

## SAMPLING SETS AND CLOSED RANGE COMPOSITION OPERATORS ON THE BLOCH SPACE

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*Dedicated to Chandler Davis for his 75th birthday*

ABSTRACT. We give a necessary and sufficient condition for a composition operator  $C_\phi$  on the Bloch space to have closed range. We show that when  $\phi$  is univalent, it is sufficient to consider the action of  $C_\phi$  on the set of Möbius transforms. In this case the closed range property is equivalent to a specific sampling set satisfying the reverse Carleson condition.

### 1. INTRODUCTION

An analytic function  $f$  on  $D$  is said to belong to the Bloch space if  $\sup\{(1 - |z|^2)|f'(z)|\}$  over  $D$  is finite. Such functions form a complex Banach space  $B$  under the norm  $\|f\|_B = \sup\{(1 - |z|^2)|f'(z)|, z \in D\} + |f(0)|$ . Functions belonging to the little Bloch space  $B_0$  (consisting of the closure of polynomials in  $B$ ) are characterized by the property:  $\lim_{|z| \rightarrow 1} (1 - |z|^2)f'(z) = 0$ .

Observe that  $\sup\{(1 - |z|^2)|f'(z)|, z \in D\}$  is a pseudonorm, which coincides with the Bloch-norm on the closed subspace of functions that vanish at the origin. In general it coincides with the quotient norm on  $B/\mathcal{C}$  where  $\mathcal{C}$  denotes the closed subspace of constant functions.

The following concept is what all our criteria are based on.

We say that a subset  $H$  of  $D$  is called a sampling set for the Bloch space  $B$  if  $\exists k > 0$  such that  $\sup\{(1 - |z|^2)|f'(z)|, z \in D\} \leq k \sup\{(1 - |z|^2)|f'(z)|, z \in H\}$  holds  $\forall f \in B$ .

This is equivalent to  $H$  being a sampling set for the  $L^\infty$  version of the weighted Bergman space, denoted by  $A^{-1}$  [8, p. 22]. There are other definitions of sampling set for the Bloch space, but this one suits our purpose the best.

For each  $z$  belonging to the unit disk  $D$ , let  $\phi_z$  denote the Möbius transformation of  $D$ , given by

$$\phi_z(w) = \frac{z - w}{1 - \bar{z}w},$$

for  $w \in D$ . The pseudohyperbolic distance (between  $z$  and  $w$ ) on  $D$  is defined by

$$\rho(z, w) = |\phi_z(w)|.$$

$D(a, s)$  stands for the set  $\{z \in D, \rho(z, a) < s\}$ .

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This metric is Möbius-invariant and has the following property:

$$1 - \rho(z, w)^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2} = (1 - |z|^2)|\phi'_z(w)|.$$

If  $\phi$  is a holomorphic self-map of  $D$ , and  $\tau_\phi(z) = \frac{(1-|z|^2)\phi'(z)}{1-|\phi(z)|^2}$ , then it is a simple consequence of the Schwarz-Pick lemma that  $0 \leq |\tau_\phi(z)| \leq 1$  [1, p. 2]. Hence the composition operator  $C_\phi$  defined by  $C_\phi(f) = f \circ \phi$  is a bounded operator from  $B$  into  $B$ . Moreover, if  $\phi \in B_0$ , then  $C_\phi$  maps  $B_0$  into  $B_0$ .

The function  $\tau_\phi$  figures strongly in the study of compact composition operators on the Bloch space. In particular, if  $C_\phi$  is compact on  $B_0$ , then  $\tau_\phi(z) \rightarrow 0$  as  $|z| \rightarrow 1$  and if it is compact on  $B$ , then  $\tau_\phi(z) \rightarrow 0$  as  $|\phi(z)| \rightarrow 1$  [6].

## 2. A NECESSARY AND SUFFICIENT CONDITION

The following fact is easy to check, but since it is pivotal to our investigation, we state it formally.

**Theorem 0.** *If  $\phi(0) = 0$ , the composition operator  $C_\phi$  is bounded below on  $B$  (equivalently, has closed range on  $B$ ) if and only if it is bounded below on the subspace of functions that vanish at the origin. This is equivalent to the condition that  $\|f \circ \phi\|_{B/C} \geq k\|f\|_{B/C}$ .*

*Remark 1.* If  $\phi(0) = a$  and  $\psi = \phi_a \circ \phi$ , then  $C_\phi$  is bounded below on  $B$  if and only if  $C_\psi$  is bounded below on  $B$ . Moreover,  $\tau_\psi = \tau_\phi$ .

