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S-INVARIANT SUBSPACES OF $L^p(\mathbf{T})$

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ABSTRACT. In this note, we give a new proof of the characterization of the S-invariant subspaces of $L^p(\mathbf{T})$ for p in $\mathcal{P} \equiv \{p : 1 < p < \infty, p \neq 2\}$ using ideas from approximation theory.

In this short note we give a new proof of the characterization of the S-invariant subspaces of $L^p(\mathbf{T}) = L^p(\mathbf{T}, m)$ for p in $\mathcal{P} \equiv \{p : 1 < p < \infty, p \neq 2\}$ where S denotes the operator of multiplication by the coordinate function. Here, $\mathbf{T}$ denotes the unit circle in the complex plane and m denotes Lebesgue measure on $\mathbf{T}$ normalized so that $m(\mathbf{T}) = 1$. The proof is based on well-known facts from approximation theory.

Henceforth, any time we make reference to $L^p(\mathbf{T})$ (or $H^p(\mathbf{T})$) we assume p is in $\mathcal{P}$. We point out first that $L^p(\mathbf{T})$ and all of the subspaces of $L^p(\mathbf{T})$ are uniformly convex Banach spaces. Hence we may employ the following result found in [5].

Lemma 1. *Let $\mathcal{X}$ be a uniformly convex Banach space and $\mathcal{K}$ a subspace of $\mathcal{X}$. Then for all x in $\mathcal{X}$ there corresponds a unique y in $\mathcal{K}$ satisfying $\|x-y\| = \inf_{z\in\mathcal{K}} \|x-z\|$. We call such a y the best approximate of x in $\mathcal{K}$.*

We say w is orthogonal to $\mathcal{K}$ and write $w \perp \mathcal{K}$ if $\|w\| \leq \|w + k\|$ for all k in $\mathcal{K}$. For $\mathcal{K}$ a subspace of $L^p(\mathbf{T})$, then $f \perp \mathcal{K}$ if

$$\int_{\mathbf{T}} g\, |f|^{p-1}\, \overline{sgn\, f}\, dm = 0$$

for all g in $\mathcal{K}$, where $sgn\, f$ is a complex measurable function of modulus 1 such that $f = sgn\, f|f|$. A proof of this can be found in [5].

If f is in $L^p(\mathbf{T})$ and f^* is the best approximate of f in $\mathcal{K}$, then $g = f - f^*$ is orthogonal to $\mathcal{K}$. This remark suggests the following lemma, which is a corollary of our first lemma.

Lemma 2. *Let $\mathcal{X}$ be a uniformly convex Banach space and $\mathcal{K}$ a subspace of $\mathcal{X}$. Then there exists an x in $\mathcal{X}$ such that $x \perp \mathcal{K}$. If $\mathcal{K}$ is a proper subspace, then x may be chosen such that $x \neq 0$.*

We say an S-invariant subspace of $L^p(\mathbf{T})$ is S-simply invariant if $S(\mathcal{M})$ is a proper subspace of $\mathcal{M}$. We also recall that $H^p(\mathbf{T}) = \{f \in L^p(\mathbf{T}) : \int_{\mathbf{T}} z^n f\, dm = 0\, \forall\, n > 0\}$.

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Theorem 1. *$\mathcal{M}$ is an S-simply invariant subspace of $L^p(\mathbf{T})$ if and only if $\mathcal{M} = \phi H^p(\mathbf{T})$ with ϕ unimodular.*

Proof. If $\mathcal{M} = \phi H^p(\mathbf{T})$ with ϕ unimodular, then it is clear that $\mathcal{M}$ is an S-simply invariant subspace of $L^p(\mathbf{T})$. It remains to show the converse. Since $S(\mathcal{M})$ is a proper subspace of $\mathcal{M}$ by Lemma 2, there exists a nonzero ϕ in $\mathcal{M}$ such that $\phi \perp S(\mathcal{M})$. There is no loss of generality if we choose ϕ such that $\|\phi\|_p = 1$. So in particular, $\phi \perp z^n\phi$ for all $n > 0$. That is,

$$\int_{\mathbf{T}} z^n |\phi|^p \, dm = 0$$

for all $n > 0$. Taking complex conjugates we get

$$\int_{\mathbf{T}} z^n |\phi|^p \, dm = 0$$

for all $n \neq 0$. So, $|\phi| = 1$ a.e. on $\mathbf{T}$. That is, ϕ is unimodular. Since ϕ is in $\mathcal{M}$, so is $z^n\phi$ for all $n \geq 0$. Therefore, ϕP is in $\mathcal{M}$ for every polynomial P. Since polynomials are dense in $H^p(\mathbf{T})$ and $|\phi| = 1$ we get that $\phi H^p(\mathbf{T}) \subseteq \mathcal{M}$. It remains to show that $\phi H^p(\mathbf{T})$ is all of $\mathcal{M}$. Let ψ be an element of $\mathcal{M}$ orthogonal to $\phi H^p(\mathbf{T})$. Since ϕ is unimodular, we get that $\psi\overline{\phi}$ is in $L^p(\mathbf{T})$. By the way we chose ϕ we get that $\phi \perp z^n\psi$ for all $n > 0$. That is,

$$\int_{\mathbf{T}} z^n \psi\overline{\phi} \, dm = 0$$

for all $n > 0$. These two facts together give us that $\psi\overline{\phi}$ is in $H^p(\mathbf{T})$. That is, ψ is in $\phi H^p(\mathbf{T})$. This can only happen if $\psi = 0$. Therefore, $\mathcal{M} = \phi H^p(\mathbf{T})$ as desired. □

Corollary 1. *$\mathcal{M}$ is an S-invariant subspace of $H^p(\mathbf{T})$ if and only if $\mathcal{M} = \phi H^p(\mathbf{T})$ with ϕ inner.*

This is easy to see since every S-invariant subspace of $H^p(\mathbf{T})$ is S-simply invariant and since unimodular plus analytic implies inner.

