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SOME CHARACTERIZATIONS OF MINIMALLY THIN SETS IN A CYLINDER AND BEURLING-DAHLBERG-SJÖGREN TYPE THEOREMS

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(Communicated by Juha M. Heinonen)

Dedicated to Professor Hidenobu Yoshida on his 60th birthday

ABSTRACT. This paper shows that some characterizations of minimally thin sets connected with a domain having smooth boundary and a half-space in particular can also be given for a minimally thin set at infinity of a cylinder.

1. INTRODUCTION

We denote by $\mathbf{R}^n$ ($n \geq 2$) the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, y), X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary of a set S in $\mathbf{R}^n$ is denoted by ∂S. The half-space

$$\{(X, y) \in \mathbf{R}^n; y > 0\}$$

will be denoted by $\mathbf{T}_n$.

As an extension of a result of Beurling [5, Lemma 1], Dahlberg proved

Theorem 1.1 (Dahlberg [8, Theorem 4]). *Suppose that $E \subset \mathbf{T}_n$ is measurable and that*

$$\int_E \frac{dP}{(1+|P|)^n} = \infty.$$

If u is a non-negative superharmonic function in $\mathbf{T}_n$ and m is a positive number such that $u(P) \geq my$ for all $P = (X, y) \in E$, then $u(P) \geq my$ for all $P = (X, y) \in \mathbf{T}_n$.

Sjögren also gave Theorem 1.1 in the following form with an ingenious proof of Dahlberg's result.

Theorem 1.2 (Sjögren [16, Theorem 2]). *Let $u(P)$ be a positive superharmonic function on $\mathbf{T}_n$ such that*

$$u(P) = \int_{\mathbf{T}_n} G(P, Q) d\mu(Q) + \int_{\partial \mathbf{T}_n} \Pi(P, Q) d\lambda(Q)$$

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with non-negative measures μ and λ on $\mathbf{T}_n$ and $\partial\mathbf{T}_n$, respectively, where $G(P,Q)$ $(P,Q\in\mathbf{T}_n)$ and

$$\Pi(P,Q)=y|P-Q|^{-n}\quad(P=(X,y)\in\mathbf{T}_n, Q\in\partial\mathbf{T}_n)$$

are the Green function and the Poisson kernel for $\mathbf{T}_n$, respectively. Then

$$\int_{E_u}\frac{dP}{(1+|P|)^n}<\infty,$$

where

$$E_u=\{P=(X,y)\in\mathbf{T}_n; u(P)>y\}.$$

Let $K(P,Q)$ $(P\in\mathbf{T}_n, Q\in\partial\mathbf{T}_n)$ be the Martin function with the reference point $(0,0,\dots,0,1)\in\mathbf{T}_n$. Then $K(P,\infty)=y$ for any $P=(X,y)\in\mathbf{T}_n$. A subset E of $\mathbf{T}_n$ is said to be minimally thin at ∞ with respect to $\mathbf{T}_n$ if there exists a point $P=(X,y)\in\mathbf{T}_n$ such that

$$\hat{R}^E_{K(\cdot,\infty)}(P)\neq y,$$

where $\hat{R}^E_{K(\cdot,\infty)}$ is the regularized reduced function of $K(P,\infty)=y$ $(P=(X,y)\in\mathbf{T}_n)$ relative to E (Helms [12, p.134]).

We remark that the conclusion of Theorem 1.1 is equivalent to the fact that E is not minimally thin at ∞ and E_u in Theorem 1.2 is minimally thin at ∞. Hence Theorems 1.1 and 1.2 say

Theorem 1.3. *If $E\subset\mathbf{T}_n$ is measurable and minimally thin at ∞ with respect to $\mathbf{T}_n$, then*

$$\int_E\frac{dP}{(1+|P|)^n}<\infty. \tag{1.1}$$

The following Theorem 1.4 shows that (1.1) characterizes the minimal thinness of E in a special case.