So we assume from now on that  $\phi(0) = 0$  and that  $C_\phi$  is acting on the subspace of functions that vanish at the origin. It is natural that the sets  $\Omega_\varepsilon = \{z, |\tau_\phi(z)| \geq \varepsilon\}$  and  $G_\varepsilon = \phi(\Omega_\varepsilon)$  play a pivotal role in our investigation.

**Theorem 1.** *The composition operator  $C_\phi$  is bounded below on  $B$  if and only if  $\exists \varepsilon > 0$  such that if  $\Omega_\varepsilon = \{z \in D, |\tau_\phi(z)| \geq \varepsilon\}$ , then  $G_\varepsilon$  is a sampling set for  $B$ .*

*Proof.* Assume that  $G_\varepsilon$  is a sampling set for  $B$ , for some  $\varepsilon > 0$ . Then  $\forall f \in B$  with  $f(0) = 0$ ,

$$\begin{aligned} \|f\|_B &\leq k \sup \{(1 - |\phi(z)|^2)|f'(\phi(z))|, z \in \Omega_\varepsilon\} \\ &\leq k \sup \{|\tau_\phi(z)|^{-1}(1 - |z|^2)|f'(\phi(z))\phi'(z)|, z \in \Omega_\varepsilon\} \\ &\leq k\varepsilon^{-1} \sup \{(1 - |z|^2)|(f \circ \phi)'(z)|, z \in D\} \\ &\leq k\varepsilon^{-1}\|f \circ \phi\|_B. \end{aligned}$$

Conversely suppose that  $C_\phi$  is bounded below on  $B$ . Then  $\exists k > 0$  such that whenever  $f(0) = 0$  and  $\|f\|_B = 1$ ,  $\sup\{(1 - |z|^2)|(f \circ \phi)'(z)| \geq k$ .

Suppose  $\|f\|_B = 1$ ,  $f(0) = 0$  and choose  $z_f$  such that  $(1 - |z_f|^2)|(f \circ \phi)'(z_f)| \geq k/2$ , i.e.  $|\tau_\phi(z_f)|[(1 - |\phi(z_f)|^2)|f'(\phi(z_f))|] \geq k/2$ . But each of the two factors is no larger than 1. Hence each is at least as large as  $k/2$ . Thus if  $\varepsilon = k/2$ , then  $G_\varepsilon$  is a sampling set for  $B$ . □

A subset  $H$  of  $D$  is said to satisfy the reverse Carleson condition if  $\exists s > 0$  and  $c > 0$  such that  $|D(a, s) \cap H| \geq c|D(a, s)|$  for all  $a \in D$ , or equivalently if  $\int_H |f(z)|^2 dA(z) \geq c \int_D |f(z)|^2 dA(z) \forall f$ , which are analytic and square integrable on  $D$ .

This definition and the techniques we use are found in [5].

We show that if  $G_\epsilon$  satisfies the reverse Carleson condition, then  $G_\epsilon$  is a sampling set for the Bloch space. In order to do that we need to note an equivalent form for the Bloch-norm. We include a short proof for completeness.

**Observation 1.** For an analytic function  $f$  on  $D$  with  $f(0) = 0$ ,

$$\|f\|_B^2 \approx \sup \left\{ \int_D |f'(z)|^2 (1 - |\phi_a(z)|^2)^2 dA(z), a \in D \right\}.$$

*Proof.* If  $a \in D$ , then,

$$\begin{aligned} \int_D |f'(z)|^2 (1 - |\phi_a(z)|^2)^2 dA(z) &= \int_D |f'(z)|^2 (1 - |z|^2)^2 |\phi'_a(z)|^2 dA(z) \\ &\leq (\|f\|_B^2) \int_D |\phi'_a(z)|^2 dA(z) \leq \|f\|_B^2. \end{aligned}$$

In order to prove the reverse inequality, first choose  $s > 0$  such that  $\rho(z, w) < s$  implies  $|(1 - |z|^2)f'(z) - (1 - |w|^2)f'(w)| \leq \frac{1}{4} \forall f \in B$  with  $\|f\|_B \leq 1$ . See [2, Proposition 2]. Now given  $f \in B$  with  $\|f\|_B = 1$  and  $f(0) = 0$ , we have  $\sup\{(1 - |z|^2)|f'(z)|, z \in D\} \geq 1$ . Choose  $z_f$  such that  $\forall z \in D(z_f, s), (1 - |z|^2)|f'(z)| \geq \frac{1}{4}$ .

