SOME REMARKS ON AN EXISTENCE PROBLEM FOR DEGENERATE ELLIPTIC SYSTEMS

OLLI MARTIO, VLADIMIR MIKLYUKOV, AND MATTI VUORINEN

(Communicated by Richard A. Wentworth)

Abstract. The system $au_x + bu_y = v_y$, $cu_x + du_y = -v_x$, which yields Beltrami's system if $b = c$, is considered. Under a condition for the coefficients $a, b, c, d$ a non-existence theorem is proved.

1. Main results

Let $D, D \subset \mathbb{R}^2$ be simply connected domains of the $z = (x, y)$ and the $w = (u, v)$ planes, respectively. Below we consider the problem of existence of homeomorphic $ACL^2_{\text{loc}}$ mappings $w = w(z)$ from $D$ onto $D$ satisfying the system

\begin{equation}
au_x + bu_y = v_y, \quad cu_x + du_y = -v_x.
\end{equation}

A function $h$ is, by definition, in the class $ACL^2_{\text{loc}}$ iff $h$ is absolutely continuous along a.e. horizontal and vertical lines and its partial derivatives $h_x, h_y$ belong to $L^2_{\text{loc}}$. Here the coefficients $a, b, c, d$ are measurable functions in $D$. We set

$$
\delta \equiv ad - (b + c)^2/4.
$$

In what follows we assume that $a > 0$ and

\begin{equation}
\text{ess inf}_D (\delta) > 0 \quad \text{for every} \quad D' \subset\subset D.
\end{equation}

In the particular case $a = d = 1$ and $b = c = 0$, (1.1) is the classic Cauchy-Riemann system and the solution of the existence problem is given by the Riemann mapping theorem. In the case $b = c$, we have the well-known Beltrami system. For more on the existence theorems given here see, for example, G. David [5], M.A. Brakalova and J.A. Jenkins [2], U. Srebro and E. Yakubov [14], V. Gutiérras, O. Martio, T. Sugawa and M. Vuorinen [8], O. Martio and V. M. Miklyukov [10].

In the general case, the global mapping problem is much more complicated than in the aforementioned particular cases. Under the assumption

\begin{equation}
\text{ess inf}_D \delta > 0,
\end{equation}

the existence problem was solved by B. Bojarski [11]. If this condition is not fulfilled, then there are isolated results only. The case in which (1.3) is violated at a finite number of boundary points was considered by A. Džuraev [6] and E.A. Chlcherbakov [3]. On the other hand, I.S. Ovchinnikov [13] and A.P. Mihailov [12] proved some results pertaining to the solvability problem of the system...
with \((1.2)\) under some special conditions, which allow degeneration close to a boundary arc.

Here we give a condition for the coefficients of \((1.1)\) under which the coordinate function \(u(z)\) of \(w(z)\) is monotone close to the boundary \([11]\). We use this condition for a non-existence theorem.

Let \(D \subset \mathbb{R}^2\) be a domain. By \(\partial D\) we denote the boundary of \(D\) in the extended plane \(\mathbb{R}^2 = \mathbb{R}^2 \cup \{\infty\}\). For an arbitrary subdomain \(\Delta \subset D\), we set

\[
\partial' \Delta = \partial \Delta \setminus \partial D \quad \text{and} \quad \partial'' \Delta = \partial \Delta \cap \partial D.
\]

Let \(\Gamma\) be a subset of \(\partial D\). A continuous function \(f : D \to \mathbb{R}\) is called monotone close to \(\Gamma\) if for every subdomain \(\Delta \subset D\) with \(\partial'' \Delta \subset \Gamma\),

\[
osc(f, \Delta) \leq osc(f, \partial' \Delta).
\]

Here the symbol \(osc(f, E)\) means the oscillation of \(f\) along the set \(E\).

