,q} = j(p,q) \in \partial M_p$ and $1 \in I_{p,q}$ is being identified with $b_{r,s} = j'(p,q) \in \partial N_r$, for each $(p,q) \in A$.

Finally, in order to obtain the desired 3-manifold with the same proper homotopy type as $\tilde{P}$, we attach three-dimensional 1-handles $H_{p,q} \equiv B^2 \times B^1$ $(p,q \in \mathbf{N})$ to this quotient space whose cores run along each $I_{p,q}$ and so that $B^2 \times \{-1\}, B^2 \times \{1\} \subset H_{p,q}$ get identified homeomorphically with disks $D \subset \partial M_p, D' \subset \partial N_r$ about the points $a_{p,q}$ and $b_{r,s}$ respectively, where $(r,s) = \varphi(p,q)$. □

Proof of Theorem 1.1. First, suppose G is properly 3-realizable, and let X be a compact 2-polyhedron with $\pi_1(X) \cong G$ and whose universal cover $\tilde{X}$ has the proper homotopy type of a 3-manifold. Since $[G : H] < \infty$, the intermediate covering space $\tilde{X}_H$ of X corresponding to the subgroup H is a compact 2-polyhedron with $\pi_1(\tilde{X}_H) \cong H$ and whose universal cover is $\tilde{X}$, whence H is properly 3-realizable.

Next, we concentrate on the "sufficient" part. Suppose H is properly 3-realizable with $H \leq G$ and $[G : H] < \infty$, and let X be a compact 2-polyhedron with $\pi_1(X) \cong G$. We show that there exists a finite bouquet of spheres $\bigvee_{i\in I} S^2_i$ such that the universal cover of $X \vee (\bigvee_{i\in I} S^2_i)$ has the proper homotopy type of a 3-manifold. As H is properly 3-realizable, there is a compact 2-polyhedron Y with $\pi_1(Y) \cong H$ and whose universal cover $\tilde{Y}$ has the proper homotopy type of a 3-manifold. Also, since $[G : H] < \infty$, the intermediate covering space $\tilde{X}_H$ of X is a compact 2-polyhedron with $\pi_1(\tilde{X}_H) \cong H$. Therefore, by ([13], Thm. 14), there exist finite bouquets of spheres $\bigvee_{i\in I} S_i^2$ and $\bigvee_{j\in J} S_j^2$ such that $\tilde{X}_H \vee (\bigvee_{i\in I} S_i^2)$ and $Y \vee (\bigvee_{j\in J} S_j^2)$ are homotopy equivalent. Assume the bouquet $\bigvee_{i\in I} S_i^2$ is being attached to $\tilde{X}_H$ through a vertex $\tilde{v} \in \tilde{X}_H$, and let $p : \tilde{X}_H \longrightarrow X$ be the corresponding (finite-sheeted) cellular covering projection. Let $\tilde{Z}_H$ be the compact 2-polyhedron obtained from $\tilde{X}_H$ by attaching a copy of $\bigvee_{i\in I} S_i^2$ through every vertex in $p^{-1}(p(\tilde{v})) \subset \tilde{X}_H$. Then, $p : \tilde{X}_H \longrightarrow X$ extends to a cellular covering projection $\tilde{Z}_H \longrightarrow Z = X \vee (\bigvee_{i\in I} S_i^2)$ with $\pi_1(Z) \cong G$ and $\tilde{Z}_H$ being the covering space of Z corresponding to the subgroup H. One can readily check that there exists a finite bouquet of spheres $\bigvee_{k\in K} S_k^2$ such that $\tilde{Z}_H$ and $Y \vee (\bigvee_{k\in K} S_k^2)$ are homotopy equivalent. Therefore, the universal cover of $\tilde{Z}_H$ (which is in turn the universal cover of Z) is proper homotopy equivalent to the universal cover of $Y \vee (\bigvee_{k\in K} S_k^2)$, the latter being proper homotopy equivalent to a 3-manifold, by Lemma 3.2. □