Hence

$$\begin{aligned} &\int_D |f'(z)|^2 (1 - |\phi_{z_f}(z)|^2)^2 dA(z) \\ &\geq \int_{D(z_f, s)} |f'(z)|^2 (1 - |\phi_{z_f}(z)|^2)^2 dA(z) \\ &\geq \frac{1}{16} \int_{D(z_f, s)} \left( \frac{1 - |\phi_{z_f}(z)|^2}{1 - |z|^2} \right)^2 dA(z). \end{aligned}$$

Note that  $z \in D(z_f, s) \Rightarrow (1 - |z|^2) \approx c_s(1 - |z_f|^2)$  and  $|\phi_{z_f}(z)| = |\rho(z_f, z)| < s$ . Thus  $\int_D |f'(z)|^2 (1 - |\phi_{z_f}(z)|^2)^2 dA(z) \geq c_s$ . □

**Proposition 1.** If  $H \subseteq D$  and  $H$  satisfies the reverse Carleson condition, then  $H$  is a sampling set for the Bloch space.

*Proof.* Suppose  $\|f_n\|_B \leq 1, f_n(0) = 0$  and  $\delta_n = \sup\{(1 - |z|^2)|f'_n(z)|, z \in H\} \rightarrow 0$  as  $n \rightarrow \infty$ .

If  $a \in D$ , then

$$\begin{aligned} &\int_D |f'_n(z)|^2 (1 - |\phi_a(z)|^2)^2 dA(z) \\ &= \int_D (1 - |z|^2)^2 |f'_n(z)|^2 |\phi'_a(z)|^2 dA(z) \\ &\leq c \int_H (1 - |z|^2)^2 |f'_n(z)|^2 |\phi'_a(z)|^2 dA(z) \text{ [6, p. 10]} \\ &\leq c \delta_n \int_H |\phi'_a(z)|^2 dA(z) \leq c \delta_n \int_D |\phi'_a(z)|^2 dA(z) \leq c \delta_n. \end{aligned}$$

Hence,  $\|f_n\|_B^2 \leq c \delta_n$  (by the observation above), and  $\|f_n\|_B \rightarrow 0$ . □

Our next proposition is an expanded form of the main result in [2]. We include a short proof of a part of it for completeness.

**Proposition 2.** Suppose  $\phi$  is an analytic self-map of  $D$  and assume that  $\|\phi_w \circ \phi\|_{B/C} \geq k \forall w \in D$ . Then the following conditions hold.

- (1) Whenever  $\varepsilon < k, \rho(z, G_\varepsilon) \leq \sqrt{1 - \varepsilon} = r \ \forall z \in D$ .
- (2) Moreover,  $\exists$  constants  $s$  and  $r', 0 < s < 1$  and  $r' \in [r, 1)$  such that given  $w \in D \ \exists z_w \in D$  such that  $\phi(D(z_w, s)) \subseteq D(w, r') \cap G_\varepsilon$ .

*Proof.* (1) Suppose that  $\varepsilon < k$  and  $w \in D$ . Choose  $z_w \in D$  such that

$$(1 - |z_w|^2)|\phi'_w(\phi(z_w))| |\phi'(z_w)| \geq \varepsilon.$$

But  $(1 - |z_w|^2)|\phi'_w(\phi(z_w))\phi'(z_w)| = |(\tau_\phi(z_w)|(1 - \rho^2(w, \phi(z_w))))$ . Each factor on the right-hand side is no larger than 1; hence each is at least  $\varepsilon$ . Thus  $z_w \in \Omega_\varepsilon$  and  $\rho(w, \phi(z_w)) \leq r < 1$  where  $r = \sqrt{1 - \varepsilon}$ .

