SURFACES, SUBMANIFOLDS, AND ALIGNED FOX REIMBEDDING IN NON-HAKEEN 3-MANIFOLDS

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ABSTRACT. Understanding non-Haken 3-manifolds is central to many current endeavors in 3-manifold topology. We describe some results for closed orientable surfaces in non-Haken manifolds, and extend Fox’s theorem for submanifolds of the 3-sphere to submanifolds of general non-Haken manifolds. In the case where the submanifold has connected boundary, we show also that the $\partial$-connected sum decomposition of the submanifold can be aligned with such a structure on the submanifold’s complement.

1. INTRODUCTION

A closed orientable irreducible 3-manifold $N$ is called Haken if it contains a closed orientable incompressible surface; otherwise $N$ is non-Haken. In Section 2 we describe some results for surfaces in non-Haken manifolds. Generalizing a theorem of Fox ([F]), we show in Section 3 that a 3-dimensional submanifold of a non-Haken manifold $N$ is homeomorphic either to a handlebody complement in $N$ or the complement of a handlebody in $S^3$. Sections 2 and 3 are independent, but both represent progress towards understanding submanifolds of non-Haken manifolds. In Section 4 we combine the techniques from Section 2 with the results from Section 3 to show that if the submanifold $M \subset N$ is $\partial$-reducible and has connected boundary, then the embedding can be chosen to align a full collection of separating $\partial$-reducing disks in $M$ with similar disks in the complement of $M$.

2. HANDLEBODIES IN NON-HAKEEN MANIFOLDS

Let $N$ be a closed orientable 3-manifold, $F$ a closed orientable surface of non-trivial genus imbedded in $N$. Recall that $F$ is compressible if there exists an essential simple closed curve on $F$ that bounds an imbedded disk $D$ in $N$ with interior disjoint from $F$. $D$ is a compressing disk for $F$.

Definition 1. Suppose $F$ is a separating closed surface in an orientable irreducible closed 3-manifold $N$. $F$ is reducible if there exists an essential simple closed curve on $F$ that bounds compressing disks on both sides of $F$. The union of the two compressing disks is a reducing sphere for $F$. 

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Suppose $S$ is a collection of disjoint reducing spheres for $F$. A reducing sphere $S \in S$ is redundant if a component of $F - S$ that is adjacent to $S \cap F$ is planar. $S$ is complete if, for any disjoint reducing sphere $S'$, $S'$ is redundant in $S \cup S'$.

Let $\sigma(S)$ denote the number of components of $F - S$ that are not planar surfaces.

Since $N$ is irreducible, any sphere in $N$ is necessarily separating. Suppose a reducing sphere $S'$ is added to a collection $S$ of disjoint reducing spheres. If $S'$ is redundant, the number of non-planar complementary components in $F$ is unchanged, since $S'$ necessarily separates the component of $F - S$ that it intersects and the union of two planar surfaces along a single boundary component is still planar. If $S'$ is not redundant, then the number of non-planar complementary components in $F$ increases by one. Thus we have:

Lemma 2. Suppose $S \subset S'$ are two collections of disjoint reducing spheres for $F$ in $N$. Then $\sigma(S) \leq \sigma(S')$. Equality holds if and only if each sphere $S'$ in $S' - S$ is redundant in $S' \cup S$. In particular, $S$ is complete if and only if for every collection $S'$ such that $S \subset S'$, $\sigma(S) = \sigma(S')$.

Let $H$ be a handlebody imbedded in $N$. $H$ has an unknotted core if there exists a pair of transverse simple closed curves $c, d \subset \partial H$ such that $c \cap d$ is a single point, $d$ bounds an imbedded disk in $H$ and $c$ (the core) bounds an imbedded disk in $N$ transverse to $\partial H$. Note that the interior of the latter imbedded disk may intersect $H$.

Lemma 3. Let $F$ be a connected, closed, separating, orientable surface in a closed orientable irreducible 3-manifold $N$. Suppose that $F$ has compressing disks to both sides. Then at least one of the following must hold:

1. $F$ is a Heegaard surface for $N$.
2. $N$ is Haken.
3. There exist disjoint compressing disks for $F$ on opposite sides of $F$.