Every function \(f\) monotone close to \(\Gamma\) is monotone in the classical sense of Lebesgue since for domains \(\Delta \subset \subset D\), \((1.1)\) reduces to \(osc(f, \Delta) \leq osc(f, \partial \Delta)\). On the other hand, if \(\Gamma = \partial D\), then every function monotone close to \(\Gamma\) is a constant function. This is obvious since choosing \(\Delta = D \setminus \{z_0\}\), where \(z_0 \in D\) is an arbitrary point, by \((1.4)\) we obtain

\[
osc(f, \Delta) \leq osc(f, \{z_0\}) = 0.
\]

We shall describe the behaviour of the coefficients of the system close to a set of degeneracy by means of a special exhaustion function \(h\). Fix a set \(\Gamma \subset \partial D\) and a positive locally Lipschitz function \(h : D \to \mathbb{R}\) such that \(\lim_{z \to \Gamma} h(z) = 0\) and

\[
0 < h_0 \leq \text{ess inf}_D |\nabla h(z)| \leq \text{ess sup}_D |\nabla h(z)| \leq h_1 < \infty,
\]

where \(h_0, h_1\) are some constants.

Below we let \(E_t = \{z \in D : h(z) = t\}\) denote the level curve of \(h\).

**Theorem 1.6.** Let \(\Gamma \subset \partial D\) be an arbitrary set, and let \(w = (u, v)\) be an ACL$^2_{loc}$ homeomorphic solution of \((1.1)\) from \(D\) onto \(D\) satisfying \((1.2)\) such that \(|u| < M\) in \(D\). If

\[
\int_0^1 dt \left( \int_{D \cap E_t} (a + d) \frac{ad - bc}{\delta} d\mathcal{H}^1(E_t) \right)^{-1} = \infty,
\]

then \(u\) is monotone close to \(\Gamma\).

For a homeomorphism \(w : D \to D\) and for an arbitrary \(\Gamma \subset \partial D\) we set

\[w(\Gamma) = \{y \in \partial D : \exists \text{ a sequence } z_n \in D, z_n \to \Gamma, \text{ such that } w(z_n) \to y\} .\]

Clearly, if \(\Gamma \subset \partial D\) is connected and \(\partial D\) is a simple Jordan curve, then \(w(\Gamma)\) is also connected.

A set \(L \subset \partial D\) is called \(u\)-forked if there are at least two points \(w' = (u', v'), \ w'' = (u'', v'') \in L\) such that the segment

\[l = \{(u, v) \in \mathbb{R}^2 : u = u', v' < v < v''\}\]

lies in \(D\) and separates from \(\partial D \setminus L\) some subdomain \(U \subset D\) with \(\partial U \subset \Gamma \cup L\).

For example, let \(D\) be a disk and let \(\partial D\) be its boundary circle. Then the right and left semicircles are \(u\)-forked; however, the upper and lower semicircles are not \(u\)-forked.
Theorem 1.8. Let $M > 0$ be a constant. Suppose that $D$ is a subdomain of \{(u, v) : |u| < M\}, the coefficients of $1.1$ satisfy $1.7$ and $L \subset \partial D$ is $u$-forked. Then there is no $\text{ACL}^2_{\text{loc}}$ homeomorphic solution $w = w(z)$ of $1.1$ from $D$ onto $D$ such that $w(\Gamma) \supset L$.

2. Proof of Theorem 1.6

Let $\Delta$ be a subdomain of $D$ with $\partial'\Delta \subset \Gamma$. We shall prove that
\[(2.1) \sup_{\Delta} u(z) = \sup_{\partial'\Delta} u(z).\]
Assume the contrary, that is, there exists a point $z_0 \in \Delta$ such that
\[u(z_0) > \sup_{\partial'\Delta} u(z) = A.\]
Choose $\epsilon > A$ such that $u(z_0) > \epsilon$. Fix the connected component $U$ of the set $\{z \in \Delta : u(z) > \epsilon\}$ containing $z_0$. By [15, Theorem 5.4.4] for almost all $\epsilon > A$, the sets $\{z \in \Delta : u(z) = \epsilon\}$ are locally rectifiable. Therefore, without loss of generality, we may assume that $\partial'U$ is locally rectifiable.