(2) In [3, Theorem 6] it is shown that  $\tau_\phi$  is Lipschitz with respect to the pseudo-hyperbolic metric on the domain and the Euclidean one on the range. We denote the Lipschitz constant by  $\alpha$ . Fix  $\varepsilon' < \varepsilon$ , choose  $s < \frac{\varepsilon - \varepsilon'}{\alpha}$  and let  $s < \varepsilon/(2\alpha)$ . If  $\lambda \in D(z_w, s)$ , then  $|\tau_\phi(\lambda)| \geq \varepsilon'$  and (by the Schwarz-Pick lemma)  $\rho(\phi(z_w), \phi(\lambda)) \leq \rho(z_w, \lambda) < s$ . Thus for  $\lambda$  in  $D(z_w, s)$  we have

$$\rho(w, \phi(\lambda)) \leq \frac{\rho(\phi(z_w), w) + \rho(\phi(z_w), \phi(\lambda))}{1 + \rho(w, \phi(z_w))\rho(\phi(z_w), \phi(\lambda))} < \frac{r + s}{1 + rs}$$

since  $\frac{r+s}{1+rs}$  is an increasing function of  $s$  and  $r$  if they both lie in  $(0, 1)$ . Let  $r' = \frac{r+s}{1+rs}$ . We have shown that  $\phi(D(z_w, s)) \subseteq D(w, r')$ . By the choice of  $s, |\tau_\phi(\lambda)| \geq \varepsilon' \ \forall \lambda \in D(z_w, s)$ , i.e.  $D(z_w, s) \subseteq \Omega_{\varepsilon'}$  and hence  $\phi(D(z_w, s)) \subseteq G_{\varepsilon'}$ .

We conclude that  $\phi(D(z_w, s)) \subseteq G_{\varepsilon'} \cap D(w, r')$ . □

In [2] it is shown that condition (1) of the previous proposition implies that  $C_\phi$  is bounded below on the set of Möbius transforms, and that in case  $C_\phi$  is close to being an isometry on the set of Möbius transforms ( $k > \frac{15}{16}$ ), then  $C_\phi$  is bounded below on the Bloch space.

We now show that in case  $\phi$  is univalent, then no lower bound on  $k$  (except  $k > 0$ ) is necessary.

### 3. UNIVALENCE

**Corollary 1.** *If  $\phi$  is a univalent self-map of  $D$  and  $\|\phi_w \circ \phi\|_{B/C} \geq k \ \forall w \in D$ , then  $\forall \varepsilon < k, G_\varepsilon$  satisfies the reverse Carleson condition.*

*Proof.* Let  $\varepsilon < k$ . Pick  $\varepsilon' \in (\varepsilon, k)$  and apply the conclusion of Proposition 2 to  $\varepsilon'$ .  $\exists s, r' \in (0, 1)$  such that given  $w \in D \ \exists z_w$  such that  $\phi(D(z_w, s)) \subseteq D(w, r') \cap G_\varepsilon$ .

We use the fact that  $\forall \lambda \in D(z_w, s), |\tau_\phi(\lambda)| \geq \varepsilon$  and the univalence of  $\phi$  to conclude that  $|\phi(D(z_w, s))| = \int_{D(z_w, s)} |\phi'(\lambda)|^2 A(\lambda) \geq \varepsilon^2 \int_{D(z_w, s)} \left(\frac{1 - |\phi(\lambda)|^2}{1 - |\lambda|^2}\right)^2 dA(\lambda)$ .

But  $1 - |\lambda|^2 \approx c_s(1 - |z_w|^2)$  and  $1 - |\phi(\lambda)|^2 \approx c_s(1 - |\phi(z_w)|^2) \approx c_r c_s(1 - |w|^2)$ . Since  $|D(a, r)| \approx c_r(1 - |a|^2)$ , we have  $|D(w, r') \cap G_\varepsilon| \geq |\phi(D(z_w, s))| \geq c|D(w, r')|$  where  $c$  is independent of  $w$ . □

We summarize the main result for the univalent case in the following proposition.

**Theorem 2.** *Suppose  $\phi$  is a univalent self-map of  $D$ . Then the following are equivalent.*

- (1)  $C_\phi$  is bounded below on  $B$ .
- (2)  $\|\phi_w \circ \phi\|_{B/C} \geq k \ \forall w \in D$ .
- (3)  $\forall \varepsilon < k, \rho(G_\varepsilon, z) \leq r < 1 \ \forall z \in D, r$  depending only on  $\varepsilon$ .
- (4)  $\forall \varepsilon < k, G_\varepsilon$  satisfies the reverse Carleson condition.

