EXAMPLES OF NON-FORMAL CLOSED \((k - 1)\)-CONNECTED MANIFOLDS OF DIMENSIONS \(\geq 4k - 1\)

ALEX N. DRANISHNIKOV AND YULI B. RUDYAK

(Communicated by Paul Goerss)

Abstract. We construct closed \((k - 1)\)-connected manifolds of dimensions \(\geq 4k - 1\) that possess non-trivial rational Massey triple products. We also construct examples of manifolds \(M\) such that all the cup-products of elements of \(H^k(M)\) vanish, while the group \(H^{3k-1}(M; \mathbb{Q})\) is generated by Massey products: such examples are useful for the theory of systols.

For every \(k\) we construct closed \((k - 1)\)-connected manifolds of dimensions \(\geq 4k - 1\) that possess non-trivial rational Massey triple products and therefore are non-formal. For \(k = 1\) such manifolds can be obtained as the products of Heisenberg manifold with circles. For \(k = 2\) such examples are also known (see e.g. [4, 2]), but even in this case our construction seems more direct and simple.

Miller [3] proved that every closed \((k - 1)\)-connected manifold \(M\) of dimension \(\leq 4k - 2\) is formal. In particular, all rational Massey products in \(M\) vanish. So, neither Miller’s nor our results can be improved.

Recall that a subset \(S\) of a space \(\mathbb{R}^m\) is called radial if, for all points \(s \in S\), the linear segment \([0, s]\) contains precisely one point of \(S\) (namely, \(s\)).

1. Proposition. Let \(B\) be a finite polyhedron in \(\mathbb{R}^m, m > 1\), let \(A\) be a subpolyhedron of \(B\) such that \(A \setminus \{0\}\) is radial in \(\mathbb{R}^m\), and let \(Y\) be a finite polyhedron in \(\mathbb{R}^n\). Then the double cylinder \(Z_f\) of any simplicial map \(f : A \rightarrow Y\) admits a PL embedding in \(\mathbb{R}^{m+n}\).

Proof. We denote by \(0_m\) and \(0_n\) the origins of spaces \(\mathbb{R}^m\) and \(\mathbb{R}^n\), respectively. We first consider the case when \(0_m \notin A\). We assume that \(Y\) is far away from \(0_n\). Let \(\Gamma \in \mathbb{R}^m \times \mathbb{R}^n\) be the graph of the map \(f\). We join every point \((x, f(x)) \in \Gamma, x \in A\) with the point \((0_m, f(x)) \in \mathbb{R}^m \times Y \subset \mathbb{R}^{m+n}\) by the linear segment. Then, since \(A\) is radial, we get an embedding of the mapping cylinder \(M_f\) of \(f\) to \(\mathbb{R}^{m+n}\). Moreover, if we join the points \((x, 0_n)\) with \((x, f(x))\) by the linear segment, we still have an embedding \(M_f \hookrightarrow \mathbb{R}^{m+n}\). Here (the image of) \(M_f\) is formed by segments \([x, 0_n], (x, f(x)]\) and \([(x, f(x)), (0_m, f(x)]\). Finally, we get an embedding of the double mapping cylinder \(Z_f\) to \(\mathbb{R}^{m+n}\) by adding the space \(B\) to the embedded mapping cylinder \(M_f\).

Received by the editors November 17, 2003 and, in revised form, January 9, 2004.

2000 Mathematics Subject Classification. Primary 55S30; Secondary 55P62, 57Q35.
The case \( 0_m \in A \) can be considered similarly. We can assume that there is a point \( y_0 \in Y \) that is the closest to \( 0_n \in \mathbb{R}^n \), i.e. \( ||y_0|| < ||y|| \) if \( y \neq y_0 \) and \( y \in Y \). We can also assume that \( f(0_m) = y_0 \). Consider the map \( f' = f(\{A \setminus \{0\}\}) \) and the embedding \( i : Z_{f'} \to \mathbb{R}^{m+n} \) as above. Then \( i(Z_{f'}) \cup [0_m, y_0] \) is an embedding of \( Z_f \). \( \square \)

2. Corollary. Let \( Y \) be a finite polyhedron in \( \mathbb{R}^n \), and let \( f : \bigvee_i S_i^{m-1} \to Y, i = 1, \ldots, k \), be a simplicial map, where \( S_i^{m-1} \) is the copy of the sphere \( S^{m-1} \). Then the cone \( C_f \) of \( f \) can be simplicially embedded in \( \mathbb{R}^{m+n} \).

Proof. Choose a base point on the boundary of each disc \( D_i^m, i = 1, \ldots, k \), and consider the wedge \( \bigvee_i D_i^m \). We can regard this wedge as a polyhedron in \( \mathbb{R}^n \) such that the base point is the origin and \( \bigvee S_i^{m-1} \setminus \{0\} \) is a radial set. Now the claim follows from Proposition \( \text{[1]} \). \( \square \)

Consider the wedge \( K = S^{k_1} \vee S^{k_2} \vee S^{k_3} \) of spheres with \( k_i \geq 2 \), and let \( \iota_r \in \pi_{k_r}(K) \) be represented by the inclusion map \( S^{k_r} \subset K \). Set \( m = k_1 + k_2 + k_3 - 1, \) let \( f : S^{m-1} \to K \) represent the element \( [\iota_1, [\iota_2, \iota_3]] \), and let \( X \) be the cone of the map \( f \). Let \( \alpha_i \in H^{k_i}(X) \) be the cohomology class that takes the value 1 on the cell \( S_i^k \) of \( X \) and 0 on the other cells. We recall the following classical result.

