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A COUNTEREXAMPLE TO THE EXISTENCE OF A LOCAL PLURISUBHARMONIC PEAK FUNCTION AT INFINITY

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Abstract. We give an example of an unbounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$, $n \geq 4$, which is Kobayashi complete but admits no local plurisubharmonic peak function at infinity.

1. Introduction

Let $G \subset \mathbb{C}^n$ be open. We denote by $\mathcal{PSH}(G)$ the set of all plurisubharmonic functions on G. For $R > 0$ we put $U_R(\infty) = U_R^n(\infty) := \{z \in \mathbb{C}^n : \|z\| > R\}$. We say that a domain $G \subset \mathbb{C}^n$ has:

• a *local plurisubharmonic peak function* φ *at infinity* whenever there exists an $R > 0$ such that $\varphi \in \mathcal{C}(\bar{G} \cap U_R(\infty)) \cap \mathcal{PSH}(G \cap U_R(\infty))$ and

$$\lim_{G \ni w \to \infty} \varphi(w) = 0 > \varphi(z), \quad z \in \bar{G} \cap U_R(\infty);$$

• a *local plurisubharmonic antipeak function* φ *at infinity* whenever there exists an $R > 0$ such that $\varphi \in \mathcal{C}(\bar{G} \cap U_R(\infty)) \cap \mathcal{PSH}(G \cap U_R(\infty))$ and

$$\lim_{G \ni w \to \infty} \varphi(w) = -\infty < \varphi(z), \quad z \in \bar{G} \cap U_R(\infty).$$

These notions were introduced by Gaussier [3]. Recently, some authors showed that the existence of such functions on a given unbounded domain is useful in studying geometric properties of the domain; see e.g. [3], [8], [4], [1]. In particular, the following result can be found in [3].

Theorem 1.1. *Suppose that an unbounded domain $G \subset \mathbb{C}^n$ admits local plurisubharmonic peak and antipeak functions at infinity. Then G is Kobayashi hyperbolic. If, moreover, every point $p \in \partial G$ has an open neighborhood $U_p \subset \mathbb{C}^n$ such that any connected component of $G \cap U_p$ is taut, then G is taut.*

Our aim in this article is to prove that the converses of the statements in Theorem 1.1 may not be true, even though the domain is Kobayashi complete. More precisely, we shall show the following result.

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Theorem 1.2. *For any $n \geq 4$ there exists an unbounded pseudoconvex Reinhardt Hartogs domain in $\mathbb{C}^n$ that is Kobayashi complete and does not admit a local plurisubharmonic peak function at infinity.*

Finally, in order to understand the (non-)existence of local plurisubharmonic peak and antipeak functions at infinity, we shall shortly give some other examples.

2. Preliminaries

Let us recall some basic notation and results that will be needed in the sequel.

By $\|\cdot\|$ we denote the Euclidean norm on $\mathbb{C}^n$, $|\cdot| := \|\cdot\|_1$. For $r > 0$ we put $E_r := \{\lambda \in \mathbb{C} : |\lambda| < 1/r\}$ and $E := E_1$. Let $u, v : G \to [-\infty, +\infty)$ be upper semicontinuous with $u + v < 0$ on G, and set

$$\Sigma_{u,v}(G) := \{(z, w) \in G \times \mathbb{C}^1 : e^{v(z)} < |w| < e^{-u(z)}\}.$$

Such a domain is called a *Hartogs-Laurent domain over G*. A domain $G \subset \mathbb{C}^m$ is called *Reinhardt* if $(\lambda_1 w_1, \cdots, \lambda_m w_m) \in G$, $\lambda_1, \cdots, \lambda_m \in \partial E$, $(w_1, \cdots, w_m) \in G$. Obviously, $\Sigma_{u,v}(G)$ is Reinhardt iff G is Reinhardt, $u(z) = u(|z_1|, \cdots, |z_n|)$, $v(z) = v(|z_1|, \cdots, |z_n|)$, $z \in G$. Recall that $\Sigma_{u,v}(G)$ is pseudoconvex iff G is pseudoconvex and $u, v \in \mathcal{PSH}(G)$. This property can be found in e.g. [11], [7].