*Proof.* (1)  $\Rightarrow$  (2) is obvious.  
 (2)  $\Rightarrow$  (3) by Proposition 2.  
 (3)  $\Rightarrow$  (4) by Corollary 1.  
 (4)  $\Rightarrow$  (1) by Proposition 1. □

**Corollary 2.** (1) *If  $\phi$  is univalent and  $C_\phi$  is bounded below on BMOA, then it is bounded below on the Bloch space.*

(2) *If  $\phi$  is univalent and  $C_\phi$  is bounded below on the Bloch space, then it is bounded below on the Dirichlet space.*

*Proof.* (1) If  $\phi$  is univalent, then  $\forall w \in D, \phi_w \circ \phi$  is univalent and in this case

$$\|\phi_w \circ \phi\|_{BMOA} \approx \|\phi_w \circ \phi\|_B$$

[10]. Hence if  $C_\phi$  is bounded below on BMOA, then  $\|\phi_w \circ \phi\|_B \geq k \forall w \in D$ . Hence by Proposition 3,  $C_\phi$  is bounded below on the Bloch space.

(2) By Theorem 2, if  $C_\phi$  is bounded below on the Bloch space, then  $\exists \varepsilon > 0$  such that  $G_\varepsilon$  satisfies the reverse Carleson condition; hence so does  $G = \phi(D)$ . By [4]  $C_\phi$  is bounded below on the Dirichlet space. □

*Remark.* W. Smith has given an example of a univalent map  $\phi$  with  $\overline{\phi(D)} = \overline{D}$  such that  $\tau_\phi(z) \rightarrow 0$  as  $|z| \rightarrow 1$  [9, 6.5]. Hence  $C_\phi$  is compact on  $B_0$  [6]. But  $C_\phi$  has closed range on the Dirichlet space [4].

We now show that in case  $\phi$  is univalent and  $C_\phi$  is bounded below, then  $G$  has no generalized cusps [7, p. 256] and in fact a somewhat stronger condition holds. We also give an example to show that it is not sufficient.

**Observation 2.** *It is a simple consequence of Koebe’s one-quarter theorem [7, p. 9] that if  $\phi$  is univalent, then  $\tau_\phi \approx \frac{\text{dist}(\phi(z), \partial G)}{\text{dist}(\phi(z), \partial D)}$ .*

**Proposition 3.** *Suppose  $\phi$  is univalent and there exists  $\varepsilon > 0$  such that  $G_\varepsilon$  satisfies the reverse Carleson condition. Then there exists  $\delta > 0$  such that  $\forall \omega \in \partial D$ ,*

$$\overline{\lim}_{\phi(z) \rightarrow \omega} \frac{\text{dist}(\phi(z), \partial G)}{|\phi(z) - \omega|} \geq \delta.$$

*Proof.* If  $\eta < 1$  let  $\Delta(a, \eta) = \{z \in D, |z - a| \leq \eta(1 - |a|)\}$  for  $a \in D$ . By [6, p. 4],  $\exists \eta < 1$  such that the following holds:

$$G_\varepsilon \cap \Delta(a, \eta) \neq \emptyset \forall a \in D.$$

Suppose  $\omega \in \partial D$  and choose  $\{a_n\}$  along the radius ending in  $\omega$  such that  $a_n \rightarrow \omega$ . Choose  $z_n \in \Omega_\varepsilon$  such that  $\phi(z_n) \in \Delta(a_n, \eta)$ . Then  $|\phi(z_n)| \leq |a_n| + \eta(1 - |a_n|) \leq \eta + |a_n|(1 - \eta)$  and hence  $1 - |\phi(z_n)| \geq (1 - \eta)(1 - |a_n|)$ . Note that  $|\omega - a_n| = 1 - |a_n|$ . On the other hand,  $|\phi(z_n) - \omega| \leq |\phi(z_n) - a_n| + |a_n - \omega| \leq \eta(1 - |a_n|) + 1 - |a_n| = (1 + \eta)(1 - |a_n|)$ .

Now by observation 2,

$$\begin{aligned} \frac{\text{dist}(\phi(z_n), \partial G)}{|\phi(z_n) - \omega|} &\geq \frac{1}{4} \frac{\tau_\phi(z_n)(1 - |\phi(z_n)|)}{|\phi(z_n) - \omega|} \\ &\geq \frac{\varepsilon}{4} \frac{1 - \eta}{1 + \eta}. \end{aligned} \quad \square$$

Next we give an example to show that the above condition is not sufficient for  $C_\phi$  to be bounded below.