Proof. The proof is an application of the generalized Heegaard decomposition described in [ST]. Since $F$ is compressible to both sides, we can construct a handle decomposition of $N$ starting at $F$ so that $F$ appears as a “thick” surface in the decomposition. If $F$ is not a Heegaard surface, then this decomposition contains a “thin” surface $G$ adjacent to $F$. If $G$ is incompressible in $N$, then $N$ is Haken. If $G$ is compressible we apply [CG] to obtain the required disjoint compressing disks for $F$.

Theorem 4. Let $H$ be a handlebody of genus $g$ imbedded in a closed orientable irreducible non-Haken 3-manifold $N$. Let $G$ be the complement of $H$ in $N$. Let $F = \partial H = \partial G$. Suppose $F$ is compressible in $G$. Then at least one of the following must hold:

1. The Heegaard genus of $N$ is less than or equal to $g$.
2. $F$ is reducible.
3. $H$ has an unknotted core.

Proof. The proof is by induction on the genus of $H$. If $g = 1$, then the result of compressing $F$ into $G$ is a 2-sphere, necessarily bounding a ball in $N$. If a ball it bounds lies in $G$, then the Heegaard genus of $N$ is $\leq 1$. If a ball it bounds contains $H$, then $H$ is an unknotted solid torus in $N$, and so it has an unknotted core.
Suppose then that \( \text{genus}(H) = g > 1 \) and assume inductively that the theorem is true for handlebodies of genus \( g - 1 \). Suppose that \( G \), the complement of \( H \), has compressible boundary. If \( G \) is a handlebody, then \( G \cup_F H \) is a Heegaard splitting of genus \( g \) and we are done. So suppose \( G \) is not a handlebody. Then by Lemma 3 there are disjoint compressing disks on opposite sides of \( F \), say \( D \) in \( H \) and \( E \) in \( G \). Without loss of generality, we can assume that \( D \) is non-separating. Compress \( H \) along \( D \) to obtain a new handlebody \( H_1 \) with boundary \( F_1 \); let \( G_1 \) be the complement of \( H_1 \).

If \( \partial E \) is inessential in \( F_1 \), then it bounds a disk in \( H_1 \subset H \) as well, so \( F \) is reducible.

If \( \partial E \) is essential in \( F_1 \), then \( E \) is a compressing disk in \( G_1 \) and so we can apply the inductive hypothesis to \( H_1 \). If 1 or 3 holds, then it holds for \( H \), and we are done. Suppose instead \( F_1 \) is reducible. Let \( S \) be a collection of disjoint reducing spheres for \( F_1 \) chosen to maximize \( \sigma \) among all possible such collections and then, subject to that condition, further choose \( S \) to minimize \( |E \cap S| \). Clearly \( E \cap S \) contains no closed curves, else replacing a subdisk lying in the disk collection \( S \cap G_1 \) with an innermost disk of \( E - S \) would reduce \( |E \cap S| \). Similarly, we have

**Claim 1.** Suppose \( \epsilon \) is an arc component of \( \partial E - S \) and \( F_0 \) is the component of \( F_1 - S \) in which \( \epsilon \) lies. If \( \epsilon \) separates \( F_0 \) (so the ends of \( \epsilon \) necessarily lie on the same component of \( \partial F_0 \)), then neither component of \( F_0 - \epsilon \) is planar.

**Proof of Claim 1.** Let \( c_0 \) be the closed curve component of \( \partial F_0 \subset S \cap F_1 \) on which the ends of \( \epsilon \) lie and, of the two arcs into which the ends of \( \epsilon \) divide \( c_0 \), let \( \alpha \) be adjacent to a planar component of \( F_0 - \epsilon \). Then the curve \( \epsilon \cup \alpha \) clearly bounds a disk in both \( G_1 \) and \( H_1 \), and then so does the curve \( \epsilon' = \epsilon \cup (c_0 - \alpha) \). Let \( S' \) be a sphere in \( N \) intersecting \( F_1 \) in \( \epsilon' \) and \( S_0 \) be the reducing sphere in \( S \) containing \( c_0 \). Replacing \( S_0 \) with \( S' \) (or just deleting \( S_0 \) if \( \epsilon' \) is inessential in \( F_1 \)) gives a new collection \( S' \) of disjoint reducing spheres, intersecting \( \partial E \) in at least two fewer points. Moreover \( \sigma(S') = \sigma(S) \) since the only change in the complementary components in \( F_1 \) is to add to one component and delete from another a planar surface along an arc in the boundary. Then the collection \( S' \) contradicts our initial choice for \( S \), a contradiction that proves the claim.

