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EXT VANISHING AND INFINITE AUSLANDER-BUCHSBAUM

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Abstract. A vanishing theorem is proved for Ext groups over non-commutative graded algebras. Along the way, an "infinite" version is proved of the non-commutative Auslander-Buchsbaum theorem.

0. Introduction

Let R be a noetherian local commutative ring, and let X be a finitely generated R-module of finite projective dimension. The classical Auslander-Buchsbaum theorem states

$$\text{pd}\ X = \operatorname{depth} R - \operatorname{depth} X.$$

This can also be phrased as an Ext vanishing theorem, namely, if M is any R-module, then

$$\operatorname{Ext}^i_R(X, M) = 0 \quad \text{for} \quad i > \operatorname{depth} R - \operatorname{depth} X. \tag{1}$$

A surprising variation of this is proved in [1]: Suppose that R is complete in the $\mathfrak{m}$-adic topology. Then equation (1) remains true if X is *any* R-module of finite projective dimension, provided M is finitely generated. In other words, the condition of being finitely generated is shifted from X to M.

In Theorem 2.3 below, this result will be generalized to the situation of a non-commutative noetherian $\mathbb{N}$-graded connected algebra.

The route goes through an "infinite" version of the non-commutative Auslander-Buchsbaum theorem, given in Theorem 1.4. This result is a substantial improvement of the original non-commutative Auslander-Buchsbaum theorem, as given in [3, thm. 3.2], in that the condition of dealing only with finitely generated modules is dropped.

The notation of this paper is standard and is already on record in several places such as [2] or [3]. So I will not say much, except that throughout, k is a field, and A is a noetherian $\mathbb{N}$-graded connected k-algebra. However, let me give one important word of caution: Everything in sight is graded. So for instance, $\mathsf{D}(A)$ stands for $\mathsf{D}(\mathsf{Gr}\,\mathsf{A})$, the derived category of the abelian category $\mathsf{Gr}(A)$ of $\mathbb{Z}$-graded A-left-modules and graded homomorphisms of degree zero.

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1. Auslander-Buchsbaum

Definition 1.1. For X in $\mathsf{D}(A)$, define the k-flat dimension by

$$\operatorname{k.fd} X \quad = \quad -\inf k \overset{\mathrm{L}}{\otimes}_A X.$$

Remark 1.2. Using a minimal free resolution, it is easy to see that if the cohomology of X is bounded and finitely generated, then

$$\operatorname{k.fd} X = \operatorname{fd} X = \operatorname{pd} X,$$

where fd stands for flat dimension and pd stands for projective dimension.

In the following lemma, $\mathsf{D}^{\mathrm{b}}(A)$, the full subcategory of $\mathsf{D}(A)$ consisting of complexes with bounded cohomology, and $\mathsf{D}^{+}(A^{\mathrm{op}})$, the full subcategory of $\mathsf{D}(A^{\mathrm{op}})$ consisting of complexes whose cohomology vanishes in low cohomological degrees, are used.

Lemma 1.3. *Let X in $\mathsf{D}^{\mathrm{b}}(A)$ have $\operatorname{fd} X < \infty$, and let T in $\mathsf{D}^{+}(A^{\mathrm{op}})$ be so that $\mathrm{h}^i(T)$ is a graded torsion module for each i. Then*

$$\inf T \overset{\mathrm{L}}{\otimes}_A X = \inf T - \operatorname{k.fd} X.$$

Proof. Observe that $\operatorname{fd} X < \infty$ implies

$$\inf k \overset{\mathrm{L}}{\otimes}_A X > -\infty; \tag{2}$$

hence

$$\operatorname{k.fd} X < \infty. \tag{3}$$

If T is zero, then $\inf T = \inf T \overset{\mathrm{L}}{\otimes}_A X = \infty$, and then the inequality (3) implies that the lemma's equation trivially reads $\infty = \infty$. So for the rest of the proof let us assume that T is non-zero and hence $\inf T < \infty$. Note that T is in $\mathsf{D}^{+}(A^{\mathrm{op}})$, so $\inf T > -\infty$, so $\inf T$ is a finite number.