Fix numbers $0 < \delta' < \delta'' < h(z_0)$ and an absolutely continuous function $\psi_0 : [\delta', \delta''] \to [0, 1]$ such that $\psi_0(\delta'') = 1$ and $\psi_0(\delta') = 0$. We shall specify $\psi_0$ later.
Define $\psi : (0, \infty) \to \mathbb{R}$ as
\[\psi(\tau) = \begin{cases} 1 & \text{for } \delta'' < \tau < \infty, \\ \psi_0(\tau) & \text{for } \delta' \leq \tau \leq \delta'', \\ 0 & \text{for } 0 < \tau < \delta'. \end{cases}\]
Then $\psi$ is an absolutely continuous function. Write $\phi(z) = \psi(h(z))^2 (u(z) - \epsilon)$ for $z \in U$ and $\phi \equiv 0$ for $z \in D \setminus U$. Using [9, Theorem 1.20] we conclude that $\phi \in \text{ACL}^2_{\text{loc}}(D)$. Because $\text{supp } \phi \subset \subset D$ we have by Green’s formula
\[\int_U d(\phi dv) = 0\]
and hence
\[\int_U d\phi \wedge dv = 0.\]
Because
\[d\phi \wedge dv = \psi^2 du \wedge dv + 2\psi (u - \epsilon) d\psi \wedge dv,\]
we obtain
\[\int_U \psi^2 (u_x v_y - u_y v_x) dxdy = -2 \int_U \psi (u - \epsilon) (\psi_x v_y - \psi_y v_x) dxdy.\]
This is clear for $u, v \in C^2(D)$. In the general case we can easily prove it if we use a standard approximation procedure (see, for example, [9, Lemma 14.15]).

From here
\[\int_U \psi^2 (u_x v_y - u_y v_x) dxdy \leq 2 \int_U |u - \epsilon| |\nabla \psi| |\nabla v| dxdy.\]
Using (1.1) we find
\[
\int_U \psi^2 \left( a u_x^2 + (b + c) u_x u_y + d u_y^2 \right) \, dxdy \\
\leq 4M \int_U \psi |\nabla \psi| \sqrt{(a u_x + bu_y)^2 + (cu_x + du_y)^2} \, dxdy
\]
\[
= 4M \int_U \psi |\nabla \psi| \sqrt{(a^2 + c^2)u_x^2 + 2(ab + cd)u_x u_y + (b^2 + d^2)u_y^2} \, dxdy
\]
\[
\leq 4M \left( \int_U |\nabla \psi|^2 \frac{(a^2 + c^2)u_x^2 + 2(ab + cd)u_x u_y + (b^2 + d^2)u_y^2}{au_x^2 + (b + c)u_x u_y + du_y^2} \, dxdy \right)^{1/2}
\times \left( \int_U \psi^2 (au_x^2 + (b + c)u_x u_y + du_y^2) \, dxdy \right)^{1/2}.
\]
Thus we obtain
\[
\int_U \psi^2 \left( a u_x^2 + (b + c) u_x u_y + d u_y^2 \right) \, dxdy \\
\leq 16M^2 \int_U |\nabla \psi|^2 \frac{(a^2 + c^2)u_x^2 + 2(ab + cd)u_x u_y + (b^2 + d^2)u_y^2}{au_x^2 + (b + c)u_x u_y + du_y^2} \, dxdy.
\]
We write
\[
I(\xi, \eta) = (a^2 + c^2)\xi^2 + 2(ab + cd)\xi\eta + (b^2 + d^2)\eta^2
\]
and
\[
II(\xi, \eta) = (a\xi^2 + (b + c)\xi\eta + d\eta^2).
\]
From (1.2) it follows that the quadratic form \( I(\xi, \eta) \) is positive definite, and hence the bundle of quadratic forms
\[
H(\xi, \eta; \lambda) = I(\xi, \eta) - \lambda II(\xi, \eta)
\]
is regular in the sense of [7, Chapter X].
Let \( \lambda_{\text{max}} \) be a maximal eigenvalue of \( H(\xi, \eta; \lambda) \). Then using well-known properties of quadratic forms (see [7, Theorem 13, Chapter X]) for every \( \xi, \eta \neq 0 \), we have
\[
\frac{I(\xi, \eta)}{II(\xi, \eta)} \leq \lambda_{\text{max}}.
\]
The characteristic equation of \( H(\xi, \eta; \lambda) \) has the form
\[
\begin{vmatrix}
    a^2 + c^2 - \lambda a & ab + cd - \lambda(b + c)/2 \\
    ab + cd - \lambda(b + c)/2 & b^2 + d^2 - \lambda d
\end{vmatrix} = 0,
\]
i.e.,
\[
\delta \lambda^2 - (a + d)(ad - bc) \lambda + (ad - bc)^2 = 0.
\]
Solving this equation for $\lambda$ and using (1.2) we find that