Theorem 1.4. *Let E be a union of Whitney cubes of $\mathbf{T}_n$. Then* (1.1) *is also sufficient for E to be minimally thin at ∞ with respect to $\mathbf{T}_n$.*

These Theorems 1.1, 1.2, 1.3 and 1.4 follow from the results of Dahlberg [8, Theorem 2], Sjögren [16, Theorem 2], Aikawa [1, Corollary 7 and Corollary 8], Aikawa and Essén [2, Corollary 7.4.6 on p.158], which are all connected with a Liapunov-Dini domain in $\mathbf{R}^n$, because $\mathbf{T}_n$ is mapped onto a ball by a suitable Kelvin transformation. All these results are connected to a boundary point of domains with smooth boundary. So we can ask what are the results similar to these results with respect to a corner or a cusp of a bounded domain. If we map a cone or a cylinder into a bounded domain by a Kelvin transformation, the infinite boundary points ∞ of a cone and $+\infty$ of a cylinder are mapped to a corner and a cusp of a bounded domain, respectively. In [15], with respect to ∞ of a cone we generalized Theorems 1.1, 1.2, 1.3 and 1.4. In this paper, with respect to $+\infty$ of a cylinder, we shall show that the same type of theorems as Theorems 1.3 and 1.4 are still true. Then we shall also give the same type of theorems as Theorems 1.1 and 1.2 for a positive superharmonic functions on a cylinder.

2. Preliminaries

Let D be a domain on $\mathbf{R}^{n-1}$ $(n \geq 2)$ with smooth boundary. Consider the Dirichlet problem

$$(2.1) \qquad \begin{aligned} (\Delta_n + \tau)f &= 0 \text{ on } D, \\ f &= 0 \text{ on } \partial D. \end{aligned}$$

We denote the least positive eigenvalue of (2.1) by τ_D and the normalized positive eigenfunction corresponding to τ_D by $f_D(X)$;

$$\int_D f_D^2(X)dX = 1,$$

where dX is the $(n-1)$-dimensional volume element.

To simplify our consideration in the following, we shall assume that if $n \geq 3$, then D is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{R}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see Gilbarg and Trudinger [11, pp. 88-89] for the definition of a $C^{2,\alpha}$-domain).

By $\Gamma_n(D)$, we denote the set

$$\{(X,y) \in \mathbf{R}^n; X \in D, -\infty < y < +\infty\},$$

which is usually called a cylinder. It is known that the Martin boundary of $\Gamma_n(D)$ is the set $\partial\Gamma_n(D) \cup \{+\infty, -\infty\}$. When we denote the Martin kernel by $K(P,Q)$ $(P \in \Gamma_n(D), Q \in \partial\Gamma_n(D) \cup \{+\infty, -\infty\})$, we know

$$K(P,+\infty) = e^{\sqrt{\tau_D}y} f_D(X),\ K(P,-\infty) = \kappa e^{-\sqrt{\tau_D}y} f_D(X) \quad (P = (X,y) \in \Gamma_n(D)),$$

where κ is a positive constant.

A subset E of $\Gamma_n(D)$ is said to be minimally thin at $+\infty$ with respect to $\Gamma_n(D)$ (Brelot [6, p.122], Doob [9, p.208]) if there exists a point $P \in \Gamma_n(D)$ such that

$$\hat{R}^E_{K(\cdot,+\infty)}(P) \neq K(P,+\infty),$$

where $\hat{R}^E_{K(\cdot,+\infty)}(P)$ is the regularized reduced function of $K(\cdot,+\infty)$ relative to E (Helms [12, p.134]). As far as we are concerned with minimal thinness in the following, we shall restrict a subset E of $\Gamma_n(D)$ to the set located in the half cylinder

$$\Gamma_n(D,0,+\infty) = \{(X,y) \in \mathbf{R}^n; X \in D, 0 < y < +\infty\},$$

because the part of E separated from $+\infty$ is unessential to minimal thinness.

