PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 6, Pages 1665–1675
S 0002-9939(04)07714-7
Article electronically published on December 20, 2004

REAL INTERPOLATION OF VECTOR-VALUED SPACES IN NON-DIAGONAL CASE

IRINA ASEKRITOVA AND NATAN KRUGLJAK

(Communicated by N. Tomczak-Jaegermann)

Abstract. It is shown that the formula

$$(l_{p_0}^{s_0}(A_0),...,l_{p_n}^{s_n}(A_n))_{\vec{\theta},q} = l_q^s((A_0,...,A_n)_{\vec{\theta},q}),$$

where $\vec{\theta} = (\theta_0,...,\theta_n)$ and $s = \theta_0 s_0 + ... + \theta_n s_n$ is correct under the restrictions $A_{n-1} = A_n$ and $s_{n-1} \neq s_n$. It is also true if we suppose that

$$A_n = (A_0, A_1,...,A_{n-1})_{\vec{\lambda},p}, s_n \neq \lambda_0 s_0 + \lambda_1 s_1 + ... + \lambda_{n-1} s_{n-1},$$

and the spaces $A_0, A_1,...,A_{n-1}$ are functional Banach or quasi-Banach lattices on the same measure space (Ω,μ).

1. Introduction

It is well known that interpolation of vector-valued spaces is a powerful tool for several different areas, and it is especially important for modern analysis (see, for example, the survey [GKKT] and the book [BL], Chapter 6). The first vector-valued result for the real method was obtained in 1964 by J.-L. Lions and J. Peetre (see [LP]) who proved that in the diagonal case, i.e. $\frac{1}{q} = \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ for any couple (A_0, A_1), the following holds:

$$(L_{p_0}(A_0), L_{p_1}(A_1))_{\theta,q} = L_q((A_0, A_1)_{\theta,q}). \tag{1.1}$$

The K-functional for the couple $(L_{p_0}(A_0), L_{p_1}(A_1))$ was calculated by G. Pisier (see [P]) in connection with some problems in analysis.

Of course, it would be nice to have an analogue of the formula (1.1) in the non-diagonal case, i.e. when $q \neq p_\theta$, but unfortunately such a formula is not true (see M. Cwikel [C]). The only known result in the non-diagonal case is the following (see [BL]):

$$(l_{p_0}^{s_0}(A), l_{p_1}^{s_1}(A))_{\theta,q} = l_q^{s_\theta}(A), \qquad s_\theta = (1-\theta)s_0 + \theta s_1\ , s_0 \neq s_1, \tag{1.2}$$

where $l_p^s(A)$ denotes the space of all sequences $a = \{a_k\}_{k\in\mathbb{Z}}\, , a_k \in A$, such that

$$\|a\|_{l_p^s(A)} = (\sum_{k\in\mathbb{Z}} (2^{sk}\|a_k\|_A)^p)^{\frac{1}{p}} \tag{1.3}$$

Received by the editors September 16, 2003.
2000 *Mathematics Subject Classification.* Primary 46B70; Secondary 46E30.
Key words and phrases. Real interpolation, vector-valued spaces.

with the usual change for $p = \infty$. In the scalar case, i.e. when $A = \mathbb{R}$, the formula (1.2) was obtained by J. Peetre (see [Pe]). The vector-valued case is not far beyond because spaces are constructed on the base of the same space A.

In the present paper we will consider an analogue of the formula (1.2) for more than two spaces. One of our results, which gives (1.2) in the case of couples, is the following formula:

$$(1.4) \qquad (l_{p_0}^{s_0}(A_0), ..., l_{p_n}^{s_n}(A_n))_{\vec{\theta},q} = l_q^s((A_0, ..., A_n)_{\vec{\theta},q}), \ s = \theta_0 s_0 + ... + \theta_n s_n$$

under the restrictions

$$(1.5) \qquad A_{n-1} = A_n, \quad s_{n-1} \neq s_n.$$

An interesting feature of (1.4) is the absence of restrictions for the spaces and parameters with indices less than n. The proof of (1.4–1.5) is much more complicated than the proof of (1.2) and is based on geometrical decreasing of some terms in the expression for the norm of the interpolation space $(l_{p_0}^{s_0}(A_0), ..., l_{p_n}^{s_n}(A_n))_{\vec{\theta},q}$.

The restriction (1.5) can be considered as the limiting case of the restriction:

$$(1.6) \qquad A_n = (A_0, A_1, ..., A_{n-1})_{\vec{\lambda},p}, s_n \neq \lambda_0 s_0 + \lambda_1 s_1 + ... + \lambda_{n-1} s_{n-1}.$$

In the present paper we will prove a general result from which we derive the formula (1.4) with the restrictions (1.6) under the additional condition that the spaces $A_0, A_1, ..., A_{n-1}$ are functional Banach or quasi-Banach lattices on the same measure space (Ω, μ). The last condition appeared because our proof is based on the reiteration theorem from [AK]. It can be weakened, but we do not know if the result is correct without any conditions on the spaces $A_0, A_1, ..., A_{n-1}$.

The starting point for this investigation is the problem of interpolation of several smooth function spaces. It is known that the interpolation of couples of smooth spaces is not stable in non-diagonal interpolation; for example, if we interpolate two Besov spaces, then we go out of the scale of Besov spaces. In [AKNMP] it was shown that contrary to the case of couples the non-diagonal interpolation of three smooth function spaces has a stability property. This result was based on the wavelet description of Besov spaces and the description of the interpolation space $(L_{p_0}(\omega_0), ..., L_{p_n}(\omega_n))_{\vec{\theta},q}$, which was done in [AKNMP] for $n = 2$ under rather heavy restrictions on weights and parameters of integration. In our next paper [AKN] we will use our new results to provide the description of the spaces $(L_{p_0}(\omega_0), ..., L_{p_n}(\omega_n))_{\vec{\theta},q}$ in general without any conditions on n, weights and parameters of integration.