$$
\lambda_{\text{max}} \leq \mu \equiv \frac{(ad - bc)(a + d)}{\delta}.
$$

Clearly, $ad - bc > 0$ since from (1.2) it follows that

$$
0 < 4ad - (b + c)^2 \leq 4ad - 4bc.
$$

By (2.3) for every $(x, y) \in U$, we obtain

$$
(2.4) \quad \frac{(a^2 + c^2)u_x^2 + 2(ab + cd)u_xu_y + (b^2 + d^2)u_y^2}{au_x^2 + (b + c)u_xu_y + du_y^2} \leq \mu.
$$

Thus from (2.2) we arrive at the estimate

$$
(2.5) \quad \int_U \psi^2 \left( au_x^2 + (b + c)u_xu_y + du_y^2 \right) dxdy \leq 16M^2 \int_U \mu |\nabla \phi|^2 dxdy.
$$

Now we shall estimate the integral in the right side of (2.5). We have

$$
I \equiv \int_U \mu |\nabla \phi|^2 dxdy = \int_{\delta''}^{\delta'} \int_{E_t \cap U} \mu \psi_0^2(h(z)) |\nabla h|^2 dxdy
$$

and next, by the well-known Kronrod-Federer coarea formula [3, Chapter 3, §2.4],

$$
I = \int_{\delta''}^{\delta'} \int_{E_t \cap U} \mu |\nabla h| d\mathcal{H}^1(E_t).
$$

Using (1.5) we find

$$
(2.6) \quad I \leq h_1 \int_{\delta'}^{\delta''} \int_{E_t \cap U} \mu d\mathcal{H}^1(E_t).
$$

Since $\psi_0(\delta') = 1$ and $\psi_0(\delta'') = 0$, we have

$$
1 \leq \left( \int_{\delta'}^{\delta''} |\psi_0'(t)| dt \right)^2 \leq \int_{\delta'}^{\delta''} |\psi_0'|^2 dt \int_{E_t \cap U} \mu d\mathcal{H}^1(E_t)
$$

$$
\times \int_{\delta'}^{\delta''} dt \left( \int_{E_t \cap U} \mu d\mathcal{H}^1(E_t) \right)^{-1}.
$$

Therefore, for every absolutely continuous function $\psi_0 : [\delta', \delta''] \to [0, 1]$ such that $\psi_0(\delta'') = 1$ and $\psi_0(\delta') = 0$, we have

$$
\left( \int_{\delta'}^{\delta''} dt \int_{E_t \cap U} \mu d\mathcal{H}^1(E_t) \right)^{-1} \leq \int_{\delta'}^{\delta''} |\psi_0'|^2 dt \int_{E_t \cap U} \mu d\mathcal{H}^1(E_t).
$$
Choosing for \( \delta' \leq \tau \leq \delta'' \),

\[
\psi_0(\tau) = \int_{\delta'}^{\tau} dt \left( \int_{E_t \cap U} \mu d\mathcal{H}^1(E_t) \right)^{-1} \int_{\delta'}^{\delta''} dt \left( \int_{E_t \cap U} \mu d\mathcal{H}^1(E_t) \right)^{-1},
\]

we have

\[
\min_{\psi_0} \int_{\delta'}^{\delta''} dt \int_{E_t \cap U} \mu d\mathcal{H}^1(E_t) = \left( \int_{\delta'}^{\delta''} dt \left( \int_{E_t \cap U} \mu d\mathcal{H}^1(E_t) \right)^{-1} \right)^{-1}.
\]

Thus from (2.5) and (2.6) we see that

\[
\int_{\delta''}^{\delta''} \left( au_x^2 + (b + c)u_xu_y + du_y^2 \right) dxdy \leq 16M^2h_1 \left( \int_{\delta'}^{\delta''} dt \left( \int_{E_t \cap U} \mu d\mathcal{H}^1(E_t) \right)^{-1} \right)^{-1}.
\]