Remark 3.3. Note that if a finitely presented group G has a finitely generated free subgroup H with $[G : H] < \infty$, then G is indeed properly 3-realizable by Theorem 1.1, since finitely generated free groups are easily proved to be properly 3-realizable. In particular, this applies to all 0-ended and 2-ended groups. Observe that if $rank(H) \geq 2$, then G is ∞-ended. Corollary 1.2 follows then from this observation. Moreover, given any compact 2-polyhedron X with $\pi_1(X) \cong G$, the covering space $\tilde{X}_H$ of X corresponding to the subgroup H is a compact 2-polyhedron with free fundamental group H. Thus, by ([12], Prop. 3.3), $\tilde{X}_H$ is in fact homotopy equivalent to a finite bouquet $(\bigvee_{i\in I} S_i^1) \vee (\bigvee_{j\in J} S_j^2)$, and hence the universal cover of $\tilde{X}_H$ (and hence of X) is proper homotopy equivalent to a 3-manifold.

Proof of Proposition 1.3. Let G be a properly 3-realizable group and Y be a compact 2-polyhedron with $\pi_1(Y) \cong G$ and whose universal cover is proper homotopy equivalent to a 3-manifold. Let X be any other compact 2-polyhedron with $\pi_1(X) \cong G$. Again, by ([13], Thm. 14), there exist finite bouquets of spheres $\bigvee_{i\in I} S^2$ and $\bigvee_{j\in J} S^2$ such that $X \vee (\bigvee_{i\in I} S^2)$ and $Y \vee (\bigvee_{j\in J} S^2)$ are homotopy equivalent, and hence their universal covers are proper homotopy equivalent. Assume the bouquet $\bigvee_{i\in I} S^2$ is being attached to X through a vertex $v \in X$, and let $p : \tilde{X} \longrightarrow X$ be the universal covering projection. Let $T \subset \tilde{X}$ be a tree containing all the vertices in $p^{-1}(v)$. Finally, according to the classification of spherical objects under T described in ([1], Prop. 4.5(b)) and the corresponding Gluing Lemma in proper homotopy theory (see [1]), the universal cover of $X \vee (\bigvee_{i\in I} S^2)$ is proper homotopy equivalent to the space obtained from $\tilde{X}$ by attaching only one sphere S^2 at a time through every vertex in $p^{-1}(v)$. The latter space is the universal cover of $X \vee S^2$, and the conclusion follows. The second part of the statement follows from Remark 3.3 above. □

Proof of Theorem 1.4. Let G_0, G_1 be properly 3-realizable groups and F be a finite cyclic group, say $F = \langle t; t^n \rangle$. Consider monomorphisms $\varphi_i : F \longrightarrow G_i$ $(i = 0, 1)$, and denote by $G_0 *_F G_1 = \langle G_0, G_1; \varphi_0(a) = \varphi_1(a), a \in F\rangle$ the corresponding amalgamated free product. Notice that if $F = \{1\}$, then the conclusion follows from Lemma 3.2. We will argue similarly in the general case. Let X_0, X_1 be compact 2-polyhedra with $\pi_1(X_i) \cong G_i$ and such that their universal covers have the proper homotopy type of 3-manifolds M_0, M_1 respectively. Let $f_i : S^1 \longrightarrow X_i$ $(i = 0, 1)$ be p.l. maps such that $Im\, f_{i_*} \subseteq \pi_1(X_i)$ corresponds to the subgroup $Im\, \varphi_i \subseteq G_i$.

