BOUNDEDNESS OF ADMISSIBLE AREA FUNCTION ON NONISOTROPIC LIPSCHITZ SPACE

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Abstract. Let $B$ be the unit ball in $\mathbb{C}^n$, let $S$ be the unit sphere, and let $S_\beta(f)$ be the admissible area function. In this paper, we show that if $f \in \text{Lip}_\alpha(S)$, then $S_\beta(f) \in \text{Lip}_\alpha(S)$ and there exists a constant $C$ such that $\|S_\beta(f)\|_{\text{Lip}_\alpha} \leq C\|f\|_{\text{Lip}_\alpha}$.

1. Introduction

Let $B$ be the unit ball in $\mathbb{C}^n$, $d\nu$ the normalized Lebesgue measure on $B$, $d\sigma$ the normalized surface measure on the boundary $S$ of $B$.

For $\xi \in S$ and $0 < \delta \leq 2$, let $Q_\delta(\xi) = \{\eta \in S : d(\xi, \eta) = |1 - \langle \xi, \eta \rangle| < \delta\}$ be a nonisotropic ball of $S$.

We denote by $f \in \text{Lip}_\alpha(S)$ the nonisotropic Lipschitz space of order $0 < \alpha < 1$ if
\[ \sup_{\xi, \eta \in S} \frac{|f(\xi) - f(\eta)|}{d(\xi, \eta)^\alpha} := \|f\|_{\text{Lip}_\alpha} < \infty. \]

We shall follow the convention of identifying the function $f$ on the unit sphere with invariant harmonic extensions into the unit ball defined via the Poisson formula:
\[ f(z) = \int_S P(z, \xi) f(\xi) d\sigma(\xi), \]
where $P(z, \xi)$ is the Poisson-Szegő kernel
\[ P(z, \xi) = \frac{(1 - |z|^2)^n}{|1 - \langle z, \xi \rangle|^{2n}}. \]

For $F \in C^1(B)$, let $DF = (\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial y_1}, \ldots, \frac{\partial F}{\partial x_n}, \frac{\partial F}{\partial y_n})$ ($k = 1, 2, \ldots, n$) be the real gradient of $F$ and $\nabla F = (\frac{\partial F}{\partial z_1}, \frac{\partial F}{\partial z_2}, \ldots, \frac{\partial F}{\partial z_n})$ the complex gradient of $F$, $z_k = x_k + iy_k$.

Let $\tilde{\nabla}$ denote the invariant gradient on $B$, that is,
\[ \tilde{\nabla} f(z) = \nabla (f \circ \varphi_z)(0), \]

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where \( \varphi_z \) is the involution automorphism of \( B \) satisfying \( \varphi_z(0) = z, \varphi_z(z) = 0 \). It has been shown in [1] that for \( a \in B \),
\[
\varphi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{\frac{1}{2}} Q_a z}{1 - \langle z, a \rangle}
\]
where \( P_a z = \frac{\langle z, a \rangle}{|a|^2} a, P_0 z = 0, Q_a = I - P_a \) and
\[
\varphi'_a(0) = -(1 - |a|^2) P_a - \sqrt{1 - |a|^2} Q_a.
\]

Now, by a simple computation we get that
\[
|\nabla f(z)|^2 = |\nabla (f \circ \varphi_z)(0)|^2 = |\nabla f(z)\varphi'_z(0)|^2 = |\varphi'_z(0)\nabla f(z)|^2
\]
\[
= (1 - |z|^2)^2 |P_z \nabla f(z)|^2 + (1 - |z|^2) |Q_z \nabla f(z)|^2
\]
\[
= (1 - |z|^2)(|\nabla f(z)|^2 - |\nabla f(z, \bar{z})|^2)
\]
\[
= (1 - |z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2)
\]
where \( Rf \) is the radial derivative of \( f \).

If \( u \in C^1(B) \), the admissible area function is defined on \( S \) by
\[
S_{\beta} u(z) = \frac{1}{\int_{D_\beta(z)} |\nabla u|^2 (1 - |z|^2)^{-\langle \beta \rangle} d\nu(z)}
\]
where \( D_\beta(z) \) denotes the admissible approach region with \( \beta > 1 \):
\[
D_\beta(z) = \{ z \in B : |1 - \langle z, \xi \rangle| < \frac{\beta}{2} (1 - |z|^2) \}.
\]

In [2] S. Y. Chang proved the boundedness of \( S_{\beta} \) on \( L^p (1 < p < +\infty) \); P. Ahern and J. Bruna [3] studied the characterization of Hardy-Sobolev spaces by \( S_{\beta}(f) \).

The aim of this paper is to study the behavior of \( S_{\beta}(f) \) where \( f \in Lip_{\alpha}(S) (0 < \alpha < 1) \).

