PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 6, Pages 1727–1731
S 0002-9939(04)07705-6
Article electronically published on November 19, 2004

REFLECTION SYMMETRY AND SYMMETRIZABILITY OF HILBERT SPACE OPERATORS

ZOLTÁN SEBESTYÉN AND JAN STOCHEL

(Communicated by Joseph A. Ball)

ABSTRACT. A general factorization theorem for symmetrizable operators relating their spectra to spectra of selfadjoint operators induced by minimal factorizations is established. Its modified version essentially improves and completes a theorem of Jorgensen, which concerns diagonalizing operators with reflection symmetry.

The problem of changing the spectrum of an operator with a view to getting a new spectrum with physical desiderata has been studied by many authors including Segal [8] and Jorgensen (see [3] and references therein). Jorgensen has proposed in [3] an axiomatic approach to solving these kinds of problems based on a notion of reflection symmetry. The aim of this note is to emphasize the relationship between reflection symmetry and symmetrizability, a notion invented at the beginning of the last century (cf. [4], [5], [2], [9] and [6] as well as references therein). Recapitulating some general factorization theorems for symmetrizable operators due to the first named author [6] enables us to improve and complete the main result of [3], Theorem 3.1. In particular, we strengthen part (v) of this theorem by replacing the spectral radius inequality by a more general spectral inclusion (cf. part (v) of our Corollary 2). This means that the new spectrum, not only its spectral radius, can be controlled in a general situation.

We begin by recalling some basic concepts from [6]. Let A be a positive bounded (linear) operator on a (complex) Hilbert space $\mathcal{K}$ with inner product $(\cdot,\cdot)$. The range $\operatorname{ran} A^{1/2}$ of $A^{1/2}$ becomes a Hilbert space $\mathcal{M}(A^{1/2})$ under the inner product

$$\langle A^{1/2}x, A^{1/2}y\rangle := (u,v),$$

where u and v are unique vectors from $\overline{\operatorname{ran} A}$, the closure of the range of A, such that $A^{1/2}x = A^{1/2}u$ and $A^{1/2}y = A^{1/2}v$. This is a well-known de Branges space (cf. [1]). The unitary operator $V_A : \mathcal{M}(A^{1/2}) \to \overline{\operatorname{ran} A}$ arises from a densely defined one as follows:

$$V_A(Ax) = A^{1/2}x, \quad x \in \mathcal{K}. \tag{1}$$

Received by the editors January 22, 2004 and, in revised form, February 14, 2004.
2000 *Mathematics Subject Classification.* Primary 47A05, 47A10; Secondary 47B32.
Key words and phrases. Operators in Hilbert space, de Branges space, factorizations of operator products, symmetrizable operators, operators with reflection symmetry.
The research of the second author was supported by KBN grant 2 P03A 037 024.

The adjoint operator $V_A^*: \overline{\operatorname{ran} A} \to \mathcal{M}(A^{1/2})$ satisfies the ensuing equality

$$V_A^*(A^{1/2}x) = Ax, \quad x \in \mathcal{K}. \tag{2}$$

One more important bounded operator $W_A: \mathcal{K} \to \mathcal{M}(A^{1/2})$ can be defined via

$$W_A(x) = Ax, \quad x \in \mathcal{K}. \tag{3}$$

It is easily seen that the norm of W_A is equal to the square root of the norm of A and that the range of W_A is dense in $\mathcal{M}(A^{1/2})$. Therefore the kernel of the adjoint operator $W_A^*: \mathcal{M}(A^{1/2}) \to \mathcal{K}$ is trivial. In other words, W_A^* is an imbedding. In fact, the operator W_A^* acts as the identity map, because

$$W_A^*(A^{1/2}x) = A^{1/2}x, \quad x \in \mathcal{K}. \tag{4}$$

Consequently, the operator A can be factorized as follows:

$$W_A^* W_A = A. \tag{5}$$

A pair $(\mathcal{L}, W)$ is said to be a *minimal factorization* of A if $\mathcal{L}$ is a Hilbert space, $W: \mathcal{K} \to \mathcal{L}$ is a bounded operator with dense range, and $A = W^*W$ (cf. [7]). If $(\mathcal{L}', W')$ is another minimal factorization of A, then there exists a (unique) unitary isomorphism $T: \mathcal{L} \to \mathcal{L}'$ such that $TW = W'$. Note that the above-defined pair $(\mathcal{M}(A^{1/2}), W_A)$ is a particular minimal factorization of A. It follows from (5) (or from (3)) that the kernels of W_A and A are equal to each other. Hence W_A is injective if and only if A is injective.

