

## EULER'S INTEGRALS AND MULTIPLE SINE FUNCTIONS

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ABSTRACT. We show that Euler's famous integrals whose integrands contain the logarithm of the sine function are expressed via multiple sine functions.

### 1. INTRODUCTION

Euler studied the definite integrals  $\int_0^{\frac{\pi}{2}} x^n \log(\sin x) dx$  for  $n = 0$  and  $1$ . In [E1] (1769), he proved the famous result

$$(1) \quad \int_0^{\frac{\pi}{2}} \log(\sin x) dx = -\frac{\pi}{2} \log 2,$$

which is frequently explained as an example of tricky integrals in analysis courses. A bit later, Euler [E2] (1772) stated that

$$(2) \quad \begin{aligned} \int_0^{\frac{\pi}{2}} x \log(\sin x) dx &= \frac{1}{2} \left( \sum_{\substack{n=1 \\ n:\text{odd}}}^{\infty} \frac{1}{n^3} - \frac{\pi^2}{4} \log 2 \right) \\ &= \frac{7}{16} \zeta(3) - \frac{\pi^2}{8} \log 2. \end{aligned}$$

Euler proved (1) by using

$$(3) \quad \log(\sin x) = -\sum_{n=1}^{\infty} \frac{\cos(2nx)}{n} - \log 2$$

for  $0 < x < \pi$ . The actual integration is obvious since

$$\int_0^{\frac{\pi}{2}} \cos(2nx) dx = 0$$

for  $n = 1, 2, 3, \dots$ . Hence, the original proof of Euler is not tricky contrary to the usual explanation. In the case of (2), Euler primarily wanted to calculate the value  $\zeta(3)$ . He started from the divergent series expression

$$\begin{aligned} \zeta(3) &= -4\pi^2 \zeta'(-2) \\ &= 4\pi^2 \sum_{n=1}^{\infty} n^2 \log n \end{aligned}$$

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and by investigating it he reached (2). Thus his arguments are difficult to follow and literally invalid. Moreover Euler did not prove the functional equation  $\zeta(3) = -4\pi^2\zeta'(-2)$  conjectured by himself in [E3]. We notice that when we use (3) we can give a secure calculation for (2):

$$\int_0^{\frac{\pi}{2}} x \log(\sin x) dx = - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\frac{\pi}{2}} x \cos(2nx) dx - \frac{\pi^2}{8} \log 2$$

with integration by parts

$$\int_0^{\frac{\pi}{2}} x \cos(2nx) dx = \begin{cases} -\frac{1}{2n^2} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

It might be remarkable that Euler missed this way.

In this paper we investigate definite integrals

$$\int_0^x \theta^{r-2} \log(\sin \theta) d\theta$$

for  $r = 2, 3, 4, \dots$  containing Euler's case  $x = \pi/2$  from the point of view of multiple sine functions. Let

$$\begin{aligned} \mathcal{S}_r(x) &= e^{\frac{x^{r-1}}{r-1}} \prod_{\substack{n=-\infty \\ n \neq 0}}^{\infty} P_r\left(\frac{x}{n}\right)^{n^{r-1}} \\ &= e^{\frac{x^{r-1}}{r-1}} \prod_{n=1}^{\infty} \left( P_r\left(\frac{x}{n}\right) P_r\left(-\frac{x}{n}\right)^{(-1)^{r-1}} \right)^{n^{r-1}} \end{aligned}$$

be the multiple sine function studied in [K1, K2, KK1, KK2, KOW, KW], where

$$P_r(u) = (1 - u) \exp\left(u + \frac{u^2}{2} + \dots + \frac{u^r}{r}\right).$$

For example,

$$\mathcal{S}_2(x) = e^x \prod_{n=1}^{\infty} \left( \left( \frac{1 - \frac{x}{n}}{1 + \frac{x}{n}} \right)^n e^{2x} \right),$$

$$\mathcal{S}_3(x) = e^{\frac{x^2}{2}} \prod_{n=1}^{\infty} \left( \left( 1 - \frac{x^2}{n^2} \right)^{n^2} e^{x^2} \right)$$

and

$$\mathcal{S}_4(x) = e^{\frac{x^3}{3}} \prod_{n=1}^{\infty} \left( \left( \frac{1 - \frac{x}{n}}{1 + \frac{x}{n}} \right)^{n^3} e^{2n^2x + \frac{2}{3}x^3} \right).$$

Then we show the following result.

