EXT VANISHING AND INFINITE AUSLANDER-BUCHSBAUM

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Abstract. A vanishing theorem is proved for Ext groups over non-commutative graded algebras. Along the way, an “infinite” version is proved of the non-commutative Auslander-Buchsbaum theorem.

0. Introduction

Let $R$ be a noetherian local commutative ring, and let $X$ be a finitely generated $R$-module of finite projective dimension. The classical Auslander-Buchsbaum theorem states

$$\text{pd } X = \text{depth } R - \text{depth } X.$$

This can also be phrased as an Ext vanishing theorem, namely, if $M$ is any $R$-module, then

$$\text{Ext}^i_R(X,M) = 0 \text{ for } i > \text{depth } R - \text{depth } X. \quad (1)$$

A surprising variation of this is proved in [1]: Suppose that $R$ is complete in the $m$-adic topology. Then equation (1) remains true if $X$ is any $R$-module of finite projective dimension, provided $M$ is finitely generated. In other words, the condition of being finitely generated is shifted from $X$ to $M$.

In Theorem 2.3 below, this result will be generalized to the situation of a non-commutative noetherian $\mathbb{N}$-graded connected algebra.

The route goes through an “infinite” version of the non-commutative Auslander-Buchsbaum theorem, given in Theorem 1.4. This result is a substantial improvement of the original non-commutative Auslander-Buchsbaum theorem, as given in [3, thm. 3.2], in that the condition of dealing only with finitely generated modules is dropped.

The notation of this paper is standard and is already on record in several places such as [2] or [3]. So I will not say much, except that throughout, $k$ is a field, and $A$ is a noetherian $\mathbb{N}$-graded connected $k$-algebra. However, let me give one important word of caution: Everything in sight is graded. So for instance, $\text{D}(A)$ stands for $\text{D}(\text{Gr}A)$, the derived category of the abelian category $\text{Gr}(A)$ of $\mathbb{Z}$-graded $A$-left-modules and graded homomorphisms of degree zero.

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Definition 1.1. For $X$ in $D(A)$, define the $k$-flat dimension by

$$k.fd X = -\inf k \otimes_A X.$$  

Remark 1.2. Using a minimal free resolution, it is easy to see that if the cohomology of $X$ is bounded and finitely generated, then

$$k.fd X = fd X = pd X,$$

where $fd$ stands for flat dimension and $pd$ stands for projective dimension.

In the following lemma, $D^b(A)$, the full subcategory of $D(A)$ consisting of complexes with bounded cohomology, and $D^+(A^{op})$, the full subcategory of $D(A^{op})$ consisting of complexes whose cohomology vanishes in low cohomological degrees, are used.

Lemma 1.3. Let $X$ in $D^b(A)$ have $fd X < \infty$, and let $T$ in $D^+(A^{op})$ be so that $h^i(T)$ is a graded torsion module for each $i$. Then

$$\inf T \otimes_A X = \inf T - k.fd X.$$  

Proof. Observe that $fd X < \infty$ implies

(2) \[ \inf k \otimes_A X > -\infty; \]

hence

(3) \[ k.fd X < \infty. \]

If $T$ is zero, then $\inf T = \inf T \otimes_A X = \infty$, and then the inequality (3) implies that the lemma’s equation trivially reads $\infty = \infty$. So for the rest of the proof let us assume that $T$ is non-zero and hence $\inf T < \infty$. Note that $T$ is in $D^+(A^{op})$, so $\inf T > -\infty$, so $\inf T$ is a finite number.

First, consider the special case where $T$ is concentrated in degree zero. Here $T$ is just a non-zero graded torsion $A$-right-module, and the lemma claims

(4) \[ \inf T \otimes_A X = -k.fd X = \inf k \otimes_A X. \]

Let us start by showing more modestly that

(5) \[ \inf T \otimes_A X \geq \inf k \otimes_A X. \]

If $F$ is a flat resolution of $X$, then this amounts to

(6) \[ \inf T \otimes_A F \geq \inf k \otimes_A F. \]

To prove this, note that since $T$ is a graded torsion module, it is the colimit of the system

$$T(1) \subseteq T(2) \subseteq \cdots$$

where

$$T(j) = \{ t \in T \mid A_{\geq j} t = 0 \}.$$  

Each quotient $T(j)/T(j-1)$ is annihilated by $A_{\geq 1}$ and so has the form $\prod_{\alpha} k(\ell_\alpha)$, so there are short exact sequences of the form

$$0 \to T(j-1) \to T(j) \to \prod_{\alpha} k(\ell_\alpha) \to 0.$$
Tensoring such a sequence with $F$ gives a short exact sequence of complexes, because $F$ consists of graded flat modules. The corresponding cohomology long exact sequence consists of pieces

$$h^i(T(j-1) \otimes_A F) \longrightarrow h^i(T(j) \otimes_A F) \longrightarrow \prod_{\alpha} h^i(k \otimes_A F)(\ell_\alpha).$$

Induction on $j$ now makes it clear that

$$h^i(k \otimes_A F) = 0$$

implies

$$h^i(T(j) \otimes_A F) = 0$$

for each $j$, and this further gives

$$h^i(T \otimes_A F) \cong h^i(\text{colim} T(j) \otimes_A F) \cong \text{colim} h^i(T(j) \otimes_A F) = 0,$$

so the inequality (5) follows, and hence, so does the inequality (6). Note that the proof even works for $\inf k \otimes_A X = \infty$.