We take the 2-polyhedron $Y' = D^2 \cup_h S^1$, with $h : \partial D^2 \longrightarrow S^1$ a map of degree n, and consider the adjunction spaces $Y = (S^1 \times I) \cup_{S^1 \times \{\frac{1}{2}\}} Y'$ (homotopy equivalent to Y') and $Z = Y \cup_{f_0 \times \{0\} \cup f_1 \times \{1\}} (X_0 \sqcup X_1)$. By Van Kampen's Theorem, Z is a compact 2-polyhedron with $\pi_1(Z) \cong G_0 *_F G_1$. Let $\tilde{Z}$ be the universal cover of Z with covering map $p : \tilde{Z} \longrightarrow Z$. Then, $p^{-1}(X_i)$ consists of a disjoint union of copies of the universal cover $\tilde{X}_i$ of X_i, since the inclusion $X_i \hookrightarrow Z$ induces a monomorphism $G_i \hookrightarrow G_0 *_F G_1$ between the fundamental groups, $i = 0, 1$. On the other hand, the universal cover $\tilde{Y}'$ of Y' consists of a collection of n disks attached along their boundary via a map of degree 1. Then, $p^{-1}(Y)$ consists of a disjoint union of copies of $K = (S^1 \times I) \cup_{S^1 \times \{\frac{1}{2}\}} \tilde{Y}'$. Observe that each component K of $p^{-1}(Y)$ contains a product $S^1 \times I$, and it thickens to an orientable 3-manifold $H \searrow K$ with $\partial(S^1 \times I) \times (-\epsilon, \epsilon) \subset \partial H$. Thus, $\tilde{Z}$ is constructed from $p^{-1}(Y)$ and a collection of copies of $\tilde{X}_0$ and $\tilde{X}_1$, by gluing each $S^1 \times \{i\} \subset p^{-1}(Y)$ to a copy of $\tilde{X}_i$ via liftings $\tilde{f}_i : S^1 \times \{i\} \longrightarrow \tilde{X}_i$ of the maps $f_i, i = 0, 1$. Next, for each copy of $\tilde{X}_i$ in $\tilde{Z}$ $(i = 0, 1)$, we take one of the maps $\tilde{f}_i : S^1 \times \{i\} \longrightarrow \tilde{X}_i$ and homotope it (within $\tilde{X}_i$) to an embedding $g_i : S^1 \times \{i\} \longrightarrow \tilde{X}_i$ so that $Im\ g_i$ bounds an arbitrarily small disk in $\tilde{X}_i$, and we do this equivariantly using the action of G_i on $\tilde{X}_i$. Since this action is properly discontinuous, the collection of all these homotopies gives rise to a proper homotopy equivalence between $\tilde{Z}$ and a new 2-polyhedron W obtained from $p^{-1}(Y)$ and a collection of copies of $\tilde{X}_0$ and $\tilde{X}_1$ by gluing each $S^1 \times \{i\} \subset p^{-1}(Y)$ to a copy of $\tilde{X}_i$ via the new maps g_i.

Finally, using an argument similar to that of Lemma 3.2 one can readily check that W is proper homotopy equivalent to a 3-manifold obtained as a quotient space from a collection of copies of the 3-manifolds M_0, M_1 (proper homotopy equivalent to $\tilde{X}_0, \tilde{X}_1$) and a collection of copies of the 3-manifold $H \searrow K$, one for each component K of $p^{-1}(Y)$, by attaching a copy of H along each of those components K so that $S^1 \times \{0\} \times (-\epsilon, \epsilon), S^1 \times \{1\} \times (-\epsilon, \epsilon) \subset \partial H$ get identified homeomorphically with arbitrarily small annuli $C_0 \subset \partial M_0, C_1 \subset \partial M_1$, with M_0, M_1 being the 3-manifolds associated to the corresponding copies of $\tilde{X}_0$ and $\tilde{X}_1$ joined by K.

In the case of an HNN-extension $G_k *_F = \langle G_k, t; t^{-1}\psi_0(a)t = \psi_1(a), a \in F\rangle$ (with monomorphisms $\psi_i : F \longrightarrow G_k, i = 0, 1$), let X be a compact 2-polyhedron with $\pi_1(X) \cong G_k$ and whose universal cover has the proper homotopy type of a 3-manifold, and let $f_i : S^1 \longrightarrow X$ $(i = 0, 1)$ be p.l. maps so that $Im\ f_{i_*} \subseteq \pi_1(X)$ corresponds to the subgroup $Im\ \psi_i \subseteq G_k$. Let Y be the 2-polyhedron constructed as above, and consider the adjunction space $Z = Y \cup_{f_0 \times \{0\} \cup f_1 \times \{1\}} X$, with $\pi_1(Z) \cong G_k *_F$. Then, the proof of this case just mimics that of the above. □

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