Let \( H' \) be the closed complement of \( S \) in \( H_1 \), so \( H' \) is itself a collection of handlebodies.

**Claim 2.** Either \( F \) is reducible or \( \partial H' \) is compressible in \( N - H' \).

**Proof of Claim 2.** If \( \partial E \) is disjoint from \( S \) and is inessential in \( \partial H' \), then \( \partial E \) bounds a disk in \( H' \), hence in \( H \), so \( F \) is reducible. If \( \partial E \) is disjoint from \( S \) and is essential in \( \partial H' \), then \( E \) compresses \( \partial H' \) in \( N - H' \), verifying the claim. Finally, if \( E \) intersects \( S \), consider an outermost disk \( A \) cut off from \( E \) by \( S \). According to Claim 1, this disk, together with a subdisk of \( S \), constitute a disk \( E' \) that compresses \( \partial H' \) in \( N - H' \), proving the claim.

Following Claim 2, either \( F \) is reducible or the inductive hypothesis applies to a component \( H_0 \) of \( H' \). If 2 holds for \( H_0 \), then consider a reducing sphere \( S \) for \( H_0 \), isotoped so that the curve \( c = S \cap \partial H_0 \) is disjoint from the disks \( S \cap H_0 \). The disk \( S - H_0 \) may intersect \( H_1 \); by general position with respect to the dual 1-handles, each component of intersection is a disk parallel to a component of \( S \cap H_1 \). But each such disk can be replaced by the corresponding disk in \( S - H_1 \) so that in the
end $c$ also bounds a disk in $N - H_1$. After this change, $S$ is a reducing sphere for $F_1$ in $N$ and, since $c$ is essential in $H_0$, $\sigma(S \cup S) > \sigma(S)$, contradicting our initial choice for $S$. Thus in fact 1 or 3 holds for $H_0$, hence also for $H$.

In the specific case $N = S^3$, we apply precisely the same argument, combined with Waldhausen’s theorem \cite{W} on Heegaard splittings of $S^3$, to obtain:

**Corollary 5.** Let $H$ be a handlebody imbedded in $S^3$, and suppose $G$, the complement of $H$, has compressible boundary. Then either $H$ has an unknotted core or the boundary of $H$ is reducible.

This corollary is similar to \cite{MT}, Theorem 1.1), but no reimbedding of $S^3 - H$ is required.

### 3. Complements of handlebodies in non-Haken manifolds

In \cite{F} (see also \cite{MT} for a brief version) Fox showed that any compact connected 3-dimensional submanifold $M$ of $S^3$ is homeomorphic to the complement of a union of handlebodies in $S^3$. We generalize this result to non-Haken manifolds, showing that a submanifold $M$ of a non-Haken manifold $N$ has an almost equally simple description, that is, $M$ is homeomorphic to the complement of handlebodies either in $S^3$ or in $N$.

**Definition 6.** Let $N$ be a compact irreducible 3-manifold, and let $M$ be a compact 3-submanifold of $N$. We will say the complement of $M$ in $N$ is **standard** if it is homeomorphic to a collection of handlebodies or to $N \# (\text{collection of handlebodies})$. (We regard $B^3$ as a handlebody of genus 0.)

Note that in the latter case $M$ is actually homeomorphic to the complement of a collection of handlebodies in $S^3$.

**Theorem 7.** Let $N$ be a closed orientable irreducible non-Haken 3-manifold, and let $M$ be a connected compact 3-submanifold of $N$ with non-empty boundary. Then $M$ is homeomorphic to a submanifold of $N$ whose complement is standard.