Let E be a bounded subset of $\Gamma_n(D,0,+\infty)$. Then $\hat{R}^E_{K(\cdot,+\infty)}$ is bounded on $\Gamma_n(D)$, and hence the greatest harmonic minorant of $\hat{R}^E_{K(\cdot,+\infty)}$ is zero. When we denote by $G(P,Q)$ $(P \in \Gamma_n(D), Q \in \Gamma_n(D))$ the Green function of $\Gamma_n(D)$, we see from the Riesz decomposition theorem that there exists a unique positive measure λ_E on $\Gamma_n(D)$ such that

$$\hat{R}^E_{K(\cdot,+\infty)}(P) = G\lambda_E(P)$$

for any $P \in \Gamma_n(D)$ and λ_E is concentrated on B_E, where

$$B_E = \{P \in \Gamma_n(D); E \text{ is not thin at } P\}$$

(see Brelot [6, Theorem VIII, 11] and Doob [9, XI. 14. Theorem (d)]). The (Green) energy $\gamma_D(E)$ of λ_E is defined by

$$\gamma_D(E) = \int_{\Gamma_n(D)} (G\lambda_E) d\lambda_E$$

(see Helms [12, p.223]). Let E be a Borel subset of $\Gamma_n(D)$ and $E_k = E \cap I_k(D)$ ($k = 0, 1, 2, \dots$), where

$$I_k(D) = \{(X, y) \in \Gamma_n(D); k \leq y < k + 1\}.$$

This paper is essentially based on the following Theorem 2.1 (Miyamoto [14, Theorem 1]), which gives not only a criterion of Wiener type ((*II*) of Theorem 2.1), but also another definition for a minimally thin set at $+\infty$ with respect to $\Gamma_n(D)$ ((*III*) of Theorem 2.1).

Theorem 2.1. *For a subset E of $\Gamma_n(D)$, the following statements are equivalent:*

(*I*) *E is minimally thin at $+\infty$ with respect to $\Gamma_n(D)$.*

(*II*) $\displaystyle\sum_{k=0}^{\infty} \gamma(E_k) e^{-2\sqrt{\tau_D} k} < \infty.$

(*III*) *There exists a positive superharmonic function $v(P)$ on $\Gamma_n(D)$ such that*

$$\inf_{P \in \Gamma_n(D)} \frac{v(P)}{K(P, +\infty)} = 0$$

and

$$E \subset M_v,$$

where

$$M_v = \{P \in \Gamma_n(D);\ v(P) \geq K(P, +\infty)\}.$$

3. Statements of results

We denote by $|E|$ the n-dimensional Lebesgue measure of E. The following Theorem 3.1 is the main theorem in this paper.

Theorem 3.1. *Let a Borel subset E of $\Gamma_n(D)$ be minimally thin at $+\infty$ with respect to $\Gamma_n(D)$. Then we have*

$$|E| < \infty. \tag{3.1}$$

To give the following Theorem 3.2, which shows that (3.1) characterizes the minimal thinness in the special case of E, we introduce the Whitney cubes of $\Gamma_n(D)$.

A cube is of the form

$$[l_1 2^{-k}, (l_1 + 1)2^{-k}] \times \cdots \times [l_n 2^{-k}, (l_n + 1)2^{-k}]$$

where $k, l_1, \dots, l_n$ are integers. When we denote by $M(\Gamma_n(D))$ the family of all cubes in $\Gamma_n(D)$, the Whitney cubes of $\Gamma_n(D)$ form a family of cubes W_j from $M(\Gamma_n(D))$ having the following properties:

(*i*) $\bigcup_j W_j = \Gamma_n(D)$,

(*ii*) $\text{int } W_j \cap \text{int } W_k = \emptyset \quad (j \neq k)$,

(*iii*) $\text{diam } W_j \leq \text{dist } (W_j, \mathbf{R}^n \setminus \Gamma_n(D)) \leq 4 \text{ diam } W_j$,

where int S, diam S, dist (S_1, S_2) stand for the interior of S, the diameter of S, the distance between S_1 and S_2, respectively (Stein [17, p.167, Theorem 1]).

Theorem 3.2. *If E is a union of Whitney cubes of $\Gamma_n(D)$, then* (3.1) *is also sufficient for E to be minimally thin at $+\infty$ with respect to $\Gamma_n(D)$.*

From Theorem 3.1, we obtain the following Theorems 3.3 and 3.4, which are similar to Theorems 1.1 and 1.2.

