ESSENTIAL NUMERICAL RANGE
OF ELEMENTARY OPERATORS

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Abstract. Let $A = (A_1, ..., A_p)$ and $B = (B_1, ..., B_p)$ denote two $p$-tuples of operators with $A_i \in \mathcal{L}(H)$ and $B_i \in \mathcal{L}(K)$. Let $R_{2,A,B}$ denote the elementary operators defined on the Hilbert-Schmidt class $C^2(H,K)$ by $R_{2,A,B}(X) = A_1XB_1 + ... + A_pXB_p$. We show that

$$co \left( W_e(A) \circ W(B) \right) \cup \left( W(A) \circ W_e(B) \right) \subseteq V_e(R_{2,A,B}).$$

Here $V_e(.)$ is the essential numerical range, $W(.)$ is the joint numerical range and $W_e(.)$ is the joint essential numerical range.

1. Introduction

Let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on a separable infinite-dimensional Hilbert space $H$. Let $A = (A_1, ..., A_p)$ and $B = (B_1, ..., B_p)$ denote two $p$-tuples of operators with $A_i \in \mathcal{L}(H)$ and $B_i \in \mathcal{L}(K)$. Let $R_{A,B} : \mathcal{L}(H) \longrightarrow \mathcal{L}(H)$ denote the elementary operators defined by

$$R_{A,B}(X) = A_1XB_1 + ... + A_pXB_p.$$ 

The class of Hilbert-Schmidt operators from a Hilbert space $H$ to a Hilbert space $K$ will be denoted by $C^2(H,K)$ and, of course, $C^2(H) = C^2(H,H)$; see [3]. Recall that $C^2(H,K)$ is a Hilbert space and that $A_iXB_i \in C^2(H,K)$ for every $A_i \in \mathcal{L}(H), X \in C^2(H,K)$ and $B_i \in \mathcal{L}(K)$. So the elementary operator $R_{A,B}$ is a bounded endomorphism of $C^2(H,K)$. We denote by $R_{2,A,B}$ the restriction of $R_{A,B}$ to $C^2(H,K)$.

If $\mathcal{A}$ is a Banach algebra with unit $e$, the algebraic numerical range of an arbitrary element $a \in \mathcal{A}$ is defined by

$$V(a) = \{ f(a); f \in \mathcal{A}', \|f\| = f(e) = 1 \}.$$ 

Here, of course, $\mathcal{A}'$ denotes the space of all continuous linear functionals on $\mathcal{A}$. Recall that $V(a)$ is a compact convex set.

For $T \in \mathcal{L}(H)$, the numerical range of $T$ is defined as

$$W(T) = \{ \langle Tx, x \rangle : \|x\| = 1 \}.$$
The essential numerical range, $V_e(T)$, is (by definition) the numerical range of the coset $T + K(H)$ in the Calkin algebra $\mathcal{L}(H)/K(H)$ where $K(H)$ is the ideal of all compact operators on $H$; see [1] and [2]. It is known that $V_e(T) \subseteq W(T) \cap K(H)$, the closure of $W(T)$.

For a $p$-tuple $A = (A_1, ..., A_p)$ of operators on a Hilbert space $H$ we define:

- i) the joint numerical range of $A$ by
  \[ W(A) = \{(\langle A_1 x, x \rangle, ..., \langle A_p x, x \rangle); \ x \in H, \|x\| = 1\}; \]
- ii) the joint essential numerical range of $A$ by
  \[ \lambda \in W_e(A) \text{ if } \lambda = (\lambda_1, ..., \lambda_p) \in C^p \text{ and there exists an orthonormal sequence } (x_n) \text{ in } H \text{ such that } \lambda_i = \text{Lim} (A_i x_n, x_n), \ i = 1, ..., p. \]

To simplify the statements, we shall use the following notation: for $\alpha, \beta \in C^n$, we let $\alpha \circ \beta = \sum_{i=1}^p \alpha_i \beta_i$ and for $K, L \subseteq C^n$,
\[ K \circ L = \{ \alpha \circ \beta, \ \alpha \in K, \beta \in L \}. \]

For vectors $x, y \in H$, the notation $x \otimes y$ will refer to the operator in $L(H)$ defined by $x \otimes y(z) = \langle z, y \rangle x$.

In the past, elementary operators and their restrictions to norm ideals in $L(H)$ have been studied by many authors. Up to now, their spectra and their essential spectra have been characterized; see [3] [4] [5]. In [7], B. Magajna has determined the essential numerical range of the restriction of a generalized derivation to the class of Hilbert-Schmidt.

In this paper, we give some results about the essential numerical range of the restriction of an elementary operator to the class of Hilbert-Schmidt. More precisely, we prove that
\[ co[(W_e(A) \circ W(B)) \cup (W(A) \circ W_e(B))] \subseteq V_e(R_2, A, B), \]
and we give some consequences of this inclusion.

2. THE ESSENTIAL NUMERICAL RANGE

We need the following characterization of the essential numerical range, obtained by Fillmore, Stampfl, and Williams in [9].