Setting here \( \delta' \to 0 \) we obtain

\[
\int_{\delta''}^{\delta''} \left( au_x^2 + (b + c)u_xu_y + du_y^2 \right) dxdy \leq 16M^2h_1 \left( \int_{0}^{\delta''} dt \left( \int_{E_t \cap U} \mu d\mathcal{H}^1(E_t) \right)^{-1} \right)^{-1}.
\]

Previously we chose \( \delta'' < h(z_0) \) where \( z_0 \in U \). Thus the open set \( U_1 = \{ z \in U : \delta'' < h(z_0) \} \) is not empty. Now from (1.7) we conclude that

\[
\int_{U_1} \left( au_x^2 + (b + c)u_xu_y + du_y^2 \right) dxdy = 0,
\]

that is,

\[
uu_x^2 + (b + c)u_xu_y + du_y^2 = 0 \quad \text{a.e. in} \quad U_1.
\]

The assumption (1.2) implies that \( \nabla u = 0 \) a.e. in \( U_1 \) and hence \( u = \text{const} \) in \( U_1 \). But \( \delta'' > \delta' > 0 \) can be arbitrarily small, and hence \( \nabla u = 0 \) a.e. in \( U \). There is a contradiction with the definition of the connected component \( U \) and (2.1) is true.

Analogously, we obtain

\[
\inf_{\Delta} u(z) = \inf_{\partial \Delta} u(z)
\]

and next, we arrive at (1.4). \(\square\)

### 3. Proof of Theorem 1.8

Suppose that there exists a homeomorphic solution \( w(z) \) of (1.1), (1.2) such that \( w(D) = D \) and \( L \subset w(\Gamma) \). Since \( L \) is \( u \)-forked, it follows that there is a vertical segment \( l \subset D \) with the end points in \( L \) and moreover, there exists a subdomain
Consider its preimage $\Delta = w^{-1}(\U)$.

By Theorem 1.6 the function $u$ is monotone close to $\Gamma$. Hence,

$$\text{osc}(u, \Delta) \leq \text{osc}(u, w^{-1}(l)) = 0.$$

Thus, $u \equiv \text{const}$ in $\Delta$, which is impossible. □

4. An example

Let $\gamma$ be a simple open Jordan arc lying in the upper half-plane with end points $(0,0)$ and $(1,0)$ on the $x$-axis. We set

$$\Gamma = \{(x, y) : 0 \leq x \leq 1, y = 0\},$$

and denote by $D$ the subdomain of $\mathbb{R}^2$ enclosed by $\gamma \cup \Gamma$.

Choose in Theorem 1.8 the function $h(z) = y$. Evidently, (1.5) is fulfilled.

Consider the case of (1.1) in which the functions $a = a(y)$, $d = d(y)$ are positive and $b = c = 0$. Then (1.1) admits the form

$$a(y) u_x = v_y, \quad d(y) u_y = -v_x. \quad (4.1)$$

The assumption (1.2) is fulfilled too.

The assumption (1.7) takes the form

$$\int_0^y a(y) + d(y) = \infty. \quad (4.2)$$

We obtain:

Corollary 4.3. Let $D$ be a simply connected subdomain of $\{(u, v) : |u| < M\}$, where $0 < M < \infty$ is a constant. Assume that (1.2) is fulfilled and $L \subset \partial D$ is $u$-forked. Then there are no $ACL_{\text{loc}}^2$ solutions $w = w(z)$ of (4.1) mapping $D$ homeomorphically onto $D$ such that $L \subset w(\Gamma)$.

Acknowledgements

The authors are indebted to Prof. V.A. Klyachin and to the anonymous referee for a number of useful remarks and corrections.

References


Department of Mathematics and Statistics, P.O. Box 68, FIN-00014, University of Helsinki, Finland
E-mail address: martio@cc.helsinki.fi

Department of Mathematics, Volgograd State University, 2 Prodolnaya, 30, Volgograd, 400062, Russia
E-mail address: miklyuk@mail.ru

Department of Mathematics, FIN-20014, University of Turku, Finland
E-mail address: vuorinen@csc.fi