2. Definitions and some results from real interpolation of several spaces

Let $A_0, A_1, ..., A_n$ be $n + 1$ Banach or quasi-Banach spaces. We will say that they form a compatible collection or simply a collection $\vec{A} = (A_0, A_1, ..., A_n)$ if they are linearly and continuously embedded in some (common for all) topological linear space with a Hausdorff topology. Then we can, analogously to the case of couples, define the K-functional (see [S]) by the formula:

$$(2.1) \quad K(\vec{t}, a; \vec{A}) = \inf(\|a_0\|_{A_0} + t_1 \|a_1\|_{A_1} + ... + t_n \|a_n\|_{A_n}), \vec{t} = (t_1, t_2, ..., t_n) \in \mathbb{R}_+^n ,$$

where inf is taken over all decompositions $a = a_0 + a_1 + ... + a_n$.

Let $\vec{\theta} = (\theta_0, \theta_1, ..., \theta_n)$ be a parameter vector, i.e. $\theta_i > 0$, $\theta_0 + \theta_1 + ... + \theta_n = 1$, and let $0 < q \leq \infty$.

Then the interpolation spaces $\vec{A}_{\vec{\theta},q} = (A_0, A_1, ..., A_n)_{\vec{\theta},q}$ (usually these spaces are called K-spaces and are denoted by $\vec{A}_{\vec{\theta},q;K}$, but we will omit the index K as we will only consider K-spaces) are defined by the norm (or a quasinorm, for the sake of simplicity we will always say norm)

$$\|a\|_{\vec{\theta},q} = (\int_{\mathbb{R}^n_+} (t_1^{-\theta_1} t_2^{-\theta_2} ... t_n^{-\theta_n} K(\vec{t}, a; \vec{A}))^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} ... \frac{dt_n}{t_n})^{\frac{1}{q}} \tag{2.2}$$

with the usual change for $q = \infty$. As the K-functional is a concave function on $\mathbb{R}^n_+$, therefore the norm (2.2) can be written in an equivalent form:

$$\|a\|_{\vec{\theta},q} \approx (\sum_{(i_1, i_2, ..., i_n) \in \mathbb{Z}^n} (2^{-\theta_1 i_1} 2^{-\theta_2 i_2} ... 2^{-\theta_n i_n} K(2^{i_1}, 2^{i_2}, ..., 2^{i_n}, a; \vec{A}))^q)^{\frac{1}{q}}.$$

We will use the so-called "monotonicity" properties of interpolation spaces, which follow easily from the definitions. Namely:

A) if $q_0 \leq q_1$, then

$$(A_0, A_1, ..., A_n)_{\vec{\theta},q_0} \subset (A_0, A_1, ..., A_n)_{\vec{\theta},q_1} \; ; \tag{2.3}$$

B) if we have embeddings $A_i \subset B_i$ $(i = 0, ..., n)$, then

$$(A_0, A_1, ..., A_n)_{\vec{\theta},q} \subset (B_0, B_1, ..., B_n)_{\vec{\theta},q}. \tag{2.4}$$

We will also use the following Reiteration Theorem,[1] which was proved in [AK] in the Banach case (the proof can be extended after some modification to the quasi-Banach case). It is important that this theorem is true even for the *extended parameter vectors.* The parameter vector $\vec{\theta}$ is called an *extended parameter vector* if some of its coordinates θ_i could be equal to zero. In this case in the definition of the K-functional and norms we omit the spaces A_i, parameters t_i and integrate on the set of smaller dimension. In a particular case, when $\vec{\theta}$ has one, say i, coordinate equal to one, and therefore all other coordinates are equal to zero, by $(A_0, A_1, ..., A_n)_{\vec{\theta},q}$ we will mean the space A_i.

Theorem 1 (Reiteration Theorem)**.** *Suppose that all spaces $A_0, A_1, ..., A_n$ are functional Banach or quasi-Banach lattices on the same measure space (Ω, μ) and the extended parameter vectors $\vec{\theta}^k$ $(k = 1, ..., m)$ span $\mathbb{R}^{n+1}$. Then the following holds:*

$$(\vec{A}_{\vec{\theta}^0,q_0}, \vec{A}_{\vec{\theta}^1,q_1}, ..., \vec{A}_{\vec{\theta}^m,q_m})_{\vec{\lambda},q} = \vec{A}_{\vec{\theta},q}, \;\; \vec{\theta} = \lambda_0 \vec{\theta}^0 + \lambda_1 \vec{\theta}^1 + ... + \lambda_m \vec{\theta}^m, \tag{2.5}$$

where $\vec{\lambda} = (\lambda_0, ..., \lambda_m)$ is a parameter vector and $q, q_i \in (0, \infty]$, $i = 0, ..., n$.

3. Main result

Let $\{A^{(k)}\}_{k \in \mathbb{Z}}$ be a sequence of Banach or quasi-Banach spaces $A^{(k)}$. In the quasi-Banach case we suppose the following important condition holds: constants in the triangle inequality are uniformly bounded. Let $l_p(\{A^{(k)}\})$ denote the

[1] The general reiteration theorem for several spaces was proved by G. Sparr (see [S]) with the condition that the equivalence theorem holds, and it was proved by I. Asekritova (see [A]) with the condition of weak K-divisibility; however, the class of collections of Banach spaces for which these conditions hold has not been described.

vector-valued space of all sequences $a = \left\{a^{(k)}\right\}_{k\in\mathbb{Z}}$, $a^{(k)} \in A^{(k)}$, with the norm

$$\|a\|_{l_p(\{A^{(k)}\})} = (\sum_{k\in\mathbb{Z}}(\left\|a^{(k)}\right\|_{A^{(k)}})^p)^{\frac{1}{p}} \tag{3.1}$$

with the usual change for $p = \infty$.

Let $\left\{A_i^{(k)}\right\}_{k\in\mathbb{Z}}$, $i = 0, ..., n$, be a family of $n+1$ sequences of Banach or quasi-Banach spaces. We suppose that for each $k \in \mathbb{Z}$ the spaces $A_0^{(k)}, ..., A_n^{(k)}$ form a compatible collection and therefore we can define the spaces $\vec{A}^{(k)}_{\vec{\theta},q} = (A_0^{(k)}, ..., A_n^{(k)})_{\vec{\theta},q}$.