Theorem 3.3. *Let E be a Borel measurable subset of $\Gamma_n(D)$ satisfying*

$$|E| = \infty.$$

If $v(P)$ is a non-negative superharmonic function on $\Gamma_n(D)$ and m is a positive number such that $v(P) \geq mK(P, +\infty)$ for all $P \in E$, then $v(P) \geq mK(P, +\infty)$ for all $P \in \Gamma_n(D)$.

Theorem 3.4. *Let $v(P)$ be a positive superharmonic function on $\Gamma_n(D)$ such that*

$$\inf_{P \in \Gamma_n(D)} \frac{v(P)}{K(P, +\infty)} = 0.$$

Then we have

$$|M_v \cap \Gamma_n(D; 0, +\infty)| < \infty.$$

4. Lemmas and their proofs

For a function $F(P, Q)$ for any $P, Q \in \Gamma_n(D)$ and a positive measure μ on $\Gamma_n(D)$,

$$\int_{\Gamma_n(D)} F(P, Q) d\mu(Q)$$

is simply denoted by $F\mu(P)$. We shall also write $g_1 \approx g_2$ for two positive functions g_1 and g_2 if and only if there exists a positive constant a such that $a^{-1} g_1 \leq g_2 \leq a g_1$.

Let E be a Borel subset of $\Gamma_n(D)$, and let $\delta(P) = \mathrm{dist}(P, \partial\Gamma_n(D))$ for a point $P \in \Gamma_n(D)$. We define a measure σ_D on $\Gamma_n(D)$ by

$$\sigma_D(E) = \int_E \left(\frac{K(P, +\infty)}{\delta(P)} \right)^2 dP.$$

Lemma 4.1. *Let E be a bounded Borel subset of $\Gamma_n(D)$. Then there exists a constant M_1 independent of E such that*

$$\sigma_D(E) \leq M_1 \gamma_D(E).$$

Proof. First of all, we remark that the set $\mathbf{R}^n \setminus \Gamma_n(D)$ is $(1, 2)$ uniformly fat, i.e. there is a positive constant ι such that at any $P \in \mathbf{R}^n \setminus \Gamma_n(D)$,

$$\mathrm{Cap}\ (\{P + r^{-1}(Q - P) \in \mathbf{R}^n; Q \in B(P, r) \cap (\mathbf{R}^n \setminus \Gamma_n(D))\}) \geq \iota$$

for every positive number r, where $B(P, r) = \{Q \in \mathbf{R}^n : |Q - P| < r\}$ and Cap denotes the Newtonian capacity (see Lewis [13, p.178]). Then by a result of Lewis [13, Theorem 2], there is a positive constant M_1 depending only on ι and n such that

$$\int_{\Gamma_n(D)} \left| \frac{\psi(P)}{\delta(P)} \right|^2 dP \leq M_1 \int_{\Gamma_n(D)} |\nabla \psi(P)|^2 \, dP \tag{4.1}$$

for every $\psi \in C_0^\infty(\Gamma_n(D))$.

We denote the function $G\lambda_E(P) = \hat{R}^E_{K(\cdot, +\infty)}(P)$ on $\Gamma_n(D)$ by $v_E(P)$. It is well known that the Green energy can be represented as the Dirichlet integral, i.e.

$$\gamma_D(E) = \int_{\Gamma_n(D)} |\nabla v_E|^2 dP. \tag{4.2}$$

Since

(4.3)

$$A^{-1} e^{\sqrt{\tau_D} y} e^{-\sqrt{\tau_D} y'} f_D(X) f_D(X') \leq G(P, Q) \leq A e^{\sqrt{\tau_D} y} e^{-\sqrt{\tau_D} y'} f_D(X) f_D(X')$$

for any $P = (X, y) \in \Gamma_n(D)$ and $Q = (X', y') \in \Gamma_n(D)$ satisfying $y < y' - 1$, where A is a positive constant (see Yoshida [18]) and

$$f_D(X) \approx \delta(P) \tag{4.4}$$

for any $P = (X, y) \in \Gamma_n(D)$ (see Courant and Hilbert [7]), we also have

$$\int_{\Gamma_n(D)} \left| \frac{v_E(P)}{\delta(P)} \right|^2 dP < \infty. \tag{4.5}$$