Recall (see e.g. [6], [12]) that the following implications are well known:

Kobayashi complete $\Rightarrow$ taut $\Rightarrow$ Kobayashi hyperbolic $\Rightarrow$ Brody hyperbolic.

In particular, the converse implications are true for any pseudoconvex Reinhardt domain in $\mathbb{C}^n$, due to Fu [2] and Zwonek [13].

Let us denote by $\mathcal{SH}(G)$ the family of all subharmonic functions on an open set $G \subset \mathbb{C}$. The following statement is due to Oka [9] (or [11], [5]).

Theorem 2.1. *Let $G \subset \mathbb{C}$ be open. Then for any $u \in \mathcal{SH}(G)$ and any continuous curve $\gamma : [0, 1] \to G$ the following is true:*

$$u(\gamma(0)) = \limsup_{t \to 0,\, t > 0} u(\gamma(t)).$$

On the other hand, we can easily show the following.

Lemma 2.2. *Let $G = \{z \in \mathbb{C}^2 : |z_1 z_2| < 1\}$. Then $u \in \mathcal{PSH}(G)$ is bounded from above iff there exists $\tilde{u} \in \mathcal{SH}(E)$ that is bounded from above on E such that $u(z) = \tilde{u}(z_1 z_2)$ for $z \in G$.*

In fact, this auxiliary lemma is a special case of Lemma 4.4.3 in [6].

3. Proof of Theorem 1.2

It suffices to find the desired example for $n = 4$. For this fix $G := \{z \in \mathbb{C}^3 : |z_1 z_2 z_3| < 1\}$ and put $u(z) := \max_{j=1,2,3} |z_j|$, $z \in G$. Clearly, $u \in \mathcal{PSH}(G)$ and $\Sigma := \Sigma_{u,-\infty}(G)$ is a pseudoconvex Reinhardt Hartogs-Laurent domain in $\mathbb{C}^4$. Observe that Σ does not contain a non-trivial entire curve (cf. [10]), so it is Kobayashi complete.

Suppose that there exist an $R > 0$ and a function $\varphi \in \mathcal{C}(\bar{\Sigma} \cap U) \cap \mathcal{PSH}(\Sigma \cap U)$, where $U := U_R^4(\infty)$, such that

$$\varphi(z, \lambda) < 0, \quad (z, \lambda) \in \bar{\Sigma} \cap U, \tag{1}$$

$$\lim_{\Sigma \ni (z,\lambda) \to \infty} \varphi(z, \lambda) = 0. \tag{2}$$

Fix a point $a \in \mathbb{C}$ with $|a| = 2R$. Put

$$\Omega_a := \{((z_1, z_2), \lambda) \in G_a \times \mathbb{C} : |\lambda| < \exp(-u_a(z_1, z_2))\},$$

where $G_a := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1 z_2| < 1/(2R)\}$ and $u_a := u(\cdot, a)$ on G_a. Put $\varphi_a(z_1, z_2, \lambda) := \varphi(z_1, z_2, a, \lambda)$ for $(z_1, z_2, \lambda) \in \Omega_a$. Observe that

$$\{(z_1, z_2, a, \lambda) : (z_1, z_2, \lambda) \in (\mathbb{C} \times \mathbb{C} \times \mathbb{C} \setminus \{0\}) \cap \Omega_a\} \subset \Sigma,$$
$$\varphi_a(z_1, z_2, 0) = \lim_{\zeta \to 0,\, \zeta \neq 0} \varphi(z_1, z_2, a, \zeta), \quad (z_1, z_2) \in G_a.$$