Our main result reads as follows:

Theorem 2. *Suppose that*[2]

$$A_n^{(k)} = c^k A_{n-1}^{(k)} \quad \textit{for all } k \in \mathbb{Z} \textit{ and some fixed positive number } c \neq 1. \tag{3.2}$$

Then

$$(l_{p_0}(\left\{A_0^{(k)}\right\}_{k\in\mathbb{Z}}), ..., l_{p_n}(\left\{A_n^{(k)}\right\}_{k\in\mathbb{Z}}))_{\vec{\theta},q} = l_q(\left\{(A_0^{(k)}, ..., A_n^{(k)})_{\vec{\theta},q}\right\}_{k\in\mathbb{Z}}), \tag{3.3}$$

where $p_i, q \in (0,\infty]$, $i = 0, 1, ..., n$.

Remark 1. Formula (1.4) from the Introduction immediately follows from the theorem with $c = 2^{s_n - s_{n-1}}$ if we take $A_i^{(k)} = 2^{s_i k} A_i$ $(k \in \mathbb{Z}, i = 0, ..., n)$.

Proof. As the right-hand side of (3.3) does not depend on $p_0, ..., p_n$, therefore from the monotonicity property of interpolation spaces (see (2.3–2.4)) it follows that it is enough to prove (3.3) in the case when all p_i are equal. So we will give the proof for $p_0 = ... = p_n = p$. We will consider only the case $c = 2$, since for $c > 1$ the proof is the same. For $c < 1$ the easiest way to see that the theorem holds is to change the order of spaces $A_i^{(k)}$, i.e. to consider first the spaces $B_i^{(k)} = A_i^{(-k)}$ $(k \in \mathbb{Z},\ i = 0, ..., n)$ to apply the theorem to the spaces $B_i^{(k)}$ with $c^{-1} > 1$ and then change the order back.

We will use the shorthand notation

$$\begin{aligned}(l_p(\left\{A_0^{(k)}\right\}), ..., l_p(\left\{A_n^{(k)}\right\})) &= \overrightarrow{l_p(A)},\\ l_p(\left\{A_0^{(k)}\right\}) + ... + l_p(\left\{A_n^{(k)}\right\}) &= \sum(\overrightarrow{l_p(A)})\end{aligned}$$

and also

$$(A_0^{(k)}, ..., A_n^{(k)}) = \vec{A}_n^{(k)}, \qquad (A_0^{(k)}, ..., A_{n-1}^{(k)}) = \vec{A}_{n-1}^{(k)}. \tag{3.4}$$

Let us first prove the embedding

$$(l_p(\left\{A_0^{(k)}\right\}), ..., l_p(\left\{A_n^{(k)}\right\}))_{\vec{\theta},q} \subset l_q(\left\{(A_0^{(k)}, ..., A_n^{(k)})_{\vec{\theta},q}\right\}). \tag{3.5}$$

From the definition of the K-functional we have for any element $a = \left\{a^{(k)}\right\}_{k\in\mathbb{Z}}$ from $\sum(\overrightarrow{l_p(A)})$ and any $k \in \mathbb{Z}$,

$$\begin{aligned}K(\vec{t}, a; \overrightarrow{l_p(A)}) &\geq K(\vec{t}, a^{(k)}; \vec{A}_n^{(k)}) = K(\vec{t}, a^{(k)}; A_0^{(k)}, ..., A_{n-1}^{(k)}, 2^k A_{n-1}^{(k)})\\ &\approx K(t_1, ..., t_{n-2}, \min(t_{n-1}, 2^k t_n), a^{(k)}; A_0^{(k)}, ..., A_{n-1}^{(k)}).\end{aligned} \tag{3.6}$$

[2] The equality for the sequence of Banach or quasi-Banach spaces $A_i = B_i$ $(i \in I)$ means that the spaces A_i, B_i coincide as sets, norms are equivalent for each i and the constants of equivalence are *independent* of i.

In (3.6) instead of the equivalence we have an equality when $A_{n-1}^{(k)}$ is a Banach space, but in the general quasi-Banach case we only have an equivalence with the constant independent of k as we supposed that the constants in the triangle inequality are uniformly bounded.

Therefore, if we take $t_m = 2^{i_m}, m = 1, ..., n$, and then choose k such that $t_{n-1} = 2^k t_n$ (so $k = i_{n-1} - i_n$), we obtain

$$\|a\|_{(\overrightarrow{l_p(A)})_{\vec{\theta},q}} = (\sum_{\overrightarrow{i}\in\mathbb{Z}^n} (2^{-i_1\theta_1}...2^{-i_n\theta_n} K(2^{i_1}, ..., 2^{i_n}, a; \overrightarrow{l_p(A)}))^q)^{\frac{1}{q}} \tag{3.7}$$

$$\geq \gamma(\sum_{\overrightarrow{i}\in\mathbb{Z}^n} (2^{-i_1\theta_1}...2^{-i_n\theta_n} K(2^{i_1}, .., 2^{i_{n-2}}, \min(2^{i_{n-1}}, 2^{k+i_n}), a^{(k)}; A_0^{(k)}, .., A_{n-1}^{(k)}))^q)^{\frac{1}{q}}$$

$$= \gamma(\sum_{\overrightarrow{i}\in\mathbb{Z}^n} (2^{-i_1\theta_1}...2^{-i_n\theta_n} K(2^{i_1}, ..., 2^{i_{n-1}}; a^{(i_{n-1}-i_n)}; A_0^{(i_{n-1}-i_n)}, ..., A_{n-1}^{(i_{n-1}-i_n)}))^q)^{\frac{1}{q}}.$$

Here and later γ denotes some constants which are different for different formulas.