Let us now step this up to show equation (4). Note that if $\inf k \otimes_A X = \infty$, then the inequality (5) forces

$$\inf T \otimes_A X = \infty,$$

and so equation (4) holds.

So let us assume that

$$\inf k \otimes_A X < \infty.$$ 

Because of the inequality (2), it follows that $\inf k \otimes_A X$ is a finite number. By the inequality (5), equation (4) will follow if it can be proved that

$$h^{\inf} k \otimes_A X (T \otimes_A X) \neq 0.$$

But since $T$ is non-zero and graded torsion, there is a short exact sequence $0 \rightarrow k(\ell) \rightarrow T \rightarrow \tilde{T} \rightarrow 0$ of graded $A$-right-modules. This gives a distinguished triangle $k(\ell) \rightarrow T \rightarrow \tilde{T} \rightarrow$ in $\mathcal{D}(A^{op})$, and tensoring with $X$ and taking the cohomology long exact sequence gives a sequence consisting of pieces

$$h^i(k(\ell) \otimes_A X) \rightarrow h^i(T \otimes_A X) \rightarrow h^i(T \otimes_A X),$$

Since $T$ is graded torsion, so is $\tilde{T}$. The inequality (5) applied to $\tilde{T}$ gives

$$h^i(\tilde{T} \otimes_A X) = 0$$

for $i < \inf k \otimes_A X$.

Hence there is a piece of the long exact sequence that reads

$$0 \rightarrow h^{\inf} k \otimes_A X (k(\ell) \otimes_A X) \rightarrow h^{\inf} k \otimes_A X (T \otimes_A X),$$

proving equation (7) and hence equation (4).

Secondly, consider the general case where $T$ is not necessarily concentrated in degree zero. There is a spectral sequence

$$E_2^{pq} = h^p(h^q(T) \otimes_A X) \Rightarrow h^{p+q}(T \otimes_A X),$$
which can be obtained as the second usual spectral sequence of the double complex defined by $M^{pq} = T^p \otimes_A F^q$, where $F$ is a flat resolution of $X$; cf. [1] thm. 11.19. The spectral sequence converges because $\text{fd} X < \infty$ implies that it is first quadrant up to shift. Now, $h^q(T)$ is graded torsion for each $q$, so if $h^q(T)$ is non-zero, then the special case of the lemma dealt with above applies to $h^q(T) \otimes_A X$ and shows that

(8)\[ \inf h^q(T) \otimes_A X = - \text{fd} X.\]

There are now two cases. The first case is

(9)\[ \text{fd} X = -\infty.\]

Here equation (3) gives that if $h^q(T)$ is non-zero, then $\inf h^q(T) \otimes_A X = \infty$, that is, $h^p(h^q(T) \otimes_A X)$ is zero for each $p$. Of course this also holds for $h^q(T)$ equal to zero, and so in the spectral sequence, $E_2^{pq}$ is identically zero. Therefore the limit $h^{p+q}(T \otimes_A X)$ of the spectral sequence is also zero, so $T \otimes_A X$ is zero, so

(10)\[ \inf T \otimes_A X = \infty.\]

But $\inf T$ is a finite number, and combining this with equations (9) and (10) says that the lemma’s equation reads

\[ \infty = (\text{a finite number}) - (-\infty),\]

which is true.

The second case is

(11)\[ \text{fd} X > -\infty.\]

Here equation (3) gives that if $h^q(T)$ is non-zero, then $h^p(h^q(T) \otimes_A X)$ is non-zero for $p = -\text{fd} X$, but zero for $p < -\text{fd} X$. Of course, if $h^q(T)$ is zero, then $h^p(h^q(T) \otimes_A X)$ is zero for each $p$. So in the spectral sequence, $E_2^{pq}$ is non-zero for $p = -\text{fd} X$ and $q = \inf T$, but zero for lower $p$ or $q$. Hence $E_2^{p, -\text{fd} X, \inf T}$ can be used in a standard corner argument which shows that the lowest non-zero term in the limit $h^{p+q}(T \otimes_A X)$ of the spectral sequence has degree $p+q = -\text{fd} X + \inf T$. Hence

\[ \inf T \otimes_A X = -\text{fd} X + \inf T,\]

proving the lemma’s equation. \hfill \Box

Observe that in the following theorem and the rest of the paper, depth $A$ stands for the depth of $A$ viewed as a left-module over itself.