Hence we have $v_E \in H(\Gamma_n(D))$ from (4.2) and (4.5), where

$$H(\Gamma_n(D)) = \{f \in L^2_{\text{loc}}(\Gamma_n(D)) : \nabla f \in L^2(\Gamma_n(D)), \delta^{-1} f \in L^2(\Gamma_n(D))\}$$

equipped with the norm

$$\|f\|_{H(\Gamma_n(D))} = \left(\|\nabla f\|^2_{L^2(\Gamma_n(D))} + \|\delta^{-1} f\|^2_{L^2(\Gamma_n(D))}\right)^{\frac{1}{2}},$$

and furthermore, $v_E \in H_0(\Gamma_n(D))$, where $H_0(\Gamma_n(D))$ denotes the closure of $C_0^\infty(\Gamma_n(D))$ in $H(\Gamma_n(D))$. Thus we obtain from (4.1) that

$$\int_{\Gamma_n(D)} \left| \frac{v_E(P)}{\delta(P)} \right|^2 dP \le M_1 \int_{\Gamma_n(D)} |\nabla v_E(P)|^2 \, dP$$

(see Ancona [3, p.288]). Since $v_E = K(\cdot, +\infty)$ quasi everywhere on E and hence *a.e.* on E, we have from (4.2),

$$\gamma_D(E) \ge M_1^{-1} \int_{\Gamma_n(D)} \left(\frac{v_E(P)}{\delta(P)} \right)^2 dP \ge M_1^{-1} \int_E \left(\frac{K(P, +\infty)}{\delta(P)} \right)^2 dP = M_1^{-1} \sigma_D(E),$$

which gives the conclusion.

Lemma 4.2. *Let W_j be any cube from the Whitney cubes of $\Gamma_n(D)$. Then there exists a constant M_2 independent of j such that*

$$\gamma_D(W_j) \le M_2 \sigma_D(W_j).$$

Proof. If we apply a standard result (cf. e.g. Theorem 5.6, p. 19 in Aikawa and Essén [2]) to a compact set $\overline{W}_j$, we obtain a measure μ on $\Gamma_n(D)$, supp $\mu \subset \overline{W}_j$, $\mu(\overline{W}_j) = 1$ such that

$$\begin{cases} \displaystyle\int_{\Gamma_n(D)} |P - Q|^{2-n} d\mu(Q) = \{\text{Cap}(\overline{W}_j)\}^{-1} & (n \ge 3), \\ \displaystyle\int_{\Gamma_2(D)} \log |P - Q| d\mu(Q) = \log \text{Cap}(\overline{W}_j) & (n = 2), \end{cases} \tag{4.6}$$

for any $P \in \overline{W}_j$. Also there exists a positive measure $\lambda_{\overline{W}_j}$ on $\Gamma_n(D)$ such that

$$\hat{R}^{\overline{W}_j}_{K(\cdot, +\infty)}(P) = G\lambda_{\overline{W}_j}(P) \quad (P \in \Gamma_n(D)). \tag{4.7}$$

Let $P_j = (X_j, y_j)$, ρ_j, t_j be the center of W_j, the diameter of W_j, the distance between W_j and $\partial\Gamma_n(D)$, respectively. Then we have $\rho_j \le t_j \le 4\rho_j$. For each W_j there exists an integer k_j such that the side-length of W_j is 2^{-k_j}. We can take an integer m_D such that $m_D = \min_j k_j$. So for any $Q = (X, y) \in \overline{W}_j$ we have

$$y_j - 2^{-m_D - 1} \le y \le y_j + 2^{-m_D - 1}.$$

Then from (4.4) we can find a positive constant A_1 independent of j such that

$$K(P,+\infty) \le A_1 e^{\sqrt{\tau_D} y_j} \rho_j \tag{4.8}$$

for any $P \in \overline{W}_j$. We can also prove that

$$G(P,Q) \ge \begin{cases} A_2 |P-Q|^{2-n} & (n \ge 3), \\ \log \frac{A_3 \rho_j}{|P-Q|} & (n=2), \end{cases} \tag{4.9}$$

for any $P \in \overline{W}_j$ and $Q \in \overline{W}_j$, where A_2 and A_3 are two positive constants independent of j. Hence we obtain