If we change the variable i_n by the variable $s = i_{n-1} - i_n$, then we can rewrite the expression in the right-hand side as

$$(\sum_{(i_1,...,i_{n-1},s)\in\mathbb{Z}^n} (2^{-i_1\theta_1}...2^{-i_{n-1}\theta_{n-1}} 2^{-(i_{n-1}-s)\theta_n} K(2^{i_1}, ..., 2^{i_{n-1}}, a^{(s)}; \overrightarrow{A}_{n-1}^{(s)}))^q)^{\frac{1}{q}} \tag{3.8}$$

$$= (\sum_{s\in\mathbb{Z}} \sum_{(i_1,...,i_{n-1})\in\mathbb{Z}^{n-1}} (2^{s\theta_n} 2^{-i_1\theta_1}...2^{-i_{n-1}(\theta_{n-1}+\theta_n)} K(2^{i_1}, ..., 2^{i_{n-1}}, a^{(s)}; \overrightarrow{A}_{n-1}^{(s)}))^q)^{\frac{1}{q}}$$

$$= (\sum_{s\in\mathbb{Z}} (2^{s\theta_n} \left\|a^{(s)}\right\|_{(\overrightarrow{A}_{n-1}^{(s)})_{\theta_1,...,\theta_{n-2},\theta_{n-1}+\theta_n,q}})^q)^{\frac{1}{q}}.$$

As $\left(\overrightarrow{A}_{n-1}^{(s)}\right)_{\theta_1,...,\theta_{n-2},\theta_{n-1}+\theta_n,q} = \left(A_0^{(s)}, ..., A_{n-1}^{(s)}, A_{n-1}^{(s)}\right)_{\vec{\theta},q}$, then

$$(\sum_{s\in\mathbb{Z}} (2^{s\theta_n} \left\|a^{(s)}\right\|_{(\overrightarrow{A}_{n-1}^{(s)})_{\theta_1,...,\theta_{n-2},\theta_{n-1}+\theta_n,q}})^q)^{\frac{1}{q}}$$

$$\geq \gamma(\sum_{s\in\mathbb{Z}} \left\|a^{(s)}\right\|^q_{(A_0^{(s)},...,A_{n-1}^{(s)},2^s A_{n-1}^{(s)})_{\vec{\theta},q}})^{\frac{1}{q}} \tag{3.9}$$

$$= \gamma(\sum_{s\in\mathbb{Z}} \left\|a^{(s)}\right\|^q_{(A_0^{(s)},...,A_{n-1}^{(s)},A_n^{(s)})_{\vec{\theta},q}})^{\frac{1}{q}} = \gamma \|a\|_{l_q(\left\{(A_0^{(s)},...,A_{n-1}^{(s)},A_n^{(s)})_{\vec{\theta},q}\right\})}.$$

Hence from (3.7–3.9) and the notation (3.4) it follows that $\|a\|_{(\overrightarrow{l_p(A)})_{\vec{\theta},q}} \geq \gamma \|a\|_{l_q(\left\{\vec{A}_n^{(s)}\right\}_{\vec{\theta},q})}$, and this proves the embedding (3.5).

To prove the inverse embedding

$$l_q(\left\{(A_0^{(k)}, ..., A_n^{(k)})_{\vec{\theta},q}\right\}) \subset (l_p(\left\{A_0^{(k)}\right\}), ..., l_p(\left\{A_n^{(k)}\right\}))_{\vec{\theta},q} \tag{3.10}$$

let us take an element $a = \left\{a^{(k)}\right\}_{k\in\mathbb{Z}}$ from $l_q(\left\{(A_0^{(k)}, ..., A_n^{(k)})_{\vec{\theta},q}\right\})$. Then for any fixed $\overrightarrow{2^i} = (2^{i_1}, ..., 2^{i_n})$ and $\gamma > 1$ we can find decompositions

$$a^{(k)} = a_0^{(k)} + ... + a_n^{(k)} \ (k \in \mathbb{Z}) \tag{3.11}$$

such that

$$\left\|a_0^{(k)}\right\|_{A_0^{(k)}} + ... + 2^{i_n}\left\|a_n^{(k)}\right\|_{A_n^{(k)}} \le \gamma K(\overrightarrow{2^i}, a^{(k)}; \vec{A}_n^{(k)}). \tag{3.12}$$

From the definition of the K-functional and (3.12) it follows that

$$\begin{aligned} K(\overrightarrow{2^i}, a; \overrightarrow{l_p(A)}) &\le \left\|\left\{a_0^{(k)}\right\}\right\|_{l_p(\left\{A_0^{(k)}\right\})} + ... + 2^{i_n}\left\|\left\{a_n^{(k)}\right\}\right\|_{l_p(\left\{A_n^{(k)}\right\})} \\ &= (\sum_{k\in\mathbb{Z}}\left\|a_0^{(k)}\right\|^p_{A_0^{(k)}})^{1/p} + ... + 2^{i_n}(\sum_{k\in\mathbb{Z}}\left\|a_n^{(k)}\right\|^p_{A_n^{(k)}})^{\frac{1}{p}} \\ &\le \gamma(\sum_{k\in\mathbb{Z}}(K(\overrightarrow{2^i}, a^{(k)}; \vec{A}_n^{(k)}))^p)^{1/p}, \end{aligned} \tag{3.13}$$

with some constant γ.

Hence from the definition of the norm of the space $(l_p(\left\{A_0^{(k)}\right\}), ..., l_p(\left\{A_n^{(k)}\right\}))_{\vec{\theta},q}$ we have

$$\begin{aligned} \|a\|_{(l_p(\left\{A_0^{(k)}\right\}),...,l_p(\left\{A_n^{(k)}\right\}))_{\vec{\theta},q}} &= (\sum_{\overrightarrow{i}\in\mathbb{Z}^n}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(\overrightarrow{2^i}, a; \overrightarrow{l_p(A)}))^q)^{1/q} \\ &\le \gamma(\sum_{\overrightarrow{i}\in\mathbb{Z}^n}(2^{-i_1\theta_1}...2^{-i_n\theta_n}((\sum_{k\in\mathbb{Z}}(K(\overrightarrow{2^i}, a^{(k)}; \vec{A}_n^{(k)}))^p)^{1/p})^q)^{1/q} \\ &= \gamma(\sum_{\overrightarrow{i}\in\mathbb{Z}^n}((\sum_{k\in\mathbb{Z}}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(\overrightarrow{2^i}, a^{(k)}; \vec{A}_n^{(k)}))^p)^{\frac{1}{p}})^q)^{1/q}. \end{aligned} \tag{3.14}$$