The ensuing theorem recapitulates the main results of [6] (see also [7] for extensions and generalizations to the case of unbounded operators).

Theorem. *Let A and B be bounded operators on a Hilbert space $\mathcal{K}$ such that A is positive and AB is selfadjoint.*[1] *Then there exists a unique bounded selfadjoint operator S on the Hilbert space $\mathcal{M}(A^{1/2})$ such that*

$$S(Ax) = A(Bx), \quad x \in \mathcal{K}. \tag{6}$$

The operators $V := V_A$, $W := W_A$ and S satisfy the following conditions:

1° $W^*W = A$,
2° $SW = WB$ (*equivalently:* $W^*SW = AB$),
3° $\sigma(S) \subseteq \sigma(B) \cap \mathbb{R}$, *where $\sigma(C)$ stands for the spectrum of an operator C,*
4° $VSV^*A^{1/2} = A^{1/2}B$.

The system $(\mathcal{M}(A^{1/2}), W, S)$ is unique up to unitary equivalence, i.e. if $(\mathcal{L}', W')$ is a minimal factorization of A and S' is a bounded selfadjoint operator on $\mathcal{L}'$ such that $S'W' = W'B$, then there exists a unitary isomorphism $T: \mathcal{L}' \to \mathcal{M}(A^{1/2})$ such that

5° $TW' = W$,
6° $TS' = ST$.

Proof. Notice that $\mathcal{M}(A^{1/2})$ (resp. W) corresponds to $\mathcal{H}_A$ (resp. J^*) in the Introduction of [6], S corresponds to $\hat{B}$ in Theorem 1 of [6], and finally V (resp. VSV^*) corresponds to U (resp. S) in the proof of Theorem 3 of [6]. It is now clear that

[1] Such a B is said to be *symmetrizable* with respect to A; cf. [4], [5], [2], [9] and [6].

1° is the same as (5), 2° is a direct consequence of (4) and part (i) of Theorem 1 of [6] (it can also be deduced from (3), (6) and (4)), 3° corresponds to part (ii) of Theorem 1 of [6], and 4° coincides in its essence with part (iv) of Theorem 3 of [6] (it can also be inferred from (2), (6) and (1)).

Since $(\mathcal{L}', W')$ and $(\mathcal{M}(A^{1/2}), W)$ are minimal factorizations of A, there exists a unitary isomorphism $T : \mathcal{L}' \to \mathcal{M}(A^{1/2})$ that satisfies 5°. It is now a matter of routine to show that 2°, 5° and $S'W' = W'B$ imply 6° (check 6° on $\operatorname{ran} W'$). $\square$

Regarding the Theorem, we see that if $A = W^*W$, $SW = WB$ and S is selfadjoint (A, B, W and S are bounded operators), then A is positive and AB is selfadjoint.

Corollary 1. *Let J, P, U be bounded operators on a Hilbert space $\mathcal{H}$. Assume that P is an orthogonal projection such that PJP is positive and*

$$(PJP)UP = (UP)^*PJP. \tag{7}$$

Then the compression operators $A := PJ|_{\mathcal{K}}$ and $B := PU|_{\mathcal{K}}$ acting on $\mathcal{K} := \operatorname{ran} P$ fulfill all the assumptions of the Theorem.