3. Theorem. The Massey product \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle \in H^{k_1+k_2+k_3-1}(X) \) has the zero indeterminacy and takes the value \((-1)^{k_1}\) on the \((m-1)\)-dimensional cell of \( X \).

Proof. See \( \text{[5]} \), Lemma 7. \( \square \)

Now let \( k_1 = k_2 = k_3 = k \), and consider the corresponding space \( X \). According to Proposition \( \text{[1]} \) \( X \) admits a PL embedding in \( \mathbb{R}^N \) with \( N \geq 4k \). Fix such an embedding, and let \( W \) be a closed regular neighborhood of \( X \) in \( \mathbb{R}^N \). So, \( W \) is a manifold with the boundary \( V = \partial W \). Furthermore, \( W \) has the homotopy type of \( X \). (Notice that \( W \) is a PL manifold by the construction, but without loss of generality we can assume that \( W \) is smooth.)

4. Proposition. The manifold \( V \) is \((k-1)\)-connected.

Proof. Consider a sphere \( S^i, i < k \), in \( V \). Since \( W \) is \((k-1)\)-connected, there exists a disk \( D^i+1 \) in \( W \) with \( \partial D^i+1 = S^i \). Since \( i + 1 + \dim X \leq 4k - 1 < N \), we can assume that \( D^i+1 \cap X = \emptyset \). But \( V \) is a retract of \( W \setminus X \), and thus \( S^i \) bounds a disk in \( V \). \( \square \)

5. Proposition. \( H^i(W, V) = H_{N-i}(X) \).

Proof. We have

\[ H^i(W, V) = H_{N-i}(W) = H_{N-i}(X) \]

where the first equality holds by the Poincaré duality; see e.g. Dold \( \text{[1]} \). \( \square \)

Consider the map

\[ g : V \xrightarrow{i} W \xrightarrow{r} X \]

where \( i \) is the inclusion and \( r \) is a deformation retraction.

6. Theorem. If \( N \neq 5k - 1, 6k - 2 \), then the Massey product \( \langle g^*\alpha_1, g_*\alpha_2, g^*\alpha_3 \rangle \) has zero indeterminacy and is non-zero.
Proof. Notice that \( H_i(X) = 0 \) for \( i \neq 0, k, 3k - 1 \). We have \( H^{2k-1}(W) = H^{2k-1}(X) = 0 \) and \( H^{2k}(W, V) = H_{n-2k}(X) = 0 \). Now, in view of the exactness of the sequence \( H^{2k-1}(W) \to H^{2k-1}(V) \to H^{2k}(W, V) \), we have \( H^{2k-1}(V) = 0 \), and therefore the indeterminacy of the Massey product is zero. Furthermore, the map \( i^* : H^{3k-1}(W) \to H^{3k-1}(V) \) is injective since \( H^{3k-1}(W, V) = H_{n-3k+1}(X) = 0 \). Thus, the map \( g^* : H^{3k-1}(X) \to H^{3k-1}(V) \) is injective. But \( g^*(\alpha_1, g_*\alpha_2, g^*\alpha_3) = \langle g^*\alpha_1, g_*\alpha_2, g^*\alpha_3 \rangle \) because both parts of the equality have zero indeterminacies. \( \square \)

Thus, we have examples of \((k-1)\)-connected manifolds with non-trivial triple Massey product of dimensions \( d \geq 4k - 1 \) but \( d \neq 5k - 2, 6k - 3 \). In order to construct an example in exceptional dimensions, just take the double of the manifold \( W \) (or multiple by the sphere of the corresponding dimension if \( k \neq 2 \)).

When we put the first version of the paper into the e-archive, Mikhail Katz asked us if we can construct a closed manifold \( M \) such that all the cup-products of elements of \( H^k(M) \) vanish, while the group \( H^{3k-1}(M; \mathbb{Q}) \) is generated by Massey products. Now we present such an example.

7. Lemma. Consider a wedge \( X \vee Y \) and three elements \( u, v, w \in H^*(X) \) such that \( uv = 0, uY = 0 = vY \) and \( wX = 0 \). Then all the Massey products \( \langle u, v, w \rangle \), \( \langle u, w, v \rangle \) and \( \langle w, u, v \rangle \) are trivial, i.e. they contain the zero element.