If $\frac{q}{p} \le 1$, then

$$\begin{aligned} &(\sum_{\overrightarrow{i}\in\mathbb{Z}^n}((\sum_{k\in\mathbb{Z}}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(\overrightarrow{2^i}, a^{(k)}; \vec{A}_n^{(k)}))^p)^{\frac{1}{p}})^q)^{1/q} \\ &\le (\sum_{\overrightarrow{i}\in\mathbb{Z}^n}\sum_{k\in\mathbb{Z}}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(\overrightarrow{2^i}, a^{(k)}; \vec{A}_n^{(k)}))^q)^{1/q} \\ &= (\sum_{k\in\mathbb{Z}}\sum_{\overrightarrow{i}\in\mathbb{Z}^n}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(\overrightarrow{2^i}, a^{(k)}; \vec{A}_n^{(k)}))^q)^{1/q} \\ &= (\sum_{k\in\mathbb{Z}}(\left\|a^{(k)}\right\|_{(\vec{A}_n^{(k)})_{\vec{\theta},q}})^q)^{\frac{1}{q}} = \|a\|_{l_q(\left\{(\vec{A}_n^{(k)})_{\vec{\theta},q}\right\})}, \end{aligned}$$

which proves (3.10).

The case when $\frac{q}{p} > 1$ is more complicated. As $\overset{(k)}{A_n} = 2^k A_{n-1}^{(k)}$ ($k \in \mathbb{Z}$) so

$$K(\overrightarrow{2^i}, a^{(k)}; \vec{A}_n^{(k)}) \approx K(2^{i_1}, ..., 2^{i_{n-2}}, \min(2^{i_{n-1}}, 2^{i_n+k}), a^{(k)}; \vec{A}_{n-1}^{(k)}) \tag{3.15}$$

(with uniform constants of equivalence). We can estimate the right-hand side of (3.14) as follows:

$$(\sum_{\vec{i}\in\mathbb{Z}^n}((\sum_{k\in\mathbb{Z}}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(\overrightarrow{2^{i}},a^{(k)};\vec{A}_n^{(k)}))^p)^{\frac{1}{p}})^q)^{1/q}$$

$$\leq\gamma(\sum_{\vec{i}}((\sum_{k\in\mathbb{Z}}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-2}},\min(2^{i_{n-1}},2^{i_n+k}),a^{(k)};\vec{A}_{n-1}^{(k)}))^p)^{\frac{1}{p}})^q)^{1/q}$$

$$\leq\gamma(\sum_{\vec{i}}((\sum_{k:i_{n-1}\leq i_n+k}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-2}},2^{i_{n-1}},a^{(k)};\vec{A}_{n-1}^{(k)}))^p)^{\frac{1}{p}})^q)^{1/q}$$

$$+\gamma(\sum_{\vec{i}}((\sum_{k:i_{n-1}>i_n+k}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-2}},2^{i_n+k},a^{(k)};\vec{A}_{n-1}^{(k)}))^p)^{\frac{1}{p}})^q)^{1/q}\ .$$

Therefore to prove the theorem it is enough to obtain the estimates

(3.16)

$$S_1=(\sum_{\vec{i}}((\sum_{k:i_{n-1}\leq i_n+k}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-2}},2^{i_{n-1}},a^{(k)};\vec{A}_{n-1}^{(k)}))^p)^{\frac{1}{p}})^q)^{1/q}$$
$$\leq\gamma\left\|a\right\|_{l_q(\left\{(\vec{A}_n^{(k)})_{\vec{\theta},q}\right\})};$$

(3.17)

$$S_2=(\sum_{\vec{i}}((\sum_{k:i_{n-1}>i_n+k}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-2}},2^{i_n+k},a^{(k)};\vec{A}_{n-1}^{(k)}))^p)^{\frac{1}{p}})^q)^{1/q}$$
$$\leq\gamma\left\|a\right\|_{l_q(\left\{(\vec{A}_n^{(k)})_{\vec{\theta},q}\right\})}.$$

Let us start with (3.16). If we denote $s=i_{n-1}-i_n-k$, then using the Minkovski inequality (we can use it because $\frac{q}{p}>1$) we will have

(3.18)

$$S_1=(\sum_{\vec{i}\in\mathbb{Z}^n}((\sum_{k:i_{n-1}\leq i_n+k}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-2}},2^{i_{n-1}},a^{(k)};\vec{A}_{n-1}^{(k)}))^p)^{\frac{1}{p}})^q)^{1/q}$$
$$=(\sum_{\vec{i}}(\sum_{s\leq 0}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-1}},a^{(i_{n-1}-i_n-s)};\vec{A}_{n-1}^{(i_{n-1}-i_n-s)}))^p)^{\frac{q}{p}})^{\frac{p}{q}\frac{1}{p}}$$
$$\leq(\sum_{s\leq 0}(\sum_{\vec{i}\in\mathbb{Z}^n}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-1}},a^{(i_{n-1}-i_n-s)};\vec{A}_{n-1}^{(i_{n-1}-i_n-s)}))^q)^{\frac{p}{q}})^{\frac{1}{p}}.$$

As $i_{n-1}=(i_n+s)+(i_{n-1}-i_n-s)$, then from (3.15) it follows that for a fixed s we have

$$(\sum_{\vec{i}\in\mathbb{Z}^n}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-1}},a^{(i_{n-1}-i_n-s)};\vec{A}_{n-1}^{(i_{n-1}-i_n-s)}))^q)^{\frac{1}{q}}$$
$$\leq\gamma(\sum_{\vec{i}\in\mathbb{Z}^n}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-1}},2^{i_n+s},a^{(i_{n-1}-i_n-s)};\vec{A}_{n}^{(i_{n-1}-i_n-s)}))^q)^{\frac{1}{q}}$$
$$=2^{s\theta_n}\gamma(\sum_{\vec{i}\in\mathbb{Z}^n}(2^{-i_1\theta_1}...2^{-(i_n+s)\theta_n}K(2^{i_1},...,2^{i_{n-1}},2^{i_n+s},a^{(i_{n-1}-i_n-s)};\vec{A}_{n}^{(i_{n-1}-i_n-s)}))^q)^{\frac{1}{q}}$$
$$\leq 2^{s\theta_n}\gamma(\sum_{k\in\mathbb{Z}}\left\|a^{(k)}\right\|^q_{(\vec{A}_n^{(k)})_{\vec{\theta},q}})^{\frac{1}{q}}=2^{s\theta_n}\gamma\left\|a\right\|_{l_q(\left\{(\vec{A}_n^{(k)})_{\vec{\theta},q}\right\}_{k\in\mathbb{Z}})},$$

and we obtain the required estimate of S_1 (see (3.16))

$$S_1 \le \gamma(\sum_{s\le 0}(2^{s\theta_n}\left\|a\right\|_{l_q(\left\{(\vec{A}_n^{(k)})_{\vec{\theta},q}\right\})})^p)^{\frac{1}{p}} \le \gamma\left\|a\right\|_{l_q(\left\{(\vec{A}_n^{(k)})_{\vec{\theta},q}\right\})}. \tag{3.19}$$