The next corollary[2] improves and extends Theorem 3.1 of [3]. The assumptions of Corollary 2 are essentially weaker than those of Theorem 3.1 of [3], as shown in the Remark ensuing Corollary 2. For the convenience of the reader, we follow the notation and the way of numbering which appear in Theorem 3.1 of [3]. Let us point out that the assertion (ix) of Corollary 2 does not appear in Theorem 3.1 of [3]; in turn, the condition (viii) of this theorem concerns the question of the existence of an injective W.

Corollary 2. *Let $\mathcal{K}$ be a closed linear subspace of a Hilbert space $\mathcal{H}$, and let P be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$. Assume U is a bounded operator on $\mathcal{H}$ leaving $\mathcal{K}$ invariant and J is a bounded operator on $\mathcal{H}$ such that*

(i) $JU|_{\mathcal{K}} = U^*J|_{\mathcal{K}}$ *(equivalently: $JUP = U^*JP$),*
(ii) *PJP is positive.*

Then the following statements are valid:

a) *There exist a unitary operator $V : \mathcal{M}((PJ|_{\mathcal{K}})^{1/2}) \to \overline{\operatorname{ran}(PJ|_{\mathcal{K}})}$, a bounded operator $W : \mathcal{K} \to \mathcal{M}((PJ|_{\mathcal{K}})^{1/2})$ with dense range, and a bounded selfadjoint operator S on the Hilbert space $\mathcal{M}((PJ|_{\mathcal{K}})^{1/2})$ such that*
   (iii) $SW = WU|_{\mathcal{K}}$,
   (iv) $W^*W = PJ|_{\mathcal{K}}$,
   (v) $\sigma(S) \subseteq \sigma(U|_{\mathcal{K}}) \cap \mathbb{R}$ *and* $\|S\| \leq \operatorname{sp}(U)$, *where* $\operatorname{sp}(U)$ *stands for the spectral radius of* U,
   (viii) $\ker W = \ker(PJ|_{\mathcal{K}})$, *where "*$\ker$*" is the abbreviation for "kernel",*
   (ix) $VSV^*(PJ|_{\mathcal{K}})^{1/2} = (PJ|_{\mathcal{K}})^{1/2}(U|_{\mathcal{K}})$.
b) *The system $(\mathcal{M}((PJ|_{\mathcal{K}})^{1/2}), W, S)$ is unique up to unitary equivalence, i.e. if $\mathcal{L}'$ is a Hilbert space, $W' : \mathcal{K} \to \mathcal{L}'$ is a bounded operator with dense range,*

[2] Applying Corollary 2 (or Corollary 1) to $\mathcal{K} = \mathcal{H}$, we get back to the Theorem. In turn, if we apply Theorem 3.1 of [3] to $\mathcal{K} = \mathcal{H}_0$ (our space $\mathcal{H}$ is denoted in [3] by $\mathcal{H}_0$), we arrive at a very particular situation: $J = W =$ the identity operator and $S = U = U^*$.

and S' is a bounded selfadjoint operator on $\mathcal{L}'$ such that $W'^{*}W' = PJ|_{\mathcal{K}}$ and $S'W' = W'U|_{\mathcal{K}}$, then there exists a unitary isomorphism $T : \mathcal{L}' \to \mathcal{M}((PJ|_{\mathcal{K}})^{1/2})$ such that
(vi) $TW' = W$,
(vii) $TS' = ST$.

Proof. Since $PUP = UP$, we deduce from (i) that

$$(PJP)UP = PJUP = PU^{*}JP = (UP)^{*}JP = (PUP)^{*}JP = (UP)^{*}PJP,$$

which means that (7) holds. This enables us to apply Corollary 1. In particular, we get the inclusion $\sigma(S) \subseteq \sigma(U|_{\mathcal{K}})$, which leads to

$$\|S\| = \operatorname{sp}(S) \leq \operatorname{sp}(U|_{\mathcal{K}}) = \lim_{n\to\infty} \|U^n|_{\mathcal{K}}\|^{1/n} \leq \lim_{n\to\infty} \|U^n\|^{1/n} = \operatorname{sp}(U).$$