First, consider the special case where T is concentrated in degree zero. Here T is just a non-zero graded torsion A-right-module, and the lemma claims

$$\inf T \overset{\mathrm{L}}{\otimes}_A X = -\operatorname{k.fd} X = \inf k \overset{\mathrm{L}}{\otimes}_A X. \tag{4}$$

Let us start by showing more modestly that

$$\inf T \overset{\mathrm{L}}{\otimes}_A X \geq \inf k \overset{\mathrm{L}}{\otimes}_A X. \tag{5}$$

If F is a flat resolution of X, then this amounts to

$$\inf T \otimes_A F \geq \inf k \otimes_A F. \tag{6}$$

To prove this, note that since T is a graded torsion module, it is the colimit of the system

$$T\langle 1\rangle \subseteq T\langle 2\rangle \subseteq \cdots$$

where

$$T\langle j\rangle = \{\, t \in T \mid A_{\geq j} t = 0 \,\}.$$

Each quotient $T\langle j\rangle / T\langle j-1\rangle$ is annihilated by $A_{\geq 1}$ and so has the form $\coprod_\alpha k(\ell_\alpha)$, so there are short exact sequences of the form

$$0 \to T\langle j-1\rangle \longrightarrow T\langle j\rangle \longrightarrow \coprod_\alpha k(\ell_\alpha) \to 0.$$

Tensoring such a sequence with F gives a short exact sequence of complexes, because F consists of graded flat modules. The corresponding cohomology long exact sequence consists of pieces

$$\mathrm{h}^i(T\langle j-1\rangle \otimes_A F) \longrightarrow \mathrm{h}^i(T\langle j\rangle \otimes_A F) \longrightarrow \coprod_\alpha \mathrm{h}^i(k \otimes_A F)(\ell_\alpha).$$

Induction on j now makes it clear that

$$\mathrm{h}^i(k \otimes_A F) = 0$$

implies

$$\mathrm{h}^i(T\langle j\rangle \otimes_A F) = 0 \text{ for each } j,$$

and this further gives

$$\mathrm{h}^i(T \otimes_A F) \cong \mathrm{h}^i(\operatorname{colim} T\langle j\rangle \otimes_A F) \cong \operatorname{colim} \mathrm{h}^i(T\langle j\rangle \otimes_A F) = 0,$$

so the inequality (6) follows, and hence, so does the inequality (5). Note that the proof even works for $\inf k \overset{\mathrm{L}}{\otimes}_A X = \infty$.

Let us now step this up to show equation (4). Note that if

$$\inf k \overset{\mathrm{L}}{\otimes}_A X = \infty,$$

then the inequality (5) forces

$$\inf T \overset{\mathrm{L}}{\otimes}_A X = \infty,$$

and so equation (4) holds.

So let us assume that

$$\inf k \overset{\mathrm{L}}{\otimes}_A X < \infty.$$

Because of the inequality (2), it follows that $\inf k \overset{\mathrm{L}}{\otimes}_A X$ is a finite number. By the inequality (5), equation (4) will follow if it can be proved that

$$\mathrm{h}^{\inf k \overset{\mathrm{L}}{\otimes}_A X}(T \overset{\mathrm{L}}{\otimes}_A X) \neq 0. \tag{7}$$

But since T is non-zero and graded torsion, there is a short exact sequence $0 \to k(\ell) \longrightarrow T \longrightarrow \tilde{T} \to 0$ of graded A-right-modules. This gives a distinguished triangle $k(\ell) \longrightarrow T \longrightarrow \tilde{T} \longrightarrow$ in $\mathsf{D}(A^{\mathrm{op}})$, and tensoring with X and taking the cohomology long exact sequence gives a sequence consisting of pieces

$$\mathrm{h}^i(k(\ell) \overset{\mathrm{L}}{\otimes}_A X) \longrightarrow \mathrm{h}^i(T \overset{\mathrm{L}}{\otimes}_A X) \longrightarrow \mathrm{h}^i(\tilde{T} \overset{\mathrm{L}}{\otimes}_A X).$$