For S_2 (see (3.17))the estimation is analogous. Indeed, for the same $s = i_{n-1} - i_n - k$ we have, by using the Minkovski inequality,

(3.20)

$$\begin{aligned}
S_2 &= (\sum_{\overrightarrow{i}\in\mathbb{Z}^n}((\sum_{k:i_{n-1}>i_n+k}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-2}},2^{i_n+k},a^{(k)};\vec{A}_{n-1}^{(k)}))^p)^{\frac{1}{p}})^q)^{1/q}\\
&= (\sum_{\overrightarrow{i}}(\sum_{s>0}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,,2^{i_{n-1}-s},a^{(i_{n-1}-i_n-s)};\vec{A}_{n-1}^{(i_{n-1}-i_n-s)}))^p)^{\frac{q}{p}})^{1/q}\\
&\le (\sum_{s>0}(\sum_{\overrightarrow{i}}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-1}-s},a^{(i_{n-1}-i_n-s)};\vec{A}_{n-1}^{(i_{n-1}-i_n-s)}))^q)^{\frac{p}{q}})^{\frac{1}{p}}.
\end{aligned}$$

For each s (by using (3.15)) we have

$$\begin{aligned}
&(\sum_{\overrightarrow{i}\in\mathbb{Z}^n}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-1}-s},a^{(i_{n-1}-i_n-s)};\vec{A}_{n-1}^{(i_{n-1}-i_n-s)}))^q)^{\frac{1}{q}}\\
&\le 2^{-s\theta_{n-1}}\gamma(\sum_{\overrightarrow{i}}(2^{-i_1\theta_1}...2^{-(i_{n-1}-s)\theta_{n-1}}2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-1}-s},2^{i_n},a^{(i_{n-1}-i_n-s)};\\
&\qquad\qquad\qquad\qquad\vec{A}_n^{(i_{n-1}-i_n-s)}))^q)^{\frac{1}{q}}\\
&\le 2^{-s\theta_{n-1}}\gamma(\sum_{k\in\mathbb{Z}}\left\|a^{(k)}\right\|^q_{(\vec{A}_n^{(k)})_{\vec{\theta},q}})^{\frac{1}{q}} = 2^{-s\theta_{n-1}}\gamma\left\|a\right\|_{l_q(\left\{(\vec{A}_n^{(k)})_{\vec{\theta},q}\right\})}.
\end{aligned}$$

So for S_2 we have the required estimate:

(3.21)

$$\begin{aligned}
S_2 &\le (\sum_{s>0}(\sum_{\overrightarrow{i}\in\mathbb{Z}^n}(2^{-i_1\theta_1}...2^{-i_n\theta_n}K(2^{i_1},...,2^{i_{n-1}-s},a^{(i_{n-1}-i_n-s)};\vec{A}_{n-1}^{(i_{n-1}-i_n-s)}))^q)^{\frac{p}{q}})^{\frac{1}{p}}\\
&\le \gamma(\sum_{s>0}(2^{-s\theta_{n-1}}\left\|a\right\|_{l_q(\left\{(\vec{A}_n^{(k)})_{\vec{\theta},q}\right\})})^p)^{\frac{1}{p}} \le \gamma\left\|a\right\|_{l_q(\left\{(\vec{A}_n^{(k)})_{\vec{\theta},q}\right\})}.
\end{aligned}$$

This concludes the proof of the estimates (3.16–3.17) and the theorem. □

4. Vector-valued interpolation with the "intermediate" condition

Theorem 2 was proved under the condition $\overset{(k)}{A_n} = c^k A_{n-1}^{(k)}$ for all $k \in \mathbb{Z}$ and some fixed $c \neq 1$. Here we will consider the "intermediate" condition $A_n^{(k)} = c^k(A_0^{(k)},...,A_{n-1}^{(k)})_{\vec{\lambda},r}$. We even consider a slightly more general case.

Let us suppose that for some fixed $r_0 \le r_1$, $c \neq 1$, and some parameter vector $\vec{\lambda} \in \mathbb{R}^n$ and all $k \in \mathbb{Z}$ we have embeddings with norms independent of k:

$$c^k(A_0^{(k)},...,A_{n-1}^{(k)})_{\vec{\lambda},r_0} \subset A_n^{(k)} \subset c^k(A_0^{(k)},...,A_{n-1}^{(k)})_{\vec{\lambda},r_1}. \tag{4.1}$$

Now we can formulate the problem:

Problem 1. *Does the formula*

$$(l_{p_0}(\left\{A_0^{(k)}\right\}_{k\in\mathbb{Z}}),...,l_{p_n}(\left\{A_n^{(k)}\right\}_{k\in\mathbb{Z}}))_{\vec{\theta},q} = l_q(\left\{(\overset{(k)}{A_0},...,\overset{(k)}{A_n})_{\vec{\theta},q}\right\}_{k\in\mathbb{Z}})$$

hold under the condition (4.1)*?*

We will prove the following result.