**Theorem 1.** For  $0 \leq x < \pi$  and for  $r = 2, 3, 4, \dots$ , we have

$$\int_0^x \theta^{r-2} \log(\sin \theta) d\theta = \frac{x^{r-1}}{r-1} \log(\sin x) - \frac{\pi^{r-1}}{r-1} \log \mathcal{S}_r\left(\frac{x}{\pi}\right).$$

In particular we have:

**Theorem 2.** For  $r = 2, 3, 4, \dots$ ,

$$\int_0^{\frac{\pi}{2}} \theta^{r-2} \log(\sin \theta) d\theta = -\frac{\pi^{r-1}}{r-1} \log \mathcal{S}_r \left( \frac{1}{2} \right).$$

**Examples.**

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \log(\sin \theta) d\theta &= -\pi \log \mathcal{S}_2 \left( \frac{1}{2} \right) \\ (4) \qquad \qquad \qquad &= -\pi \log \left( e^{\frac{1}{2}} \prod_{n=1}^{\infty} \left( \left( \frac{2n-1}{2n+1} \right)^n e \right) \right), \end{aligned}$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \theta \log(\sin \theta) d\theta &= -\frac{\pi^2}{2} \log \mathcal{S}_3 \left( \frac{1}{2} \right) \\ (5) \qquad \qquad \qquad &= -\frac{\pi^2}{2} \log \left( e^{\frac{1}{8}} \prod_{n=1}^{\infty} \left( \left( 1 - \frac{1}{4n^2} \right)^{n^2} e^{\frac{1}{4}} \right) \right), \end{aligned}$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \theta^2 \log(\sin \theta) d\theta &= -\frac{\pi^3}{3} \log \mathcal{S}_4 \left( \frac{1}{2} \right) \\ (6) \qquad \qquad \qquad &= -\frac{\pi^3}{3} \log \left( e^{\frac{1}{24}} \prod_{n=1}^{\infty} \left( \left( \frac{2n-1}{2n+1} \right)^{n^3} \exp \left( n^2 + \frac{1}{12} \right) \right) \right). \end{aligned}$$

We notice that we have  $\mathcal{S}_2 \left( \frac{1}{2} \right) = \sqrt{2}$  and  $\mathcal{S}_3 \left( \frac{1}{2} \right) = 2^{\frac{1}{4}} \exp \left( -\frac{7\zeta(3)}{8\pi^2} \right)$  from Euler's results (1) and (2) combined with (4) and (5). We demonstrate a calculation of the special value  $\mathcal{S}_4 \left( \frac{1}{2} \right)$  from the product expression directly as follows.

**Theorem 3.**

$$\mathcal{S}_4 \left( \frac{1}{2} \right) = 2^{\frac{1}{8}} \exp \left( -\frac{9\zeta(3)}{16\pi^2} \right)$$

and

$$\int_0^{\frac{\pi}{2}} \theta^2 \log(\sin \theta) d\theta = \frac{3\pi}{16} \zeta(3) - \frac{\pi^3}{24} \log 2.$$

In our calculation a generalization of the Stirling formula is crucial.

2. MULTIPLE SINE FUNCTIONS

To make this paper self-contained we prove some basic properties of multiple sine functions. For general background we refer to [KK1, KK2, KOW, M].