Since T is graded torsion, so is $\tilde{T}$. The inequality (5) applied to $\tilde{T}$ gives

$$\mathrm{h}^i(\tilde{T} \overset{\mathrm{L}}{\otimes}_A X) = 0 \ \text{ for } \ i < \inf k \overset{\mathrm{L}}{\otimes}_A X.$$

Hence there is a piece of the long exact sequence that reads

$$0 \longrightarrow \mathrm{h}^{\inf k \overset{\mathrm{L}}{\otimes}_A X}(k(\ell) \overset{\mathrm{L}}{\otimes}_A X) \longrightarrow \mathrm{h}^{\inf k \overset{\mathrm{L}}{\otimes}_A X}(T \overset{\mathrm{L}}{\otimes}_A X),$$

proving equation (7) and hence equation (4).

Secondly, consider the general case where T is not necessarily concentrated in degree zero. There is a spectral sequence

$$E_2^{pq} = \mathrm{h}^p(\mathrm{h}^q(T) \overset{\mathrm{L}}{\otimes}_A X) \Rightarrow \mathrm{h}^{p+q}(T \overset{\mathrm{L}}{\otimes}_A X),$$

which can be obtained as the second usual spectral sequence of the double complex defined by $M^{pq} = T^p \otimes_A F^q$, where F is a flat resolution of X; cf. [4, thm. 11.19]. The spectral sequence converges because $\operatorname{fd} X < \infty$ implies that it is first quadrant up to shift. Now, $\mathrm{h}^q(T)$ is graded torsion for each q, so if $\mathrm{h}^q(T)$ is non-zero, then the special case of the lemma dealt with above applies to $\mathrm{h}^q(T) \overset{\mathrm{L}}{\otimes}_A X$ and shows that

$$\inf \mathrm{h}^q(T) \overset{\mathrm{L}}{\otimes}_A X = -\operatorname{k.fd} X. \tag{8}$$

There are now two cases. The first case is

$$\operatorname{k.fd} X = -\infty. \tag{9}$$

Here equation (8) gives that if $\mathrm{h}^q(T)$ is non-zero, then $\inf \mathrm{h}^q(T) \overset{\mathrm{L}}{\otimes}_A X = \infty$, that is, $\mathrm{h}^p(\mathrm{h}^q(T) \overset{\mathrm{L}}{\otimes}_A X)$ is zero for each p. Of course this also holds for $\mathrm{h}^q(T)$ equal to zero, and so in the spectral sequence, E_2^{pq} is identically zero. Therefore the limit $\mathrm{h}^{p+q}(T \overset{\mathrm{L}}{\otimes}_A X)$ of the spectral sequence is also zero, so $T \overset{\mathrm{L}}{\otimes}_A X$ is zero, so

$$\inf T \overset{\mathrm{L}}{\otimes}_A X = \infty. \tag{10}$$

But $\inf T$ is a finite number, and combining this with equations (9) and (10) says that the lemma's equation reads

$$\infty = (\text{a finite number}) - (-\infty),$$

which is true.