**Proof.** The proof will be by induction on $n + g$ where $n$ is the number of components of $\partial M$ and $g$ is the genus of $\partial M$, that is, the sum of the genera of its components. If $n + g = 1$, then $\partial M$ is a single sphere. Since $N$ is irreducible, the sphere bounds a 3-ball in $N$. So either $M$ or its complement is a 3-ball and in either case the proof is immediate.

To verify the inductive step, suppose first that $\partial M$ has multiple components $T_1, \ldots, T_n, n \geq 2$. Each component $T_i$ must bound a distinct component $J_i$ of $N - M$ since each must be separating in the non-Haken manifold $N$. Let $M' = M \cup J_n$; by inductive assumption $M'$ can be reimbedded so that its complement is standard. After the reimbedding, remove $J_n$ from $M'$ to recover a homeomorph of $M$ and adjoin $J_1$ (now homeomorphic either to a handlebody or to $N \# (\text{handlebody})$) instead. Reimbed the resulting manifold so that its complement is standard and remove $J_1$ to recover $M$, now with standard complement.

Henceforth we can therefore assume that $\partial M$ is connected and not a sphere. Since $N$ is non-Haken there exists a compressing disk $D$ for $\partial M$ in $N - \partial M$; the compressing disk lies either in $M$ or in its closed complement $J$. 

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Case 1. $\partial D$ is non-separating on $\partial M$.
If $D$ lies inside $M$, compress $M$ along $D$ to obtain $M'$ and use the induction hypothesis to find an imbedding of $M'$ with standard complement. Reconstruct $M$ by attaching a trivial 1-handle to $M'$, thus simultaneously attaching a trivial 1-handle to the complement.

If $D$ lies outside $M$, attach a 2-handle to $M$ corresponding to $D$ to obtain $M'$, whose connected boundary has lower genus. Invoking the inductive hypothesis, imbed $M'$ in $N$ with standard complement. Reconstruct $M$ from $M'$ by removing a co-core of the attached 2-handle, thus adding a 1-handle to the complement of $M'$.

Case 2. $\partial D$ is separating on $\partial M$.
Suppose $D$ lies outside $M$. Then $D$ also separates the closed complement $J$ of $M$ into two components, $J_1$ and $J_2$, since $H_2(N) = 0$. Denote the components of $\partial M - \partial D$ by $\partial_1 \subset J_1$ and $\partial_2 \subset J_2$, both of positive genus. Let $M' = M \cup J_2$. Reimbed $M'$ so that its complement is standard. The boundary of $M'$ consists of $\partial_1$ together with a disk. Since the complement of $M'$ is standard, there is a non-separating compressing disk $D'$ for $\partial M'$ contained in the complement of $M'$. $D'$ is also a non-separating compressing disk for the reimbedded $\partial M$ (which is contained in $M'$). Apply case 1 to this new imbedding of $M$.

We can now suppose that the only compressing disks for $\partial M$ are separating compressing disks lying inside $M$. Choose a family $D$ of such $\partial$-reducing disks for $M$ that is maximal in the sense that no component of $M' = M - D$ is itself $\partial$-compressible. Since each compressing disk is separating, $\text{genus}(\partial M') = \text{genus}(\partial M) > 0$, so $\partial M'$ is compressible in $N$. Such a compressing disk $E$ cannot lie inside $M'$, by construction, so it lies in the connected manifold $N - M'$; let $M_1$ be the component of $M'$ on whose boundary $\partial E$ lies. Since each disk in $D$ was separating, $M$ has the simple topological description that it is the boundary-convert sum of the components of $M'$. So $M$ can easily be reconstructed from $M'$ in $N - M'$ by doing boundary connect sum along arcs connecting each component of $M' - M_1$ to $M_1$ in $N - (M' \cup E)$. After this reimbedding of $M$, $E$ is a compressing disk for $\partial M$ that lies outside $M$, so we can conclude the proof via one of the previous cases.

4. Aligned Fox reimbedding

Now we combine results from the previous two sections and consider this question: If $M$ is a connected 3-submanifold of a non-Haken manifold $N$ and $M$ is $\partial$-reducible, to what extent can a reimbedding of $M$, so that its complement is standard, have its $\partial$-reducing disks aligned with meridian disks of its complement. Obviously non-separating disks in $M$ cannot have boundaries matched with meridian disks of $N - M$, since $N$ contains no non-separating surfaces. But at least in the case when $\partial M$ is connected, this is the only restriction.