(4.10)

$$\lambda_{\overline{W}_j}(\Gamma_n(D)) \le \begin{cases} (A_1/A_2) e^{\sqrt{\tau_D} y_j} \rho_j \mathrm{Cap}(\overline{W}_j) & (n \ge 3), \\ A_1 e^{\sqrt{\tau_D} y_j} \rho_j \left\{ \log \frac{A_3 \rho_j}{\mathrm{Cap}(\overline{W}_j)} \right\}^{-1} & (n=2) \end{cases}$$

from (4.6), (4.7), (4.8) and (4.9). Since

$$\gamma_D(\overline{W}_j) = \int G \lambda_{\overline{W}_j} d\lambda_{\overline{W}_j} \le \int_{\overline{W}_j} K(P,\infty) d\lambda_{\overline{W}_j}(P) \le A_1 e^{\sqrt{\tau_D} y_j} \rho_j \lambda_{\overline{W}_j}(\Gamma_n(D))$$

from (4.7) and (4.8), we have from (4.10),

$$\gamma_D(\overline{W}_j) \le \begin{cases} A_1^2 A_2^{-1} e^{2\sqrt{\tau_D} y_j} \rho_j^2 \mathrm{Cap}(\overline{W}_j) & (n \ge 3), \\ A_1^2 e^{2\sqrt{\tau_D} y_j} \rho_j^2 \left\{ \log \frac{A_3 \rho_j}{\mathrm{Cap}(\overline{W}_j)} \right\}^{-1} & (n=2). \end{cases} \tag{4.11}$$

Since

$$\begin{cases} \mathrm{Cap}(\overline{W}_j) \approx \rho_j^{n-2} & (n \ge 3), \\ \mathrm{Cap}(\overline{W}_j) \approx \rho_j & (n=2), \end{cases}$$

we obtain from (4.11)

$$\gamma_D(W_j) \le A_4 e^{2\sqrt{\tau_D} y_j} \rho_j^n \tag{4.12}$$

with a positive constant A_4. On the other hand, we have from (4.4) that

$$\sigma_D(W_j) \approx e^{2\sqrt{\tau_D} y_j} \rho_j^n \tag{4.13}$$

for any $P = (X,y) \in W_j$. From (4.12) and (4.13) we finally have

$$\gamma_D(W_j) \le M_2 \sigma_D(W_j),$$

which is the conclusion of Lemma 2.

5. Proofs of theorems

Proof of Theorem 3.1. First of all we remark that

$$|E| = \sum_{k=0}^{\infty} |E_k|. \tag{5.1}$$

We have from (4.4),

$$A_5 \delta(P) \le f_D(X),$$

for any $P = (X,y) \in \Gamma_n(D)$, where A_5 is a positive constant. Hence

$$\begin{aligned} \sigma(E_k) \quad &= \int_{E_k} \left(\frac{K(P,+\infty)}{\delta(P)} \right)^2 dP \ge A_5^2 \int_{E_k} \left(\frac{e^{\sqrt{\tau_D} y} f_D(X)}{f_D(X)} \right)^2 dP \\ &= A_5^2 \int_{E_k} e^{2\sqrt{\tau_D} y} dP \ge A_5^2 e^{2\sqrt{\tau_D} k} |E_k|. \end{aligned}$$

By using Lemma 4.1, we obtain

$$\gamma_D(E_k) \geq M_1^{-1}\sigma_D(E_k) \geq A_6 e^{2\sqrt{\tau_D}k}|E_k|, \tag{5.2}$$

where A_6 is a positive constant.

If E is minimally thin at $+\infty$ with respect to $\Gamma_n(D)$, then by Theorem 2.1, (5.1) and (5.2), we have

$$|E| \leq |A_6^{-1}\sum_{k=0}^{\infty}\gamma_D(E_k)e^{-2\sqrt{\tau_D}k}| < \infty,$$

which is the conclusion of Theorem 3.1.

Proof of Theorem 3.2. Let $\{W_j\}$ be a family of Whitney cubes of $\Gamma_n(D)$ such that $E = \bigcup_j W_j$, and let $\{W_{k,i}\}$ be a subfamily of $\{W_j\}$ such that $W_{k,i} \subset (E_{k-1} \cup E_k \cup E_{k+1})$ $(k = 1, 2, \dots)$.