**Proposition 1.** For  $r = 2, 3, 4, \dots$ ,  $\mathcal{S}_r(x)$  is a meromorphic function in  $x \in \mathbf{C}$  and it satisfies

$$\frac{\mathcal{S}'_r(x)}{\mathcal{S}_r(x)} = \pi x^{r-1} \cot(\pi x).$$

*Proof.* The fact that  $\mathcal{S}_r(x)$  is a meromorphic function in  $x \in \mathbf{C}$  (and its order as a meromorphic function being  $r$ ) is seen from the product expression defining  $\mathcal{S}_r(x)$ . Let us calculate the logarithmic derivative. From

$$\mathcal{S}_r(x) = e^{\frac{x^{r-1}}{r-1}} \prod_{n=1}^{\infty} \left( P_r\left(\frac{x}{n}\right) P_r\left(-\frac{x}{n}\right)^{(-1)^{r-1}} \right)^{n^{r-1}}$$

we have

$$\begin{aligned} \log \mathcal{S}_r(x) &= \frac{x^{r-1}}{r-1} + \sum_{n=1}^{\infty} n^{r-1} \left( \log P_r\left(\frac{x}{n}\right) + (-1)^{r-1} \log P_r\left(-\frac{x}{n}\right) \right) \\ &= \frac{x^{r-1}}{r-1} + \sum_{n=1}^{\infty} n^{r-1} \left( \log\left(1 - \frac{x}{n}\right) + (-1)^{r-1} \log\left(1 + \frac{x}{n}\right) \right. \\ &\quad \left. + \left(\frac{x}{n} + \frac{1}{2}\left(\frac{x}{n}\right)^2 + \dots + \frac{1}{r}\left(\frac{x}{n}\right)^r\right) \right. \\ &\quad \left. + (-1)^{r-1} \left( \left(\frac{-x}{n}\right) + \frac{1}{2}\left(\frac{-x}{n}\right)^2 + \dots + \frac{1}{r}\left(\frac{-x}{n}\right)^r \right) \right). \end{aligned}$$

Hence

$$\begin{aligned} \frac{\mathcal{S}'_r(x)}{\mathcal{S}_r(x)} &= x^{r-2} + \sum_{n=1}^{\infty} n^{r-1} \left( \frac{1}{x-n} + \frac{(-1)^{r-1}}{x+n} + \left( \frac{1}{n} + \frac{x}{n^2} + \dots + \frac{x^{r-1}}{n^r} \right) \right. \\ &\quad \left. + (-1)^{r-1} \left( -\frac{1}{n} + \frac{x}{n^2} + \dots + (-1)^r \frac{x^{r-1}}{n^r} \right) \right). \end{aligned}$$

Here

$$\frac{1}{n} + \frac{x}{n^2} + \dots + \frac{x^{r-1}}{n^r} = \frac{\left(\frac{x}{n}\right)^r - 1}{x - n}$$

and

$$-\frac{1}{n} + \frac{x}{n^2} + \dots + (-1)^r \frac{x^{r-1}}{n^r} = \frac{(-1)^r \left(\frac{x}{n}\right)^r - 1}{x + n}.$$

Thus

$$\begin{aligned} \frac{\mathcal{S}'_r(x)}{\mathcal{S}_r(x)} &= x^{r-2} + \sum_{n=1}^{\infty} n^{r-1} \left( \frac{\left(\frac{x}{n}\right)^r}{x-n} - \frac{\left(\frac{x}{n}\right)^r}{x+n} \right) \\ &= x^{r-2} + \sum_{n=1}^{\infty} \frac{2x^r}{x^2 - n^2} \\ &= \pi x^{r-1} \cot(\pi x), \end{aligned}$$

where we used

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2}. \quad \square$$

**Proposition 2.** For  $0 \leq x < 1$  and for  $r = 2, 3, 4, \dots$ ,

$$\log \mathcal{S}_r(x) = \int_0^x \pi t^{r-1} \cot(\pi t) dt.$$

*Proof.* Since  $\mathcal{S}_r(0) = 1$ , both sides are 0 at  $x = 0$ . Hence it is sufficient to remark that the differentiations of both sides are  $\pi x^{r-1} \cot(\pi x)$  from Proposition 1.  $\square$

3. EULER'S INTEGRALS

Using Proposition 2 we show Theorems 1 and 2.