Lemma 2.1. Let $T \in \mathcal{L}(H)$. Each of the following conditions is necessary and sufficient in order that $\lambda \in V_e(T)$:

1. $\langle Tx_n, x_n \rangle \rightarrow \lambda$ for some sequence $(x_n)$ of unit vectors such that $x_n \rightarrow 0$ weakly.
2. $\langle Te_n, e_n \rangle \rightarrow \lambda$ for some orthonormal sequence $(e_n)$.

The main result of this paper is the following.

Theorem 2.2. Let $H, K$ be two separable Hilbert spaces and $A = (A_1, ..., A_p)$, $B = (B_1, ..., B_p)$ two $p$-tuples with $A_i \in \mathcal{L}(H)$ and $B_i \in \mathcal{L}(K)$ for $i = 1, ..., p$. Then
\[ co[(W_e(A) \circ W(B)) \cup (W(A) \circ W_e(B))] \subseteq V_e(R_2, A, B). \]

Proof. Let $\lambda \in W_e(A)$. There exists an orthonormal sequence $(x_n)$ in $H$ such that $\lambda_i = \text{Lim} (A_i x_n, x_n)$ for each $i = 1, ..., p$.

Let $\mu \in W(B)$. There exists a unit vector $y$ in $K$ such that $\mu_i = \langle B_i y, y \rangle$. It is easily verified that $(x_n \otimes y)$ is an orthonormal sequence in $C^2(H, K)$ and
\[ \langle A_i (x_n \otimes y) B_i, x_n \otimes y \rangle = tr(A_i (x_n \otimes y) B_i (y \otimes x_n)) = \langle A_i x_n, x_n \rangle \cdot \langle B_i y, y \rangle. \]
Hence,
\[ \langle R_{2,A,B}(x_n \otimes y), x_n \otimes y \rangle = \sum_{i=1}^{p} (A_ix_n, x_n) \cdot \langle B_i y, y \rangle . \]

That is, \( \lambda \circ \mu \in V_e(R_{2,A,B}) . \)

The essential numerical range of the restriction of a generalized derivation to the class of Hilbert-Schmidt has been computed in [7], by B. Magajna. He has shown that
\[ V_e(\delta_{2,A,B}) = co[(V_e(A) - W(B)^-) \cup (W(A)^- - V_e(B))] . \]

Here we give only the easiest inclusion.

**Corollary 2.3.** For \( A \in \mathcal{L}(H) \) and \( B \in \mathcal{L}(K) , \)
\[ co[(V_e(A) - W(B)^-) \cup (W(A)^- - V_e(B))] \subseteq V_e(\delta_{2,A,B}) . \]

If, in addition, \( V_e(A) = W(A)^- \) or \( V_e(B) = W(B)^- , \) then we have equality.

**Corollary 2.4.** For \( A \in \mathcal{L}(H) \) and \( B \in \mathcal{L}(K) , \)
\[ co[(V_e(A)W(B)^-) \cup (W(A)^- . V_e(B))] \subseteq V_e(M_{2,A,B}) , \]
\[ V_e(L_{2,A}) = W(A)^- \text{ and } V_e(R_{2,B}) = W(B)^-. \]

*Proof.* We have \( W(A)^- \subseteq V_e(L_{2,A}) \subseteq W(L_{2,A})^- = W(A)^- . \)

3. **Nonnegative operators and the essential numerical range**

**Lemma 3.1.** Let \( A \) be a nonnegative, selfadjoint operator and \( AB = BA . \) Then
\[ V_e(AB) \subseteq V_e(A)V_e(B) . \]

*Proof.* Let \( \lambda \in V_e(AB) . \) There exists a sequence \( (x_n) \) of unit vectors in \( H \) such that \( x_n \rightharpoonup 0 \) weakly and
\[ \lambda = \text{Lim} \langle AB(x_n), x_n \rangle . \]

Let \( y_n = A^*x_n . \) If \( y_n \rightharpoonup 0 \) for some subsequence, then 0 is in both sides of (1). If not and by passing to a subsequence if necessary, we can assume that \( y_n \neq 0 \) \( \forall n . \)

Put \( z_n = \frac{y_n}{\|y_n\|} . \) Then \( (z_n) \) is a sequence of unit vectors with \( z_n \rightharpoonup 0 \) weakly and
\[ \lambda = \text{Lim} \langle Bz_n, z_n \rangle \cdot \langle Ax_n, x_n \rangle . \]

But \( \text{Lim} \langle Bz_n, z_n \rangle \in V_e(B) . \) So \( \lambda \in V_e(A)V_e(B) . \)

*Corollary 3.2.** Let \( A \in \mathcal{L}(H) \) be a nonnegative, selfadjoint operator and \( B \in \mathcal{L}(K) . \) Then
\[ V_e(M_{2,A,B}) \subseteq W(A)^- W(B)^-. \]

*Proof.* Recall that \( L_{2,A}R_{2,B} = R_{2,B}L_{2,A} , \)
\[ V_e(L_{2,A}) = W(A)^- \text{ and } V_e(R_{2,B}) = W(B)^-. \]

The rest is fromLemma 3.1.

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References


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