The second case is

$$\operatorname{k.fd} X > -\infty.$$

Here equation (8) gives that if $\mathrm{h}^q(T)$ is non-zero, then $\mathrm{h}^p(\mathrm{h}^q(T) \overset{\mathrm{L}}{\otimes}_A X)$ is non-zero for $p = -\operatorname{k.fd} X$, but zero for $p < -\operatorname{k.fd} X$. Of course, if $\mathrm{h}^q(T)$ is zero, then $\mathrm{h}^p(\mathrm{h}^q(T) \overset{\mathrm{L}}{\otimes}_A X)$ is zero for each p. So in the spectral sequence, E_2^{pq} is non-zero for $p = -\operatorname{k.fd} X$ and $q = \inf T$, but zero for lower p or q. Hence $E_2^{-\operatorname{k.fd} X, \inf T}$ can be used in a standard corner argument which shows that the lowest non-zero term in the limit $\mathrm{h}^{p+q}(T \overset{\mathrm{L}}{\otimes}_A X)$ of the spectral sequence has degree $p+q = -\operatorname{k.fd} X + \inf T$. Hence

$$\inf T \overset{\mathrm{L}}{\otimes}_A X = -\operatorname{k.fd} X + \inf T,$$

proving the lemma's equation. □

Observe that in the following theorem and the rest of the paper, depth A stands for the depth of A viewed as a left-module over itself.

Theorem 1.4 (Infinite Auslander-Buchsbaum)**.** *Assume that A satisfies that each $\operatorname{Ext}^i_A(k, A)$ is a graded torsion A-right-module. Let X in $\mathsf{D}^{\mathrm{b}}(A)$ have $\operatorname{fd} X < \infty$. Then*

$$\operatorname{depth} X = \operatorname{depth} A - \operatorname{k.fd} X.$$

Proof. I have

$$\operatorname{RHom}_A(k, X) \cong \operatorname{RHom}_A(k, A \overset{\mathrm{L}}{\otimes}_A X) \cong \operatorname{RHom}_A(k, A) \overset{\mathrm{L}}{\otimes}_A X,$$

where the second $\cong$ holds by [3, prop. 2.1] because X is in $\mathsf{D}^{\mathrm{b}}(A)$ and has $\operatorname{fd} X < \infty$.

Thus

$$\begin{aligned}\operatorname{depth} X &= \inf \operatorname{RHom}_A(k, X) \\ &= \inf \operatorname{RHom}_A(k, A) \overset{\mathrm{L}}{\otimes}_A X \\ &\overset{(a)}{=} \inf \operatorname{RHom}_A(k, A) - \operatorname{k.fd} X \\ &= \operatorname{depth} A - \operatorname{k.fd} X,\end{aligned}$$

where (a) is by Lemma 1.3. The lemma applies because $\operatorname{RHom}_A(k, A)$ is in $\mathsf{D}^+(A^{\mathrm{op}})$, and has $\mathrm{h}^i\operatorname{RHom}_A(k, A) = \operatorname{Ext}^i_A(k, A)$ a graded torsion A-right-module for each i by assumption. □

Remark 1.5. Theorem 1.4 even holds for $\operatorname{depth} A = \infty$, where the theorem states that $\operatorname{depth} X = \infty$.

On the other hand, suppose $\operatorname{depth} A < \infty$. Then it is easy to see that it makes sense to rearrange the equation in Theorem 1.4 as

$$\operatorname{k.fd} X = \operatorname{depth} A - \operatorname{depth} X.$$

If the cohomology of X is bounded and finitely generated, then the equation of Theorem 1.4 reads

$$\operatorname{depth} X = \operatorname{depth} A - \operatorname{pd} X$$

by Remark 1.2. This is the original non-commutative Auslander-Buchsbaum theorem, as proved in [3, thm. 3.2].

2. Ext vanishing

Lemma 2.1. *Assume that A has $\operatorname{depth} A < \infty$ and satisfies that each $\operatorname{Ext}^i_A(k, A)$ is a graded torsion A-right-module.*

Let X in $\mathsf{D}^{\mathrm{b}}(A)$ have $\operatorname{fd} X < \infty$, and let T in $\mathsf{D}^+(A^{\mathrm{op}})$ be so that $\mathrm{h}^i(T)$ is a graded torsion module for each i. Then

$$\inf T \overset{\mathrm{L}}{\otimes}_A X = \inf T + \operatorname{depth} X - \operatorname{depth} A.$$

Proof. Using Lemma 1.3 and Remark 1.5 gives

$$\inf T \overset{\mathrm{L}}{\otimes}_A X = \inf T - \operatorname{k.fd} X = \inf T + \operatorname{depth} X - \operatorname{depth} A.$$

□

In the following theorem, $\mathsf{D}^-_{\mathrm{fg}}(A)$, the full subcategory of $\mathsf{D}(A)$ consisting of complexes whose cohomology vanishes in high cohomological degrees and consists of finitely generated graded modules, is used.