This completes the proof. □

Remark. Let us notice that condition (i) of Corollary 2 results from the following two conditions (at this stage we need not assume that $U(\mathcal{K}) \subseteq \mathcal{K}$):

(i-a) $JUX = U^{*}$,
(i-b) $XJ|_{\mathcal{K}}$ is the identity operator on $\mathcal{K}$ (equivalently: $XJP = P$),

where X is a bounded operator on $\mathcal{H}$. Indeed, (i-b) and (i-a) imply

$$JUx = (JUX)Jx = U^{*}Jx, \quad x \in \mathcal{K}.$$

On the other hand, neither (i-a) nor (i-b) follows from (i) because the operator $J := 0$ satisfies (i) and (ii), but it fails to satisfy (i-a) and (i-b) (provided $U \neq 0$ and $\mathcal{K} \neq \{0\}$).

Let us look at particular choices for X. If J is a bijection, then the condition (i-b) is satisfied with $X = J^{-1}$ (in fact, it is sufficient to assume that J has a left inverse X); for such an X the condition (i-a) means that the operators U and U^{*} are similar. If J is a unitary operator (or if J is an isometry), then the condition (i-b) is satisfied with $X = J^{*}$; now (i-a) can be interpreted to mean that the operators U and U^{*} are unitarily equivalent. Finally, the case $J^{-1} = J^{*} = J$ considered in Theorem 3.1 of [3] follows from any of the two mentioned above (in the terminology of [3], such a J is called a period-2 unitary operator); therefore this theorem, except for its part (c), is a consequence of Corollary 2 (the contractiveness of W results from that of J via the condition (iv) of Corollary 2). What is more, the assumption of the unitarity of J made in Theorem 3.1 of [3] is superfluous (however, now W need not be a contraction).

Acknowledgement

The first named author gratefully acknowledges the support by a bilateral agreement between the Eötvös Loránd University in Budapest and the Jagiellonian University in Kraków. This note was completed during a stay of the first named author at the Jagiellonian University, who takes this opportunity to thank professors Franciszek H. Szafraniec and Jan Stochel for their warm hospitality.

References

[1] T. Ando, *De Branges spaces and analytic operator functions*, Lecture Note, Hokkaido University, Sapporo, Japan, 1990.
[2] E. Garbe, Zur Theorie der Integralgleichung dritter Art, *Math. Annalen* **76** (1915), 409-416.
[3] P. E. T. Jorgensen, Diagonalizing operators with reflection symmetry, *J. Funct. Anal.* **190** (2002), 93-132. MR1895530 (2003e:47069)
[4] J. Marty, Valeurs singulières d'une équation de Fredholm, *C.R. Acad. Sci.,* Paris **150** (1910), 1499-1502.
[5] A. J. Pell, Applications of biorthogonal systems of functions to the theory of integral equations, *Trans. Amer. Math. Soc.* **12** (1911), 165-180.
[6] Z. Sebestyén, Positivity of operator products, *Acta Sci. Math. (Szeged)*, **66** (2000), 287-294. MR1768867 (2001b:47004)
[7] Z. Sebestyén, J. Stochel, On products of unbounded operators, *Acta Math. Hungar.*, **100** (2003), 105-129. MR1984863 (2004c:47003)
[8] I. E. Segal, Real spinor fields and the electroweak interaction, *J. Funct. Anal.* **154** (1998), 542-558. MR1612662 (99g:81075)
[9] A. C. Zaanen, *Linear analysis. Measure and integral, Banach and Hilbert space, linear integral equations,* Interscience Publishers Inc., New York; North-Holland Publishing Co., Amsterdam; P. Noordhoff N.V., Groningen, 1953. MR0061752 (15:878b)

Department of Applied Analysis, Eötvös University, H-1117 Budapest, Pázmány Péter sétány 1/c, Hungary
E-mail address: sebesty@cs.elte.hu

Instytut Matematyki, Uniwersytet Jagielloński, Reymonta 4, 30-059 Kraków, Poland
E-mail address: stochel@im.uj.edu.pl