*Proof of Theorems 1 and 2.* By integration by parts in

$$\log \mathcal{S}_r(x) = \int_0^x \pi t^{r-1} \cot(\pi t) dt$$

we have

$$\begin{aligned} \log \mathcal{S}_r(x) &= [t^{r-1} \log(\sin \pi t)]_0^x - \int_0^x (r-1)t^{r-2} \log(\sin \pi t) dt \\ &= x^{r-1} \log(\sin \pi x) - (r-1) \int_0^x t^{r-2} \log(\sin \pi t) dt. \end{aligned}$$

Hence changing the variable to  $\theta = \pi t$  in the integral, we have

$$\log \mathcal{S}_r(x) = x^{r-1} \log(\sin \pi x) - \frac{r-1}{\pi^{r-1}} \int_0^{\pi x} \theta^{r-2} \log(\sin \theta) d\theta.$$

This gives Theorem 1. Then, letting  $x = 1/2$  we have Theorem 2. □

**Examples.**

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \log(\sin \theta) d\theta &= -\frac{\pi}{8} \log 2 - \pi \log \mathcal{S}_2\left(\frac{1}{4}\right), \\ \int_0^{\frac{\pi}{3}} \log(\sin \theta) d\theta &= \frac{\pi}{3} \log \frac{\sqrt{3}}{2} - \pi \log \mathcal{S}_2\left(\frac{1}{3}\right), \\ \int_0^{\frac{\pi}{4}} \theta \log(\sin \theta) d\theta &= -\frac{\pi^2}{64} \log 2 - \frac{\pi^2}{2} \log \mathcal{S}_3\left(\frac{1}{4}\right), \\ \int_0^{\frac{\pi}{3}} \theta \log(\sin \theta) d\theta &= \frac{\pi^2}{18} \log \frac{\sqrt{3}}{2} - \frac{\pi^2}{2} \log \mathcal{S}_3\left(\frac{1}{3}\right). \end{aligned}$$

4. A CALCULATION OF THE SPECIAL VALUE

*Proof of Theorem 3.* Since

$$\mathcal{S}_4\left(\frac{1}{2}\right) = e^{\frac{1}{24}} \prod_{n=1}^{\infty} \left( \left( \frac{2n-1}{2n+1} \right)^{n^3} \exp\left(n^2 + \frac{1}{12}\right) \right),$$

we put

$$\begin{aligned} A_N &= e^{\frac{1}{24}} \prod_{n=1}^N \left( \left( \frac{2n-1}{2n+1} \right)^{n^3} \exp\left(n^2 + \frac{1}{12}\right) \right) \\ &= \exp\left(\frac{1}{24} + (1^2 + \dots + N^2) + \frac{N}{12}\right) \times \prod_{n=1}^N (2n-1)^{n^3 - (n-1)^3} \\ &\quad \times (2N+1)^{-N^3} \end{aligned}$$

and show

$$\lim_{N \rightarrow \infty} A_N = 2^{\frac{1}{8}} \exp\left(\frac{9}{4} \zeta'(-2)\right) = 2^{\frac{1}{8}} \exp\left(-\frac{9\zeta(3)}{16\pi^2}\right).$$

Then the value of the integral follows from (6). We use the Stirling formula

$$N! = \prod_{n=1}^N n \sim \sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N}$$

and its generalization

$$\prod_{n=1}^N n^{n^2} \sim \exp(-\zeta'(-2)) N^{\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}} e^{-\frac{N^3}{9} + \frac{N}{12}},$$

which follow from the Euler-Maclaurin summation formula for  $\zeta'(s)$ :

$$\begin{aligned} \zeta'(s) = \lim_{N \rightarrow \infty} \left( - \sum_{n=1}^N n^{-s} \log n + \frac{N^{1-s} \log N}{1-s} - \frac{N^{1-s}}{(1-s)^2} \right. \\ \left. + \frac{1}{2} N^{-s} \log N - \frac{s}{12} N^{-s-1} \log N + \frac{1}{12} N^{-s-1} \right) \end{aligned}$$