Moreover, $\varphi_a \leq 0$ on $\bar{\Omega}_a$, $\varphi_a \in \mathcal{PSH}(\Omega_a)$, and $\varphi_a(\cdot, 0) \in \mathcal{PSH}(G_a)$. By virtue of Lemma 2.2, we may take a function $\tilde{\varphi}_{(a,0)} \in \mathcal{SH}(E_{2R})$ such that

$$\varphi_a(z_1, z_2, 0) = \tilde{\varphi}_{(a,0)}(z_1 z_2), \quad (z_1, z_2) \in G_a.$$

On the other hand, it follows from Theorem 2.1 that

$$\tilde{\varphi}_{(a,0)}(0) = \limsup_{\mathbb{R} \ni t \to \infty} \tilde{\varphi}_{(a,0)}(\frac{1}{2Rt}) = \limsup_{\mathbb{R} \ni t \to \infty} \varphi_a(t, \frac{1}{2Rt^2}, 0) =: -C. \tag{3}$$

Obviously, $C \geq 0$. If $C = 0$, the maximum principle for subharmonic functions implies that $\tilde{\varphi}_{(a,0)} = 0$ on E_{2R} and so $\varphi(\cdot, a, 0) = 0$ on G_a, a contradiction to our assumption (1). Hence, $C > 0$.

In view of (3), we can take a constant $M' \gg 1$ so large that

$$\varphi_a(t, \frac{1}{2Rt^2}, 0) < -\frac{3}{4}C, \quad t \in \mathbb{R},\, t > M'.$$

Let $t \in \mathbb{R}$ with $t > M'' := \max\{M', |a|\}$. Then

$$\varphi_a(t, \frac{1}{2Rt^2}, 0) = \lim_{\lambda \to 0,\, \lambda \neq 0} \varphi(t, \frac{1}{2Rt^2}, a, \lambda) < -\frac{3}{4}C.$$

Hence we can take $M_t > t$ so large that

$$\varphi(t, \frac{1}{2Rt^2}, a, \lambda) < -\frac{1}{2}C, \quad 0 < |\lambda| < \exp(-M_t). \tag{4}$$

Notice that $u(t, 1/(2Rt^2), a) = t$ for any $t \in \mathbb{R}$ with $t > M''$ and $\lim_{\mathbb{R} \ni t \to \infty} M_t = \infty$. Therefore, we can take a sequence $(t_j, 1/(2Rt_j^2), a, \lambda_j) \in (\mathbb{R} \times \mathbb{R} \times \{a\} \times \mathbb{C}) \cap \Sigma$ such that $t_j > M''$, $0 < |\lambda_j| < \exp(-M_{t_j}) \leq \exp(-t_j)$, and $\lim_{j \to \infty} t_j = \infty$. From (4) it follows that

$$\lim_{j \to \infty} \varphi(t_j, \frac{1}{2Rt_j^2}, a, \lambda_j) \leq -\frac{1}{4}C < 0,$$

which is a contradiction to our assumption (2). Thus the domain Σ has no function φ as above, so we are done.

4. Some examples

As concrete examples of unbounded domains admitting a local plurisubharmonic peak (or antipeak) function at infinity, it is known that any unbounded domain $G \subset \mathbb{C}^n$ with a *local holomorphic peak function f at infinity*, which means that there is an $R > 0$ such that f is holomorphic in $G \cap U_R(\infty)$, continuous in $\bar{G} \cap U_R(\infty)$, and

$$\lim_{G \ni w \to \infty} |f(w)| = 1 > |f(z)|, \quad z \in \bar{G} \cap U_R(\infty),$$

has also local plurisubharmonic peak and antipeak functions at infinity. Such domains can be found in ([3], Example 3.2.1, Example 3.2.2).

On the other hand, in the case of non-hyperbolic domains, we have the following simple examples:

(a) *A domain with a local plurisubharmonic antipeak function at infinity, and without local plurisubharmonic peak functions at infinity:* Let $G' \Subset \mathbb{C}^n$. Then it is easy to check that any unbounded subdomain G of $G' \times \mathbb{C}$ has a local plurisubharmonic antipeak function at infinity. Moreover, according to Theorem 1.1, the domain $G' \times \mathbb{C}$ has no local plurisubharmonic peak function at infinity.