Definition 8. For $M$ a compact irreducible orientable 3-manifold, define a disjoint collection of separating $\partial$-reducing disks $D \subset M$ to be full if each component of $M - D$ is either a solid torus or is $\partial$-irreducible.

For $M$ reducible, $D \subset M$ is full if there is a prime decomposition of $M$ so that for each summand $M'$ of $M$ containing the boundary, $D \cap M'$ is full in $M'$.

$M \subset N$ a 3-submanifold is aligned to a standard complement if the complement of $M$ is standard and there is a (complete) collection of reducing spheres $S$ for $\partial M$ so that $S \cap M$ is a full collection of $\partial$-reducing disks for $M$. 

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Lemma 9. Suppose $M$ is an irreducible orientable 3-manifold with boundary and $M$ is expressed as a boundary connect sum in two ways: $M = M_1 \# M_2 \# \cdots \# M_n = M'_1 \# M'_2 \# \cdots \# M'_{n^*}$, where each $M_i, M'_i$ is either a solid torus or $\partial$-irreducible. Then, after rearrangement, $n^* = n$ and $M_i \cong M'_i$.

Proof. One can easily prove the theorem from first principles, along the lines of, e.g. [H] Theorem 3.21, the standard proof of the corresponding theorem for a connected sum. But a cheap start is to just double $M$ along its boundary to get a manifold $DM$. The decompositions above double to give connected sum decompositions of $DM$ in which each factor consists of either $S^1 \times S^2$ or the double of an irreducible, $\partial$-irreducible manifold, which is then necessarily irreducible. Then [H] Theorem 3.21 implies that $n = n^*$ and that the two original decompositions of $M$ also each contain the same number of solid tori. After removing these, we are reduced to the case in which the only $\partial$-reducing disks in $M$ are separating and $n^* = n$

Following the outline suggested by the proof of [H] Theorem 3.21, choose a disk $D$ that separates $M$ into the component $M_n$ and the component $M_1 \# M_2 \# \cdots \# M_{n-1}$. Choose disks $E_1, \ldots, E_{n-1}$ that separate $M$ into the components $M'_1, M'_2, \ldots M'_{n^*}$. Choose the disks to minimize the number of intersection components in $D \cap \bigcup \{E_i\}$. Since each manifold is irreducible and $\partial$-irreducible, a standard innermost disk, outermost arc argument (in $D$) shows that in fact $D$ is then disjoint from $\{E_i\}$, so $D \subset M^*_n$ (say). Since $M^*_n$ is $\partial$-irreducible, $D$ is $\partial$-parallel in $M^*_n$. So in fact (with no loss of generality) $M_n \cong M^*_n$ and $M_1 \# M_2 \# \cdots \# M_{n-1} \cong M'_1 \# M'_2 \# \cdots \# M'_{n-1}$. The result follows by induction.

Theorem 10. Let $N$ be a closed orientable irreducible non-Haken 3-manifold, and $M$ be a connected compact 3-submanifold of $N$ with connected boundary. Then $M$ can be reembedded in $N$ with standard complement so that $M$ is aligned to the standard complement.

Proof. The proof is by induction on the genus of $\partial M$. Unless $M$ has a separating $\partial$-reducing disk, there is nothing beyond the result of Theorem 3 to prove. So we assume that $M$ does have a separating $\partial$-reducing disk; in particular, the genus of $\partial M$ is $g \geq 2$. We inductively assume that the theorem has been proven whenever the genus of $\partial M$ is less than $g$.

The first observation is that it suffices to find an embedding of $M$ in $N$ so that there is some reducing sphere $S$ for $\partial M$ in $N$, for such a reducing sphere divides $J = N - M$ into two components $J_1$ and $J_2$. Apply the inductive hypothesis to $M \cup J_1$ to reembed it with an aligned complement $J'_1$. Notice that by a standard innermost disk argument, the reducing spheres can be taken to be disjoint from $S$. After this reembedding, apply the inductive hypothesis to $M \cup J'_2$ to reembed it so that its complement $J''_1$ is aligned. After this reembedding, $M$ has aligned complement $J'_1 \cup S - M J''_2$.