Theorem 2.2. *Assume that A has $\operatorname{depth} A < \infty$ and satisfies that each $\operatorname{Ext}^i_A(k, A)$ is a graded torsion A-right-module.*

Let X in $\mathsf{D}^{\mathrm{b}}(A)$ have $\operatorname{fd} X < \infty$, and let M be in $\mathsf{D}^-_{\mathrm{fg}}(A)$. Then

$$\sup \operatorname{RHom}_A(X, M) = \sup M - \operatorname{depth} X + \operatorname{depth} A.$$

Proof. It is easy to see that since M is in $\mathsf{D}^{-}_{\mathrm{fg}}(A)$, the Matlis dual M' is in $\mathsf{D}^{+}(A^{\mathrm{op}})$ and has $\mathrm{h}^i(M')$ a graded torsion module for each i. So

$$\begin{aligned} \sup \mathrm{RHom}_A(X, M) &= \sup \mathrm{RHom}_A(X, M'') \\ &\overset{(a)}{=} \sup((M' \overset{\mathrm{L}}{\otimes}_A X)') \\ &= -\inf\, M' \overset{\mathrm{L}}{\otimes}_A X \\ &\overset{(b)}{=} -\inf\, M' - \operatorname{depth} X + \operatorname{depth} A \\ &= \sup\, M - \operatorname{depth} X + \operatorname{depth} A, \end{aligned}$$

where (a) is by adjunction and (b) is by Lemma 2.1. □

The following is the special case of Theorem 2.2 where X and M are concentrated in degree zero, that is, where X and M are graded modules.

Theorem 2.3 (Ext vanishing)**.** *Assume that A has* $\operatorname{depth} A < \infty$ *and satisfies that each* $\mathrm{Ext}^i_A(k, A)$ *is a graded torsion A-right-module.*

Let X in $\mathsf{Gr}(A)$ have $\operatorname{fd} X < \infty$*, and let M be in $\mathsf{gr}(A)$. Then*

$$\mathrm{Ext}^i_A(X, M) = 0 \;\; \textit{for} \;\; i > \operatorname{depth} A - \operatorname{depth} X.$$

If $\operatorname{depth} X < \infty$ *and* $M \neq 0$ *also hold, then*

$$\mathrm{Ext}^i_A(X, M) \neq 0 \;\; \textit{for} \;\; i = \operatorname{depth} A - \operatorname{depth} X.$$

This says that for $\operatorname{fd} X < \infty$, the number $\operatorname{depth} A - \operatorname{depth} X$ plays the role of projective dimension of X, but only with respect to finitely generated graded modules M.

Of course, this fails when M is general, as illustrated by the following example.

Example 2.4. Let A be the polynomial algebra $k[x]$. Then the conditions of Theorem 2.3 are satisfied, and it is classical that $\operatorname{depth} A$ is 1.

Let X be $k[x, x^{-1}]$. Then $\operatorname{depth} X \geq 1$, because X is a graded torsion free module, so $\operatorname{depth} A - \operatorname{depth} X \leq 0$ and Theorem 2.3 gives

$$\mathrm{Ext}^i_A(X, M) = 0 \;\; \text{for} \;\; i > 0$$

for M in $\mathsf{gr}(A)$.

However, this must fail when M is general, for otherwise X would be a projective object of $\mathsf{Gr}(A)$, which it is certainly not.

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