(b) *A domain without local plurisubharmonic peak and antipeak functions at infinity:* Let $h : \mathbb{C}^n \to [0, \infty)$ be upper semicontinuous, $h(\lambda z) = |\lambda| h(z)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$, and put $D := \{z \in \mathbb{C}^n : h(z) < 1\}$. Assume that there exists a point $z^0 \in \mathbb{C}^n \setminus \{0\}$ such that $h(z^0) = 0$. Let $u : \mathbb{C}^m \to [-\infty, \infty)$ be upper semicontinuous. Consider the following unbounded domain:

$$\Omega := \{(z, w) \in D \times \mathbb{C}^m : h(z)u(w) < 1\}.$$

Clearly, it contains a non-trivial entire curve, but has no local plurisubharmonic peak functions at infinity. To check the last assertion, suppose that there exist a constant $R > 0$ and a function $\varphi \in \mathcal{C}(\bar{\Omega} \cap U) \cap \mathcal{PSH}(\Omega \cap U)$ with

$$\lim_{G \ni w \to \infty} \varphi(w) = 0 > \varphi(z), \quad z \in \bar{\Omega} \cup U, \tag{5}$$

where $U := U_R^{n+m}(\infty)$. Fix a point $w^0 \in \mathbb{C}^m$ with $\|w^0\| = 2R$. Since $h = 0$ on $\mathbb{C}z^0$, one has $\varphi(\lambda z^0, w^0) < 0$, $\lambda \in \mathbb{C}$. So, it follows from the Liouville type theorem for subharmonic functions that $\varphi(\lambda z^0, w^0) \equiv \text{constant} =: 2C < 0$ for any $\lambda \in \mathbb{C}$, so we have $\lim_{|\lambda| \to \infty} \varphi(\lambda z^0, w^0) \leq C < 0$, a contradiction to (5).

Therefore, for any $n \geq 3$, the domain $G_n := \{z \in \mathbb{C}^n : |z_1 \cdots z_n| < 1\}$ has no local plurisubharmonic peak function at infinity. Moreover, it has also no local plurisubharmonic antipeak function at infinity. To check this, fix $n \geq 3$ and let ψ be a function defined on $W := G_n \cap U_R(\infty)$ for some $R > 0$. Suppose that $\psi \in \mathcal{PSH}(W)$ and $\psi\big|_W > -\infty = \lim_{W \ni z \to \infty} \psi(z)$. Fix $a \in \mathbb{C}$ with $|a| = 2R$. Then $\psi(\cdot, a) \in \mathcal{PSH}(\tilde{W})$ where $\tilde{W} := \{(z_1, \cdots, z_{n-1}) \in \mathbb{C}^{n-1} : |z_1 \cdots z_{n-1}| < 1/(2R)\}$, and so $\psi_a := \psi(\cdot, 0, \cdots, 0, a) \in \mathcal{SH}(\mathbb{C})$. Our assumption gives that $\lim_{|\lambda| \to \infty} \psi_a(\lambda) = -\infty$. Hence, the maximum principle for subharmonic functions gives that $\psi_a \equiv -\infty$, which is a contradiction to the fact that $\psi_a > -\infty$ on $\mathbb{C}$.

Remark 4.1. The previous two examples imply that, in Theorem 1.1, the assumption of the existence of a local plurisubharmonic peak function at infinity cannot be removed.

Remark 4.2. Finally, we would like to mention that it would be interesting to know the existence of a domain that has a local plurisubharmonic peak function at infinity, but does not have local plurisubharmonic antipeak functions at infinity. Moreover, we do not know yet whether a domain that belongs to the same category as the domain obtained in Theorem 1.2 could be found in $\mathbb{C}^n$, $n = 2, 3$. We should notice that the method used in the proof of this theorem could not be applied to the lower-dimensional cases because of the choice of the domain G.

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