Our goal then is to find a reembedding of $M$ so that afterwards $\partial M$ has a reducing sphere. First use Theorem 3 to reembed $M$ in $N$ so that its complement $J$ is standard, i.e. either a handlebody or $N\#$ (handlebody). Since $M$ is $\partial$-reducible, Lemma 3 applies: either $M$ is itself a handlebody (in which case the required reembedding of $M$ is easy) or there are disjoint compressing disks $D$ in $J$ and $E$ in $M$. Since $J$ is standard, $D$ can be chosen to be non-separating in $J$. Then $\partial E$ is
not homologous to $\partial D$ in $\partial M$, so $\partial E$ is either separating in $\partial M$ or non-separating in $\partial M - \partial D$. In the latter case, two copies of $E$ can be banded together along an arc in $\partial M - \partial D$ to create a separating essential disk in $M$ that is disjoint from $D$. The upshot is that we may as well assume that $D \subset J$ is non-separating and $E \subset M$ is separating.

Add a 2-handle to $M$ along $D$ to get $M'$, still with standard complement $J'$. Dually, $M$ can be viewed as the complement of the neighborhood of an arc $\alpha \subset M'$. If $\partial E$ is inessential in $\partial M'$, it bounds a disk $D'$ in $J' \subset J$. Then the sphere $D' \cup E$ is a reducing sphere for $M$ as required. So we may as well assume that $\partial E$ is essential in $\partial M'$ and of course still separates $M'$. By the inductive assumption, $M'$ can be embedded in $N$ so that its complement is aligned, but note that this does not immediately mean that $\partial E$ itself bounds a disk in $N - M'$. Let $S$ be a complete collection of reducing spheres for $\partial M'$ intersecting $M'$ in a full collection of disks.

$E$ divides $M'$ into two components, $U$ and $V$ with, say, $\alpha \subset U$. If $M'$ is reducible (i.e. contains a punctured copy of $N$) an innermost (in $E$) disk argument ensures that the reducing sphere is disjoint from $E$. By possibly tubing $E$ to that reducing sphere, we can ensure that the $N$-summand, if it lies in $M'$, lies in $U \subset M'$. That is, we can arrange that $V$ is irreducible. $E$ extends to a full collection of disks in $M'$, with the new disks dividing $U$ and $V$ into $\partial$-connected sums: $U = U_1 \# \cdots \# U_m, V = V_1 \# \cdots \# V_n, m,n \geq 1$, with each $U_i, V_j$ either $\partial$-irreducible or a solid torus (with one of the $U_i$ possibly containing $N$ as a connect summand). By Lemma 10, some component $V'$ of $M' - S$ is homeomorphic to $V_n$. Tube together all components of $S$ incident to $V'$ along arcs in $\partial V'$ to get a reducing sphere $S'$ dividing $M'$ into two components, one homeomorphic to $V_n$ and the other homeomorphic to $U_1 V_1 \# V_2 \# \cdots \# V_{n-1}$. The latter homeomorphism carries $\alpha \subset U$ to an arc $\alpha'$ that is disjoint from the reducing sphere $S'$. Then $M' - \eta(\alpha')$ is homeomorphic to $M$ and admits the reducing sphere $S'$. In other words, the reembedding of $M$ that replaces $M' - \eta(\alpha)$ with $M' - \eta(\alpha')$ makes $\partial M$ reducible in $N$, completing the argument.

**Corollary 11.** Given $M \subset N$ as in Theorem 10, suppose $D$ is a full set of disks in $M$. Then, with at most one exception, each component of $M - D$ embeds in $S^3$.

**Proof.** Following Theorem 10, reembed $M$ in $N$ with the standard complement so that $M$ is aligned to the standard complement. Then there is a collection $S$ of disjoint spheres in $N$ so that, via Lemma 11, $M - S$ and $M - D$ are homeomorphic. Since $N$ is irreducible, each component but at most one of $N - S$ is a punctured 3-ball. Finally, each component of $N - S$ contains at most one component of $M - S$ since each component of $S$ is separating.

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