valid in  $\operatorname{Re}(s) > -3$ . We refer to Hardy [H], Chap. XIII, for the Euler-Maclaurin summation formula and its applications. Then, letting  $s = 0$  and  $-2$  we see

$$\begin{aligned} \log \sqrt{2\pi} = -\zeta'(0) &= \lim_{N \rightarrow \infty} \left( \sum_{n=1}^N \log n - \left( \left( N + \frac{1}{2} \right) \log N - N \right) \right) \\ &= \lim_{N \rightarrow \infty} \log \left( \frac{N!}{N^{N+\frac{1}{2}} e^{-N}} \right) \end{aligned}$$

and

$$\begin{aligned} -\zeta'(-2) &= \lim_{N \rightarrow \infty} \left( \sum_{n=1}^N n^2 \log n - \left( \left( \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \right) \log N - \frac{N^3}{9} + \frac{N}{12} \right) \right) \\ &= \lim_{N \rightarrow \infty} \log \left( \frac{\prod_{n=1}^N n^{n^2}}{N^{\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}} e^{-\frac{N^3}{9} + \frac{N}{12}}} \right). \end{aligned}$$

Using

$$\begin{aligned} \prod_{n=1}^N (2n-1)^{n^3-(n-1)^3} &= \prod_{n=1}^N (2n-1)^{3n^2-3n+1} \\ &= \left( \prod_{n=1}^N (2n-1)^{(2n-1)^2} \right)^{\frac{3}{4}} \times \left( \prod_{n=1}^N (2n-1) \right)^{\frac{1}{4}} \\ &= \left( \frac{\prod_{n=1}^{2N} n^{n^2}}{\prod_{n=1}^N (2n)^{(2n)^2}} \right)^{\frac{3}{4}} \times \left( \frac{\prod_{n=1}^{2N} n}{\prod_{n=1}^N (2n)} \right)^{\frac{1}{4}} \end{aligned}$$

and the (generalized) Stirling formulas

$$\begin{aligned} \prod_{n=1}^{2N} n^{n^2} &\sim e^{-\zeta'(-2)} (2N)^{\frac{8N^3}{3}+2N^2+\frac{N}{3}} e^{-\frac{8N^3}{9}+\frac{N}{6}}, \\ \prod_{n=1}^N (2n)^{(2n)^2} &= 2^{4(1^2+\dots+N^2)} \left( \prod_{n=1}^N n^{n^2} \right)^4 \\ &\sim 2^{\frac{2}{3}N(N+1)(2N+1)} \left( e^{-\zeta'(-2)} N^{\frac{N^3}{3}+\frac{N^2}{2}+\frac{N}{6}} e^{-\frac{N^3}{9}+\frac{N}{12}} \right)^4, \\ \prod_{n=1}^{2N} n &\sim \sqrt{2\pi} (2N)^{2N+\frac{1}{2}} e^{-2N}, \\ \prod_{n=1}^N (2n) &= 2^N \prod_{n=1}^N n \sim \sqrt{2\pi} N^{N+\frac{1}{2}} e^{-N} 2^N, \end{aligned}$$

we have

$$\begin{aligned} A_N &\sim \exp\left(\frac{1}{24} + \left(\frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6}\right) + \frac{N}{12}\right) \\ &\quad \times e^{\frac{9}{4}\zeta'(-2)} N^{N^3-\frac{N}{4}} 2^{N^3-\frac{N}{4}} e^{-\frac{N^3}{3}-\frac{N}{8}} \\ &\quad \times N^{\frac{N}{4}} 2^{\frac{N}{4}+\frac{1}{8}} e^{-\frac{N}{4}} \times (2N+1)^{-N^3}. \end{aligned}$$

