S-INVARIANT SUBSPACES OF $L^p(T)$

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Abstract. In this note, we give a new proof of the characterization of the $S$-invariant subspaces of $L^p(T)$ for $p$ in $P = \{ p : 1 < p < \infty, p \neq 2 \}$ using ideas from approximation theory.

In this short note we give a new proof of the characterization of the $S$-invariant subspaces of $L^p(T) = L^p(T, m)$ for $p$ in $P = \{ p : 1 < p < \infty, p \neq 2 \}$ where $S$ denotes the operator of multiplication by the coordinate function. Here, $T$ denotes the unit circle in the complex plane and $m$ denotes Lebesgue measure on $T$ normalized so that $m(T) = 1$. The proof is based on well-known facts from approximation theory.

Henceforth, any time we make reference to $L^p(T)$ (or $H^p(T)$) we assume $p$ is in $P$. We point out first that $L^p(T)$ and all of the subspaces of $L^p(T)$ are uniformly convex Banach spaces. Hence we may employ the following result found in [5].

Lemma 1. Let $\mathcal{X}$ be a uniformly convex Banach space and $K$ a subspace of $\mathcal{X}$. Then for all $x$ in $\mathcal{X}$ there corresponds a unique $y$ in $K$ satisfying $\| x - y \| = \inf_{z \in K} \| x - z \|$. We call such a $y$ the best approximate of $x$ in $K$.

We say $w$ is orthogonal to $K$ and write $w \perp K$ if $\| w \| \leq \| w + k \|$ for all $k$ in $K$. For $K$ a subspace of $L^p(T)$, then $f \perp K$ if

$$\int_T g \left| f \right|^{p-1} \text{sgn} f \, dm = 0$$

for all $g$ in $K$, where $\text{sgn} f$ is a complex measurable function of modulus 1 such that $f = \text{sgn} f |f|$. A proof of this can be found in [5].

If $f$ is in $L^p(T)$ and $f^*$ is the best approximate of $f$ in $K$, then $g = f - f^*$ is orthogonal to $K$. This remark suggests the following lemma, which is a corollary of our first lemma.

Lemma 2. Let $\mathcal{X}$ be a uniformly convex Banach space and $K$ a subspace of $\mathcal{X}$. Then there exists an $x$ in $\mathcal{X}$ such that $x \perp K$. If $K$ is a proper subspace, then $x$ may be chosen such that $x \neq 0$.

We say an $S$-invariant subspace of $L^p(T)$ is $S$-simply invariant if $S(M)$ is a proper subspace of $M$. We also recall that $H^p(T) = \{ f \in L^p(T) : \int_T z^n f \, dm = 0 \forall n > 0 \}$. 

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Theorem 1. $\mathcal{M}$ is an $S$-simply invariant subspace of $L^p(\mathbf{T})$ if and only if $\mathcal{M} = \phi H^p(\mathbf{T})$ with $\phi$ unimodular.

Proof. If $\mathcal{M} = \phi H^p(\mathbf{T})$ with $\phi$ unimodular, then it is clear that $\mathcal{M}$ is an $S$-simply invariant subspace of $L^p(\mathbf{T})$. It remains to show the converse. Since $S(\mathcal{M})$ is a proper subspace of $\mathcal{M}$ by Lemma 2, there exists a nonzero $\phi$ in $\mathcal{M}$ such that $\phi \perp S(\mathcal{M})$. There is no loss of generality if we choose $\phi$ such that $\|\phi\|_p = 1$. So in particular, $\phi \perp z^n\phi$ for all $n > 0$. That is,

$$\int_{\mathbf{T}} z^n|\phi|^p \, dm = 0$$

for all $n > 0$. Taking complex conjugates we get

$$\int_{\mathbf{T}} z^n|\phi|^p \, dm = 0$$

for all $n \neq 0$. So, $|\phi| = 1$ a.e. on $\mathbf{T}$. That is, $\phi$ is unimodular. Since $\phi$ is in $\mathcal{M}$, so is $z^n\phi$ for all $n \geq 0$. Therefore, $\phi P$ is in $\mathcal{M}$ for every polynomial $P$. Since polynomials are dense in $H^p(\mathbf{T})$ and $|\phi| = 1$ we get that $\phi H^p(\mathbf{T}) \subseteq \mathcal{M}$. It remains to show that $\phi H^p(\mathbf{T})$ is all of $\mathcal{M}$. Let $\psi$ be an element of $\mathcal{M}$ orthogonal to $\phi H^p(\mathbf{T})$. Since $\phi$ is unimodular, we get that $\psi\overline{\phi}$ is in $L^p(\mathbf{T})$. By the way we chose $\phi$ we get that $\phi \perp z^n\psi$ for all $n > 0$. That is,

$$\int_{\mathbf{T}} z^n\psi\overline{\phi} \, dm = 0$$

for all $n > 0$. These two facts together give us that $\psi\overline{\phi}$ is in $H^p(\mathbf{T})$. That is, $\psi$ is in $\phi H^p(\mathbf{T})$. This can only happen if $\psi = 0$. Therefore, $\mathcal{M} = \phi H^p(\mathbf{T})$ as desired. \qed

Corollary 1. $\mathcal{M}$ is an $S$-invariant subspace of $H^p(\mathbf{T})$ if and only if $\mathcal{M} = \phi H^p(\mathbf{T})$ with $\phi$ inner.