Theorem 3. *Suppose that for each $k \in \mathbb{Z}$ the spaces $A_0^{(k)}, ..., A_{n-1}^{(k)}$ are functional Banach or quasi-Banach lattices on the same measure space (Ω_k, μ_k). If the condition* (4.1) *holds, then*

$$(l_{p_0}(\Big\{A_0^{(k)}\Big\}_{k\in\mathbb{Z}}), ..., l_{p_n}(\Big\{A_n^{(k)}\Big\}_{k\in\mathbb{Z}}))_{\vec{\theta},q} = l_q(\Big\{(A_0^{(k)}, ..., A_n^{(k)})_{\vec{\theta},q}\Big\}_{k\in\mathbb{Z}}). \tag{4.2}$$

Proof. From the reiteration theorem (see Theorem 1) and (2.3–2.4) it follows that the right-hand side of (4.2) does not depend on r_0, r_1. Moreover, it also does not depend on p_i $(i = 0, ..., n)$. Therefore from the monotonicity property of interpolation spaces (see (2.3–2.4)) it follows that it is enough to give the proof in the case when

$$p_0 = ... = p_n = r_0 = r_1 = p\ .$$

So we have only to prove

$$(l_p(\Big\{A_0^{(k)}\Big\}), ..., l_p(\Big\{A_n^{(k)}\Big\}))_{\vec{\theta},q} = l_q(\Big\{(A_0^{(k)}, ..., A_n^{(k)})_{\vec{\theta},q}\Big\}_{k\in\mathbb{Z}})$$

under the condition

$$A_n^{(k)} = c^k (A_0^{(k)}, ..., A_{n-1}^{(k)})_{\vec{\lambda},p}\ .$$

Then we have

$$\begin{aligned} &(l_p(\Big\{A_0^{(k)}\Big\}), ..., l_p(\Big\{A_n^{(k)}\Big\}))_{\vec{\theta},q} \\ &= (l_p(\Big\{A_0^{(k)}\Big\}), ..., l_p(\Big\{A_{n-1}^{(k)}\Big\}, l_p(\Big\{c^k(A_0^{(k)}, ..., A_{n-1}^{(k)})_{\vec{\lambda},p}\Big\}))_{\vec{\theta},q}. \end{aligned} \tag{4.3}$$

Now we will use our Theorem 2 and the reiteration theorem (see Theorem 1). Let us first notice that we can find $\vec{\eta} = (\eta_0, \eta_1, ..., \eta_n)$ and $\tilde{c} \neq 1$ such that

$$(A_0^{(k)}, ..., A_{n-1}^{(k)}, \tilde{c}^k A_{n-1}^{(k)})_{\vec{\eta},p} = c^k (A_0^{(k)}, ..., A_{n-1}^{(k)})_{\vec{\lambda},p}. \tag{4.4}$$

Equality (4.4) is equivalent to

$$(\eta_0, \eta_1, ..., \eta_{n-2}, \eta_{n-1} + \eta_n) = \vec{\lambda} \quad \text{and} \quad \tilde{c}^{\eta_n} = c, \tag{4.5}$$

and (4.5) clearly has a solution for given $\vec{\lambda}$ and c and the solution is not unique. Moreover, from Theorem 2 it follows that

$$\begin{aligned} &(l_p(\Big\{A_0^{(k)}\Big\}), ..., l_p(\Big\{A_{n-1}^{(k)}\Big\}), l_p(\Big\{\tilde{c}^k A_{n-1}^{(k)}\Big\}))_{\vec{\eta},p} \\ &= l_p(\Big\{(A_0^{(k)}, ..., A_{n-1}^{(k)}, \tilde{c}^k A_{n-1}^{(k)})_{\vec{\eta},p}\Big\}) = l_p(\Big\{c^k (A_0^{(k)}, ..., A_{n-1}^{(k)})_{\vec{\lambda},p}\Big\}). \end{aligned} \tag{4.6}$$

Therefore, denoting $B_i = l_p(\Big\{A_i^{(k)}\Big\}), i = 0, ..., n-1$, and $B_n = l_p(\Big\{\tilde{c}^k A_{n-1}^{(k)}\Big\})$ and using (4.6) we can rewrite the right-hand side in (4.3) as

$$\begin{aligned} &(l_p(\Big\{A_0^{(k)}\Big\}), ..., l_p(\Big\{A_{n-1}^{(k)}\Big\}, l_p(\Big\{c^k(A_0^{(k)}, ..., A_{n-1}^{(k)})_{\vec{\lambda},p}\Big\}))_{\vec{\theta},q} \\ &= (B_0, B_{1,...}, B_{n-1}, (B_0, B_{1,...}, B_{n-1}, B_n)_{\vec{\eta},p})_{\vec{\theta},q}. \end{aligned} \tag{4.7}$$

As the spaces B_i are functional Banach lattices on the measure space (Ω, μ), where Ω is constructed as a disjoint union of Ω_i (disjoint means that Ω_i does not intersect with Ω_j for $j \neq i$) with the measure $\mu = \sum_i \mu_i$, therefore we can use the reiteration theorem (see Theorem 1) and Theorem 2 to obtain

$$
\begin{aligned}
(4.8) \qquad (B_0, B_{1,...}, B_{n-1}, (B_0, B_{1,...}, B_{n-1}, B_n)_{\vec{\eta},p})_{\vec{\theta},q} &= (B_0, B_{1,...}, B_{n-1}, B_n)_{\vec{\gamma},q} \\
&= (l_p(\left\{A_0^{(k)}\right\}, ..., l_p(\left\{A_{n-1}^{(k)}\right\}, l_p(\left\{\tilde{c}^k A_{n-1}^{(k)}\right\}))_{\vec{\gamma},q} \\
&= l_q(\left\{(A_0^{(k)}, ..., A_{n-1}^{(k)}, \tilde{c}^k A_{n-1}^{(k)})_{\vec{\gamma},q}\right\})
\end{aligned}
$$

where

$$\vec{\gamma} = (\theta_0 + \eta_0\theta_n, ..., \theta_{n-1} + \eta_{n-1}\theta_n, \eta_n\theta_n).$$

So we have proved the equality

$$(l_p(\left\{A_0^{(k)}\right\}), ..., l_p(\left\{A_n^{(k)}\right\}))_{\vec{\theta},q} = l_q(\left\{(A_0^{(k)}, ..., A_{n-1}^{(k)}, \tilde{c}^k A_{n-1}^{(k)})_{\vec{\gamma},q}\right\}).$$