Hence, combining with

$$\begin{aligned} (2N+1)^{-N^3} &= (2N)^{-N^3} \left(1 + \frac{1}{2N}\right)^{-N^3} \\ &= (2N)^{-N^3} \exp\left(-N^3 \log\left(1 + \frac{1}{2N}\right)\right) \\ &\sim (2N)^{-N^3} \exp\left(-N^3 \left(\frac{1}{2N} - \frac{1}{2} \left(\frac{1}{2N}\right)^2 + \frac{1}{3} \left(\frac{1}{2N}\right)^3\right)\right) \\ &= (2N)^{-N^3} \exp\left(-\frac{N^2}{2} + \frac{N}{8} - \frac{1}{24}\right), \end{aligned}$$

we obtain the desired result

$$\lim_{N \rightarrow \infty} A_N = 2^{\frac{1}{8}} \exp\left(\frac{9}{4}\zeta'(-2)\right). \quad \square$$

*Remarks.* (1) The fact that  $\mathcal{S}_2(\frac{1}{2}) = \sqrt{2}$  is also proved as Theorem 3 and we get (1) again from (4). In fact:

$$\begin{aligned} \mathcal{S}_2\left(\frac{1}{2}\right) &= e^{\frac{1}{2}} \prod_{n=1}^{\infty} \left( \left( \frac{1 - \frac{1}{2n}}{1 + \frac{1}{2n}} \right)^n e \right) \\ &= \lim_{N \rightarrow \infty} e^{\frac{1}{2}} \prod_{n=1}^N \left( \left( \frac{2n-1}{2n+1} \right)^n e \right) \\ &= \lim_{N \rightarrow \infty} \left( e^{\frac{1}{2}} \left( \frac{1}{3} \right)^1 \left( \frac{3}{5} \right)^2 \left( \frac{5}{7} \right)^3 \cdots \left( \frac{2N-1}{2N+1} \right)^N e^N \right) \\ &= \lim_{N \rightarrow \infty} \left( e^{\frac{1}{2}} \frac{3 \cdot 5 \cdots (2N-1)}{(2N+1)^N} e^N \right) \\ &= \lim_{N \rightarrow \infty} \left( e^{\frac{1}{2}} \frac{(2N)!}{N! 2^{2N} N^N (1 + \frac{1}{2N})^N} e^N \right) \\ &= \sqrt{2} \end{aligned}$$

by the (usual) Stirling formula.

(2) The case of  $\mathcal{S}_3(1/2)$  is similar by using the generalized Stirling's formula:

$$\begin{aligned} \mathcal{S}_3\left(\frac{1}{2}\right) &= e^{\frac{1}{8}} \prod_{n=1}^{\infty} \left( \left( 1 - \frac{1}{4n^2} \right)^{n^2} e^{\frac{1}{4}} \right) \\ &= \lim_{N \rightarrow \infty} e^{\frac{1}{8}} \prod_{n=1}^N \left( \left( 1 - \frac{1}{4n^2} \right)^{n^2} e^{\frac{1}{4}} \right) \\ &= \lim_{N \rightarrow \infty} e^{\frac{N}{4} + \frac{1}{8}} \prod_{n=1}^N \left( \frac{(2n-1)(2n+1)}{(2n)^2} \right)^{n^2} \\ &= \lim_{N \rightarrow \infty} e^{\frac{N}{4} + \frac{1}{8}} \frac{\left( \prod_{n=1}^{2N} n^{n^2} \right)^{1/2}}{\left( \prod_{n=1}^N n^{n^2} \right)^4} \times \left( \frac{(2N)!}{2^N N!} \right)^{\frac{1}{2}} \times \frac{(2N+1)^{N^2}}{2^{4(1^2 + \cdots + N^2)}} \\ &= 2^{\frac{1}{4}} \exp\left(\frac{7}{2} \zeta'(-2)\right). \end{aligned}$$

Thus we obtain Euler's formula (2) via (5).

(3) Except for  $\mathcal{S}_2(1/2) = \sqrt{2}$  we do not know the algebraicity of  $\mathcal{S}_r(1/2)$  for  $r \geq 2$ . In fact we cannot deny even the optimistic expectation  $\mathcal{S}_r(1/2) \in 2^{\mathbf{Q}}$ .

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