This is easy to see since every $S$-invariant subspace of $H^p(\mathbf{T})$ is $S$-simply invariant and since unimodular plus analytic implies inner.

We say an $S$-invariant subspace of $L^p(\mathbf{T})$ is $S$-doubly invariant if it is $S$-invariant but not $S$-simply invariant. That is, $S(\mathcal{M}) = \mathcal{M}$. So in particular, $\mathcal{M}$ is invariant under both $S$ and $S^{-1}$.

Theorem 2. $\mathcal{M}$ is an $S$-doubly invariant subspace of $L^p(\mathbf{T})$ if and only if $\mathcal{M} = 1_E L^p(\mathbf{T})$ where $E$ is a measurable subset of $\mathbf{T}$.

By $1_E$ we mean a function that takes the value 1 on $E$ and 0 on $E^c$.

Proof. If $\mathcal{M} = 1_E L^p(\mathbf{T})$, then it is clear that $\mathcal{M}$ is an $S$-doubly invariant subspace of $L^p(\mathbf{T})$. It remains to show the converse. If $1_{\mathbf{T}}$ is in $\mathcal{M}$, then $\mathcal{M} = L^2(\mathbf{T})$ and we are done. So we may assume $1_{\mathbf{T}}$ is not in $\mathcal{M}$, and let $q$ denote the best approximate of $1_{\mathbf{T}}$ in $\mathcal{M}$. Then $1_{\mathbf{T}} - q$ is orthogonal to $\mathcal{M}$. So in particular, $(1_{\mathbf{T}} - q) \perp z^n q$ for all $n \in \mathbf{Z}$. That is,

$$\int_{\mathbf{T}} z^n q |1_{\mathbf{T}} - q|^{p-1} sgn(1_{\mathbf{T}} - q) \, dm = 0$$

for all $n \in \mathbf{Z}$. So, $q |1_{\mathbf{T}} - q|^{p-1} sgn(1_{\mathbf{T}} - q) = 0$ a.e. Let $E = \{z \in \mathbf{T} : q(z) = 1\}$. Then on $E^c$, $q = 0$ a.e. That is, $q = 1_E$. $1_E L^p(\mathbf{T})$ is the smallest $S$-doubly invariant subspace of $L^p(\mathbf{T})$ containing $1_E$. Therefore, $1_E L^p(\mathbf{T}) \subseteq \mathcal{M}$. It remains
to show that $1_E L^p(T) = M$. Let $g$ be an element of $M$ orthogonal to $1_E L^p(T)$. So in particular, $g \perp 1_E z^n$ for all $n \in \mathbb{Z}$. That is,
\[
\int_T z^n 1_E |g|^{p-1} \text{sgn} g \, dm = 0
\]
for all $n \in \mathbb{Z}$. So, $1_E |g|^{p-1} \text{sgn} g = 0$ a.e. Therefore, $g = 0$ on $E$. It remains to show that $g = 0$ on $E^c$. Since $g$ is in $M$, so are $z^n g$ for all $n \in \mathbb{Z}$. So, $1_T - 1_E \perp z^n g$ for all $n \in \mathbb{Z}$. That is,
\[
\int_T z^n g |1_T - 1_E|^{p-1} \text{sgn} (1_T - 1_E) \, dm = 0
\]
for all $n \in \mathbb{Z}$. So, $g |1_T - 1_E|^{p-1} \text{sgn} (1_T - 1_E) = 0$ a.e. On $E^c$ we see that $g = 0$. Therefore, $g = 0$ a.e. So, $M = 1_E L^p(T)$ as desired. \(\square\)

Remark 1. The above proofs apply equally well for the case $p = 2$, in which case the best approximation operator is then just a Hilbert space orthogonal projection. However, this proof is well known (see [3]).

Remark 2. Another well-known proof for the case $p = 2$ involves the so-called “Wold Decomposition”. This approach is very useful. It applies equally well to Hilbert space shift operators of arbitrary multiplicity (see [2]) as well as for proving the characterization of certain invariant sub-Hilbert spaces of $H^p(T)$ and other Banach spaces of analytic functions (see [5]). These sub-Hilbert spaces are referred to as de Branges subspaces and have received some recent attention. The referee pointed out a generalized “Wold Decomposition” (see [1]) that applies to certain Banach spaces. Such a decomposition would be a useful tool both here and in possibly generalizing the de Branges subspace idea. However, it is unclear to us that such a decomposition may be employed here.

Remark 3. Finally, we point out that the above results are well known and the proofs known to us are found in [3]. It is shown in [3] that the $L^p$ results follow from the case $p = 2$. For $p < 2$ they use a density argument and for the case when $p > 2$ they employ a duality argument utilizing their result for $p < 2$.

References


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