**Theorem 1.4** (Infinite Auslander-Buchsbaum). Assume that $A$ satisfies that each $\text{Ext}_A^i(k, A)$ is a graded torsion $A$-right-module. Let $X$ in $D^b(A)$ have $\text{fd} X < \infty$. Then

\[ \text{depth} X = \text{depth} A - \text{fd} X.\]

**Proof.** I have

\[ \text{RHom}_A(k, X) \cong \text{RHom}_A(k, A \otimes_A X) \cong \text{RHom}_A(k, A) \otimes_A X,\]

where the second $\cong$ holds by [3] prop. 2.1 because $X$ is in $D^b(A)$ and has $\text{fd} X < \infty$. 

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Thus
\[
\text{depth } X = \inf \text{RHom}_A(k, X) \\
= \inf \text{RHom}_A(k, A) \otimes_A X \\
= \inf \text{RHom}_A(k, A) - \text{k.fd } X \\
= \text{depth } A - \text{k.fd } X,
\]
where (a) is by Lemma 1.3. The lemma applies because \( \text{RHom}_A(k, A) \) is in \( \text{D}^+(A^{\text{op}}) \), and has \( \text{h}^i \text{RHom}_A(k, A) = \text{Ext}_A^i(k, A) \) a graded torsion \( A \)-right-module for each \( i \) by assumption.

Remark 1.5. Theorem 1.4 even holds for depth \( A = \infty \), where the theorem states that depth \( X = \infty \).

On the other hand, suppose depth \( A < \infty \). Then it is easy to see that it makes sense to rearrange the equation in Theorem 1.4 as
\[
\text{k.fd } X = \text{depth } A - \text{depth } X.
\]

If the cohomology of \( X \) is bounded and finitely generated, then the equation of Theorem 1.4 reads
\[
\text{depth } X = \text{depth } A - \text{pd } X
\]
by Remark 1.2. This is the original non-commutative Auslander-Buchsbaum theorem, as proved in [3, thm. 3.2].

2. Ext vanishing

Lemma 2.1. Assume that \( A \) has depth \( A < \infty \) and satisfies that each \( \text{Ext}_A^i(k, A) \) is a graded torsion \( A \)-right-module.

Let \( X \) in \( \text{D}^b(A) \) have \( \text{fd } X < \infty \), and let \( T \) in \( \text{D}^+(A^{\text{op}}) \) be so that \( \text{h}^i(T) \) is a graded torsion module for each \( i \). Then
\[
\inf T \otimes_A X = \inf T + \text{depth } X - \text{depth } A.
\]

Proof. Using Lemma 1.3 and Remark 1.4 gives
\[
\inf T \otimes_A X = \inf T - \text{k.fd } X = \inf T + \text{depth } X - \text{depth } A.
\]
Proof. It is easy to see that since $M$ is in $D_{fg}^{-}(A)$, the Matlis dual $M'$ is in $D^{+}(A^{\text{op}})$ and has $h^{i}(M')$ a graded torsion module for each $i$. So

$$\sup R\text{Hom}_{A}(X, M) = \sup R\text{Hom}_{A}(X, M'') \quad (a) = \sup ((M' \otimes_{A} X)'') \quad (b)$$

$$= \inf M' \otimes_{A} X \quad (b) = \inf M' - \text{depth} X + \text{depth} A \quad (b) = \sup M - \text{depth} X + \text{depth} A,$$

where $(a)$ is by adjunction and $(b)$ is by Lemma 2.1. □

The following is the special case of Theorem 2.2 where $X$ and $M$ are concentrated in degree zero, that is, where $X$ and $M$ are graded modules.

**Theorem 2.3** (Ext vanishing). Assume that $A$ has depth $A < \infty$ and satisfies that each $\text{Ext}^{i}_{A}(k, A)$ is a graded torsion $A$-right-module.

Let $X$ in $\text{Gr}(A)$ have $\text{fd} X < \infty$, and let $M$ be in $\text{gr}(A)$. Then

$$\text{Ext}^{i}_{A}(X, M) = 0 \text{ for } i > \text{depth} A - \text{depth} X.$$

If depth $X < \infty$ and $M \neq 0$ also hold, then

$$\text{Ext}^{i}_{A}(X, M) \neq 0 \text{ for } i = \text{depth} A - \text{depth} X.$$

This says that for $\text{fd} X < \infty$, the number $\text{depth} A - \text{depth} X$ plays the role of projective dimension of $X$, but only with respect to finitely generated graded modules $M$.

Of course, this fails when $M$ is general, as illustrated by the following example.

**Example 2.4.** Let $A$ be the polynomial algebra $k[x]$. Then the conditions of Theorem 2.3 are satisfied, and it is classical that depth $A$ is 1.

Let $X$ be $k[x, x^{-1}]$. Then depth $X \geq 1$, because $X$ is a graded torsion free module, so depth $A - \text{depth} X \leq 0$ and Theorem 2.3 gives

$$\text{Ext}^{i}_{A}(X, M) = 0 \text{ for } i > 0$$

for $M$ in $\text{gr}(A)$.

However, this must fail when $M$ is general, for otherwise $X$ would be a projective object of $\text{Gr}(A)$, which it is certainly not.

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**References**


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