Proof. This follows from the following fact: If \( A \in C^*(X \vee Y) \) and \( B \in C^*(X \vee Y) \) are cochains with the supports in \( X \) and \( Y \), respectively, then their product is equal to zero. We leave the details to the reader. \( \square \)

Consider the wedge \( S^k_1 \vee S^k_2 \vee S^k_3 \vee S^k_4 \) of \( k \)-dimensional spheres, \( k > 1 \). Let \( \iota_m \in \pi_k(S^k_m) \) be the generator. Set

\[
(1) \quad Z = \left( \bigvee_{i=1}^{4} S^k_i \right) \cup f_{i1} e^{3k-1}
\]

where \( f_{i1} : S^{3k-2} \to \bigvee_{i=1}^{4} S^k_i \) represents the homotopy class \([\iota_1, [\iota_2, \iota_3]]\). Let \( \alpha_i \in H^k(Z) \) be the cohomology class that takes the value 1 on the cell \( S^k_i \) of \( Z \) and 0 on the other cells.

8. Corollary. If at least one of the indices \( i, j, k \) is equal to 4, then \( \langle \alpha_i, \alpha_j, \alpha_k \rangle = 0 \) in \( Z \).

Proof. This follows directly from Lemma 7 since

\[
Z = \left( \bigvee_{i=1}^{3} S^k_i \right) \cup f_{i1} e^{3k-1} \vee S^k_4.
\]

For convenience of notation, we set \( \iota_5 = \iota_1 \) and \( \iota_6 = \iota_2 \). Let \( f_m : S^{3k-2} \to \bigvee_{i=1}^{4} S^k_i \) be the map that represents \([\iota_m, [\iota_{m+1}, \iota_{m+2}]]\), \( m = 1, 2, 3, 4 \). Consider the map

\[
f : \bigvee_{i=1}^{4} S^{3k-2} \to \bigvee_{i=1}^{4} S^k_i
\]

such that \( f|S^{3k-2} = f_1 \), and set \( X = C_f\). We define \( \alpha_m \in H^k(X) \) to be the cohomology class that takes the value 1 on the cell \( S^k_i \) of \( X \) and 0 on the other cells. For convenience of notation, we set \( \alpha_5 = \alpha_1 \) and \( \alpha_6 = \alpha_2 \).
9. **Lemma.** The homology classes $\langle \alpha_m, \alpha_{m+1}, \alpha_{m+2} \rangle$ are linearly independent in $H^{3k-1}(X)$.

**Proof.** First, notice that all these Massey products are defined and have zero indeterminacies. Now, suppose that $\sum_{m=1}^{4} c_m \langle \alpha_m, \alpha_{m+1}, \alpha_{m+2} \rangle = 0$ for some $c_m \in \mathbb{R}$. Consider the space $Z$ as in \[1\] and the obvious inclusion $j : Z \to X$. Then $j^* \langle \alpha_m, \alpha_{m+1}, \alpha_{m+2} \rangle = 0$ for $m = 2, 3, 4$ by Corollary \[5\] while $j^* \langle \alpha_1, \alpha_2, \alpha_3 \rangle \neq 0$ by Theorem \[3\]. Therefore $c_1 = 0$. Similarly, we can prove that $c_m = 0$ for all $m$. □

Now, because of Proposition \[1\] $X$ can be regarded as a polyhedron in $\mathbb{R}^N$ with $N \geq 4k$. Let $W$ be a regular neighborhood of $X$ in $\mathbb{R}^N$ and set $M = \partial W$.

10. **Theorem.** If $N \neq 4k$, $5k - 1$, $6k - 2$, $6k - 1$, then $H^{3k-1}(M; \mathbb{Q})$ is generated by triple Massey products, while all the cup-products of elements of $H^k(M)$ vanish.

**Proof.** Consider the map

$$g : V \xrightarrow{i} W \xrightarrow{r} X$$

where $i$ is the inclusion and $r$ is a deformation retraction. Using the isomorphisms

$$H^i(W, M) \cong H_{N-i}(X) \quad \text{and} \quad H^i(W) \cong H^i(X),$$

and the exactness of the sequence

$$\begin{array}{ccc}
H^i(W, M) & \xrightarrow{j^*} & H^i(W) \\
\xrightarrow{i^*} & & \xrightarrow{r^*} \\
& & \Downarrow H^i(M)
\end{array}$$

we conclude that $H^{2k-1}(M) = 0$ and

$$g^* : H^{2k-1}(X) \to H^{3k-1}(M)$$

is an isomorphism. Now, the equality $H^{2k-1}(M) = 0$ implies that all the Massey products $\langle \alpha_i, \alpha_j, \alpha_k \rangle$ have zero indeterminacies. Furthermore, since $g^*$ is an isomorphism, Lemma \[9\] implies that the $g^*$-images of the classes $\langle \alpha_m, \alpha_{m+1}, \alpha_{m+2} \rangle$, $m = 1, 2, 3, 4$, in $M$ form a basis of $H^{3k-1}(M; \mathbb{Q})$. Finally, the map $i^* : H^k(W) \to H^k(M)$ is surjective for $N \neq 4k$, and so the cup-products of elements of $H^k(M)$ vanish. □

**Acknowledgment**

The first author was partially supported by NSF, grant 0305152. The second author was partially supported by MCyT, project BFM 2002-00788, Spain.

**References**


Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, Florida 32611-8105

E-mail address: dranish@math.ufl.edu

Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, Florida 32611-8105

E-mail address: rudyak@math.ufl.edu