Since from the reiteration theorem it follows (compare with (4.8)) that

$$
\begin{aligned}
(A_0^{(k)}, ..., A_{n-1}^{(k)}, c^k(A_0^{(k)}, ..., A_{n-1}^{(k)})_{\vec{\lambda},p})_{\vec{\theta},q} &= \\
(A_0^{(k)}, ..., A_{n-1}^{(k)}, (A_0^{(k)}, ..., A_{n-1}^{(k)}, \tilde{c}^k A_{n-1}^{(k)})_{\vec{\eta},p})_{\vec{\theta},q} &= (A_0^{(k)}, ..., A_{n-1}^{(k)}, \tilde{c}^k A_{n-1}^{(k)})_{\vec{\gamma},q},
\end{aligned}
$$

then

$$
\begin{aligned}
(l_p(\left\{A_0^{(k)}\right\}), ..., l_p(\left\{A_n^{(k)}\right\}))_{\vec{\theta},q} &= l_q(\left\{(A_0^{(k)}, ..., A_{n-1}^{(k)}, c^k(A_0^{(k)}, ..., A_{n-1}^{(k)})_{\vec{\lambda},p})_{\vec{\theta},q}\right\}) \\
&= l_q(\left\{(A_0^{(k)}, ..., A_{n-1}^{(k)}, A_n^{(k)})_{\vec{\theta},q}\right\}).
\end{aligned}
$$

□

Corollary 1. *Suppose that* $A_0^{(k)}, ..., A_{n-1}^{(k)}$ *are functional Banach or quasi-Banach lattices on* (Ω_k, μ_k) *and* $A_n^{(k)} = c^k(A_0^{(k)}, ..., A_{n-1}^{(k)})_{\vec{\lambda},p}$ *for all* $k \in \mathbb{Z}$ *and fixed* $\vec{\lambda}$, p *and some positive* $c \neq 1$. *Then*

$$(l_{p_0}(\left\{A_0^{(k)}\right\}_{k\in\mathbb{Z}}), ..., l_{p_n}(\left\{A_n^{(k)}\right\}_{k\in\mathbb{Z}}))_{\vec{\theta},q} = l_q(\left\{(A_0^{(k)}, ..., A_n^{(k)})_{\vec{\theta},q}\right\}_{k\in\mathbb{Z}}).$$

Proof. Apply the theorem with $r_0 = r_1 = p$. □

Remark 2. Formula (1.4) from the Introduction with the restriction (1.6) is a particular case of Corollary 1. Indeed, we only need to take $A_i^{(k)} = 2^{s_i k} A_i$, $i = 0, ..., n-1$. Then from (1.6), it follows that

$$
\begin{aligned}
A_n^{(k)} = 2^{s_n k} A_n &= 2^{s_n k}(A_0, ..., A_{n-1})_{\vec{\lambda},p} \\
&= 2^{(s_n - \lambda_0 s_0 - ... - \lambda_{n-1}s_{n-1})k}(A_0^{(k)}, ..., A_{n-1}^{(k)})_{\vec{\lambda},p} \,.
\end{aligned}
$$

As $s_n \neq \lambda_0 s_0 + ... + \lambda_{n-1}s_{n-1}$(see (1.6)), we have $c = 2^{(s_n - \lambda_0 s_0 - ... - \lambda_{n-1}s_{n-1})}$ not equal to 1.

Remark 3. Instead of the property that $A_0^{(k)}, ..., A_{n-1}^{(k)}$ are functional Banach or quasi-Banach lattices on (Ω_k, μ_k) for each $k \in \mathbb{Z}$ it is enough to have that the reiteration theorem is valid for the collections $(A_0^{(k)}, ..., A_{n-1}^{(k)})$ and for all collections of the type $(l_p(\left\{A_0^{(k)}\right\}), ..., l_p(\left\{A_{n-1}^{(k)}\right\}))$.

References

[A] I. Asekritova, *The property of K-divisibility of finite collections of Banach spaces*, Studies in the Theory of Functions of Several Real Variables, Yaroslavl, 1984, 3-9. MR0830212 (87f:46136)

[AK] I. Asekritova and N. Krugljak, *On equivalence of K- and J-methods for $(n+1)$-tuples of Banach spaces*, Studia Math. 122 (1997), 99-116. MR1432161 (98c:46157)

[AKN] I. Asekritova, N. Krugljak and L. Nikolova, *Lizorkin-Freitag formula for weighted L_p spaces and vector-valued interpolation*, (in preparation).

[AKNMP] I. Asekritova, N. Krugljak, L. Maligranda, L. Nikolova and L.-E. Persson, *Lions-Peetre reiteration formulas for triples and their applications*, Studia Math. 145 (2001), 219-254. MR1829209 (2002e:46089)

[BL] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin, 1976. MR0482275 (58:2349)

[C] M. Cwikel, *On* $(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}$, Proc. Amer. Math. Soc. 44 (1974), 286-292. MR0358326 (50:10792)

[GKKT] J. Garcia-Cuerva, K.S. Kazarian, V.I. Kolyada and J.L. Torrea, *Vector-valued Hausdorff-Young inequality and applications,* Uspekhi Mat. Nauk 53 no.3 (1998), 3-84 (English trans. Russian Math. Surveys 53 no.3 (1998), 435-513). MR1657592 (2000b:42008)

[LP] J.-L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation,* Inst. Hautes Etudes Sci. Publ. Math. 19 (1964), 5-68. MR0165343 (29:2627)

[Pe] J. Peetre, *Une propriété de stabilité par interpolation, application.* Compt. Rend. Acad. Sci. Paris 256, 1424 bis 1426(1963) MR0178381 (31:2639)

[P] G. Pisier, *The K_t-functional for the interpolation couple $L_1(A_0), L_\infty(A_1)$*, J. Approx. Theory 73 (1993), 106-117. MR1213125 (94g:46088)

[S] G. Sparr, *Interpolation of several Banach spaces*, Ann. Mat. Pura Appl. 99 (1974), 247-316. MR0372641 (51:8848)

School of Mathematics and Systems Engineering, Vaxjo University, SE 351 93, Vaxjo, Sweden

E-mail address: irina.asekritova@msi.vxu.se

Department of Mathematics, Luleå University of Technology, SE 972 33, Luleå, Sweden

E-mail address: natan@sm.luth.se

URL: http://www.math.ltu.se/~natan/