**Example 1.** Let

$$G = D \setminus \left\{ \bigcup_i D_i \cup l_i \right\}$$

where  $D_i = D(a_i, r_i)$  is the pseudohyperbolic disk with  $|a_i|$  close enough to 1,  $r_i \rightarrow 1$  and  $l_i$  is a line segment connecting  $D_i$  to  $\partial D$ .

Let  $\phi$  be the Riemann map onto  $G$ . If  $\phi(z)$  approaches a point  $\omega$  on  $\partial D$  that is not an endpoint of the line segment  $l_i$  or if  $\omega \neq 1$ , then  $\tau_\phi(z) \approx 1$  by observation 2. It is clear that the conclusion of the previous lemma holds for all  $\omega \in \partial D$  that are not endpoints of line segments  $l_i$  or 1. For  $\omega = 1$ , choose  $\phi(z_n) \in G_\varepsilon$ , approaching 1 non-tangentially, from below the  $x$ -axis.

For  $\omega_i =$  endpoint of the line segment  $l_i$ ,  $G_\varepsilon$  has an extra non-tangential region taken away from  $G$  but with the same angle opening for every  $i$ . Thus we may pick  $\phi(z_n)$  approaching  $\omega_i$  through an angle with a slightly larger opening. So the conclusion of the previous proposition is satisfied, but no pseudohyperbolic neighbourhood of  $G_\varepsilon$  covers  $D$ . Hence  $C_\phi$  is not bounded below on  $B$ .

The next example deals with a non-automorphic univalent  $\phi$  that induces a closed-range composition operator on the Bloch space.

**Example 2.** Let  $G = D \setminus [0, 1)$ , and let  $\phi$  be the Riemann map onto  $G$ . Since

$$\tau_\phi(z) \approx \frac{\text{dist}(\phi(z), \partial G)}{1 - |\phi(z)|}$$

and the ratio on the right approaches 1 as  $\phi(z)$  approaches any point  $\omega \neq 1$  on the unit circle,  $G_\varepsilon$  includes all of  $D$ , except a pseudohyperbolic neighbourhood of  $[0, 1)$ . Hence with a suitable value of  $r$ , every point of  $D$  is within pseudohyperbolic distance  $r$  of  $G_\varepsilon$  and hence  $C_\phi$  is bounded below.

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## REFERENCES

1. J. Garnett, *Bounded Analytic Functions*, Academic Press, 1981. MR0628971 (83g:30037)
2. P. Ghatage, J. Yan and D. Zheng, *Composition operators with closed range on the Bloch space*, Proc. Amer. Math. Soc. **129** (2001), 2039-2044. MR1825915 (2002a:47034)
3. P. Ghatage and D. Zheng, *Hyperbolic derivatives and generalized Schwarz-Pick Estimates*, To appear in Proc. Amer. Math Soc..
4. M. Jovovic and B. MacCluer, *Composition operators on Dirichlet spaces*, Acta. Sci. Math. (Szeged) **63** (1997), 229-297. MR1459789 (98d:47067)
5. D. Luecking, *Inequalities on Bergman Spaces*, Ill. Jour. of Math. **25** (1981), 1-11. MR0602889 (82e:30072)
6. K. Madigan and A. Matheson, *Compact composition operators on the Bloch space*, Trans. Amer. Math. Soc. **347** (1995), 2679-2687. MR1273508 (95i:47061)
7. C. Pommerenke, *Boundary Behaviour of Conformal Maps*, vol. 299, Springer-Verlag, 1992. MR1217706 (95b:30008)
8. K. Seip, *Beurling type density theorems in the unit disk*, Invent. Math. **113**, 21-29. MR1223222 (94g:30033)

9. W. Smith, *Composition operators between Bergman and Hardy spaces*, Trans. Amer. Math. Soc. **348** (1996), 2331-2348. MR1357404 (96i:47056)
10. D. Stegenga and K. Stephenson, *A geometric characterization of analytic functions with bounded mean oscillation*, J. London Math. Soc. 2 **24** (1981), 243-254. MR0631937 (82m:30036)

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