We say an S-invariant subspace of $L^p(\mathbf{T})$ is S-doubly invariant if it is S-invariant but not S-simply invariant. That is, $S(\mathcal{M}) = \mathcal{M}$. So in particular, $\mathcal{M}$ is invariant under both S and S^{-1}.

Theorem 2. *$\mathcal{M}$ is an S-doubly invariant subspace of $L^p(\mathbf{T})$ if and only if $\mathcal{M} = \mathbf{1}_E L^p(\mathbf{T})$ where E is a measurable subset of $\mathbf{T}$.*

By $\mathbf{1}_E$ we mean a function that takes the value 1 on E and 0 on E^c.

Proof. If $\mathcal{M} = \mathbf{1}_E L^p(\mathbf{T})$, then it is clear that $\mathcal{M}$ is an S-doubly invariant subspace of $L^p(\mathbf{T})$. It remains to show the converse. If $\mathbf{1}_{\mathbf{T}}$ is in $\mathcal{M}$, then $\mathcal{M} = L^2(\mathbf{T})$ and we are done. So we may assume $\mathbf{1}_{\mathbf{T}}$ is not in $\mathcal{M}$, and let q denote the best approximate of $\mathbf{1}_{\mathbf{T}}$ in $\mathcal{M}$. Then $\mathbf{1}_{\mathbf{T}} - q$ is orthogonal to $\mathcal{M}$. So in particular, $(\mathbf{1}_{\mathbf{T}} - q) \perp z^n q$ for all $n \in \mathbf{Z}$. That is,

$$\int_{\mathbf{T}} z^n q |\mathbf{1}_{\mathbf{T}} - q|^{p-1} \, \overline{sgn\,(\mathbf{1}_{\mathbf{T}} - q)} \, dm = 0$$

for all $n \in \mathbf{Z}$. So, $q|\mathbf{1}_{\mathbf{T}} - q|^{p-1}\,\overline{sgn\,(\mathbf{1}_{\mathbf{T}} - q)} = 0$ a.e. Let $E = \{z \in \mathbf{T} : q(z) = 1\}$. Then on E^c, $q = 0$ a.e. That is, $q = \mathbf{1}_E$. $\mathbf{1}_E L^p(\mathbf{T})$ is the smallest S-doubly invariant subspace of $L^p(\mathbf{T})$ containing $\mathbf{1}_E$. Therefore, $\mathbf{1}_E L^p(\mathbf{T}) \subseteq \mathcal{M}$. It remains to show that $\mathbf{1}_E L^p(\mathbf{T}) = \mathcal{M}$. Let g be an element of $\mathcal{M}$ orthogonal to $\mathbf{1}_E L^p(\mathbf{T})$. So in particular, $g \perp \mathbf{1}_E z^n$ for all $n \in \mathbf{Z}$. That is,

$$\int_{\mathbf{T}} z^n \mathbf{1}_E \, |g|^{p-1} \, \overline{sgn\, g} \, dm = 0$$

for all $n \in \mathbf{Z}$. So, $\mathbf{1}_E \, |g|^{p-1} \, \overline{sgn\, g} = 0$ a.e. Therefore, $g = 0$ on E. It remains to show that $g = 0$ on E^c. Since g is in $\mathcal{M}$, so are $z^n g$ for all $n \in \mathbf{Z}$. So, $\mathbf{1}_{\mathbf{T}} - \mathbf{1}_E \perp z^n g$ for all $n \in \mathbf{Z}$. That is,

$$\int_{\mathbf{T}} z^n g \, |\mathbf{1}_{\mathbf{T}} - \mathbf{1}_E|^{p-1} \, \overline{sgn\,(\mathbf{1}_{\mathbf{T}} - \mathbf{1}_E)} \, dm = 0$$

for all $n \in \mathbf{Z}$. So, $g \, |\mathbf{1}_{\mathbf{T}} - \mathbf{1}_E|^{p-1} \, \overline{sgn\,(\mathbf{1}_{\mathbf{T}} - \mathbf{1}_E)} = 0$ a.e. On E^c we see that $g = 0$. Therefore, $g = 0$ a.e. So, $\mathcal{M} = \mathbf{1}_E L^p(\mathbf{T})$ as desired. □

Remark 1. The above proofs apply equally well for the case $p = 2$, in which case the best approximation operator is then just a Hilbert space orthogonal projection. However, this proof is well known (see [3]).

Remark 2. Another well-known proof for the case $p = 2$ involves the so-called "Wold Decomposition". This approach is very useful. It applies equally well to Hilbert space shift operators of arbitrary multiplicity (see [2]) as well as for proving the characterization of certain invariant sub-Hilbert spaces of $H^p(\mathbf{T})$ and other Banach spaces of analytic functions (see [6]). These sub-Hilbert spaces are referred to as de Branges subspaces and have received some recent attention. The referee pointed out a generalized "Wold Decomposition" (see [1]) that applies to certain Banach spaces. Such a decomposition would be a useful tool both here and in possibly generalizing the de Branges subspace idea. However, it is unclear to us that such a decomposition may be employed here.

Remark 3. Finally, we point out that the above results are well known and the proofs known to us are found in [3]. It is shown in [3] that the L^p results follow from the case $p = 2$. For $p < 2$ they use a density argument and for the case when $p > 2$ they employ a duality argument utilizing their result for $p < 2$.

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