Since γ_D is a countably subadditive set function (Essén and Jackson [10, Lemma 2.1]), we have

$$\gamma_D(E_k) \leq \sum_i \gamma_D(W_{k,i}) \quad (k = 1, 2, \dots). \tag{5.3}$$

Hence we see from Lemma 4.2,

$$\sum_i \gamma_D(W_{k,i}) \leq M_2 \sum_i \sigma_D(W_{k,i}) \quad (k = 1, 2, \dots). \tag{5.4}$$

Since we see from (4.4) that

$$f_D(X) \leq A_7\delta(P)$$

for any $P = (X, y) \in \Gamma_n(D)$, where A_7 is a positive constant, we have

$$\begin{aligned}\sum_i \sigma_D(W_{k,i}) &\leq A_7^2\left(\int_{E_{k-1}} e^{2\sqrt{\tau_D}y}dP + \int_{E_k} e^{2\sqrt{\tau_D}y}dP + \int_{E_{k+1}} e^{2\sqrt{\tau_D}y}dP\right)\\ &\leq A_7^2 e^{2\sqrt{\tau_D}k}\left(|E_{k-1}| + e^{2\sqrt{\tau_D}}|E_k| + e^{4\sqrt{\tau_D}}|E_{k+1}|\right) \quad (k = 1, 2, \dots).\end{aligned} \tag{5.5}$$

Thus (5.3), (5.4) and (5.5) give

$$\gamma_D(E_k) \leq M_2 \cdot A_7^2 e^{2\sqrt{\tau_D}k}\left(|E_{k-1}| + e^{2\sqrt{\tau_D}}|E_k| + e^{4\sqrt{\tau_D}}|E_{k+1}|\right) \quad (k = 1, 2, \dots).$$

Finally we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\gamma_D(E_k)e^{-2\sqrt{\tau_D}k} &\leq M_2A_7^2\sum_{k=1}^{\infty}\left(|E_{k-1}| + e^{2\sqrt{\tau_D}}|E_k| + e^{4\sqrt{\tau_D}}|E_{k+1}|\right)\\ &\leq A_8|E| < \infty,\end{aligned}$$

where A_8 is a positive constant, which shows from Theorem 2.1 that E is minimally thin at $+\infty$ with respect to $\Gamma_n(D)$.

Proof of Theorem 3.3. Let E be a Borel measurable subset of $\Gamma_n(D)$, $v(P)$ be a positive superharmonic function on $\Gamma_n(D)$ and m be a positive number such that $v(P) \geq mK(P, +\infty)$ $(P \in E)$. We shall prove that if there exists $P_0 \in \Gamma_n(D)$ satisfying $v(P_0) < mK(P_0, +\infty)$, then $|E| < \infty$.

If we put

$$\inf_{P\in\Gamma_n(D)}\frac{v(P)}{K(P, +\infty)} = c_\infty(v)$$

and

$$u(P) = v(P) - c_\infty(v)K(P, +\infty),$$

then we have

$$\inf_{P \in \Gamma_n(D)} \frac{u(P)}{K(P, +\infty)} = 0.$$

Since $v(P_0) < mK(P_0, +\infty)$, we note that

$$c_\infty(u) < m.$$

Now we obtain

$$\begin{aligned} u(P) &\geq mK(P, +\infty) - c_\infty(v)K(P, +\infty) \\ &= (m - c_\infty(v))K(P, +\infty) \end{aligned}$$

for any $P \in E$. Hence by Theorem 2.1, E is minimally thin at $+\infty$ with respect to $\Gamma_n(D)$. Therefore from Theorem 3.1 we have $|E| < \infty$.

Proof of Theorem 3.4. We use Theorem 2.1 in the case where a set E in (III) is $M_v \cap \Gamma_n(D; 0, +\infty)$. Then we see that $M_v \cap \Gamma_n(D; 0, +\infty)$ is minimally thin at $+\infty$ with respect to $\Gamma_n(D)$. Hence by Theorem 3.1 we easily have the conclusion.

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