**Theorem 1.1.** If \( f \in Lip_{\alpha}(S) (0 < \alpha < 1) \) and \( S_{\beta}(f) < +\infty \) a.e. on \( S \), then \( S_{\beta} f \in Lip_{\alpha}(S) \) and there exists a constant \( C \) such that
\[
||S_{\beta}(f)||_{Lip_{\alpha}} \leq C ||f||_{Lip_{\alpha}}.
\]
Throughout this paper we shall use the letter \( C \) to denote constants, and it may change from line to line.

2. Preliminaries

**Lemma 2.1.** Let \( z \in B, \xi \in S \). We have
\[
|\nabla P(z, \xi)| \leq n \left( \frac{(1 - |z|^2)^{n-1}}{|1 - \langle z, \xi \rangle|^{2n}} - \frac{(1 - |z|^2)^n (1 - \overline{\langle z, \xi \rangle})}{|1 - \langle z, \xi \rangle|^{2n+2}} \right).
\]

**Proof.** Notice firstly that \( P(z, \xi) = \overline{P}(z, \xi) \) and
\[
|\overline{P}(z, \xi)| = |\nabla P(z, \xi)|.
\]
To calculate \( |\nabla P(z, \xi)| \) we will use formula (1.1). We have
\[
\nabla P(z, \xi) = -n \left( \frac{(1 - |z|^2)^{n-1}}{|1 - \langle z, \xi \rangle|^{2n}} - \frac{(1 - |z|^2)^n (1 - \overline{\langle z, \xi \rangle})}{|1 - \langle z, \xi \rangle|^{2n+2}} \right).
\]
Lemma 2.3. Let $f \in L^1(S)$ and $0 < \alpha < 1$. Then the norm $\|f\|_{\text{Lip}_\alpha}$ is equivalent to

$$\sup_Q \frac{1}{|Q|^{1+\frac{\alpha}{2}}} \int_Q |f - f_Q|d\sigma$$

where $f_Q = \frac{1}{|Q|} \int_Q f(\xi)d\sigma(\xi)$.

Lemma 2.2. Let $f \in \text{Lip}_\alpha(S)$, $\gamma \geq 0$. Let $Q_0$ be a nonisotropic ball of radius $\delta > 0$ and center $\eta^*$. If $|1 - \langle \eta, \eta^* \rangle| < \frac{\delta}{16}$, then there exists a constant $C$ depending only on $n, \gamma$ so that for any $z \in B$ that satisfies $|1 - \langle z, \eta \rangle| < \frac{\delta}{16}$, we have

$$\int_{S \setminus Q_0} \frac{|f(\xi) - f_Q|}{|1 - \langle z, \xi \rangle|^{2n+\gamma}} d\sigma(\xi) \leq C \delta^{-(n+\gamma-\alpha)}\|f\|_{\text{Lip}_\alpha}.$$

Proof. For $\xi \in S \setminus Q_0$ and $z \in B$ that satisfies $|1 - \langle z, \eta \rangle| < \frac{\delta}{16}$, $|1 - \langle z, \eta^* \rangle| < \frac{\delta}{16}$, we have

$$|f(\xi) - f_Q| \leq \frac{1}{|Q_0|} \int_{Q_0} |f(\xi) - f(\eta')|d\sigma(\eta') \leq \frac{1}{|Q_0|} \int_{Q_0} |1 - \langle \xi, \eta' \rangle|^\alpha d\sigma(\eta')\|f\|_{\text{Lip}_\alpha} \leq C |1 - \langle z, \xi \rangle|^\alpha \|f\|_{\text{Lip}_\alpha}.$$

Hence

$$\int_{S \setminus Q_0} \frac{|f(\xi) - f_Q|}{|1 - \langle z, \xi \rangle|^{2n+\gamma}} d\sigma(\xi) \leq C \|f\|_{\text{Lip}_\alpha} \int_{S \setminus Q_0} \frac{1}{|1 - \langle z, \xi \rangle|^{2n+\gamma-\alpha}} d\sigma(\xi) \leq C \delta^{-(n+\gamma-\alpha)}\|f\|_{\text{Lip}_\alpha}.$$

To be convenient, we denote $r(z) = (1 - |z|^2)$ ($z \in B$).
Lemma 2.4. Let $Q_0$ be a nonisotropic ball of radius $\delta > 0$ and center $\eta^*$, let $R$ be a nonisotropic ball of radius $r(z)$ and center at $\eta^*$, and let $f \in \text{Lip}_\alpha(S)$. Then
\begin{align}
(2.1) \quad |f_R - f_{Q_0}| &\leq C(1 + |\ln \frac{r(z)}{\delta}|)|r(z)|^\alpha \|f\|_{\text{Lip}_\alpha} \quad (r(z) > \delta), \\
(2.2) \quad |f_R - f_{Q_0}| &\leq C(1 + |\ln \frac{r(z)}{\delta}|)|\delta|^\alpha \|f\|_{\text{Lip}_\alpha} \quad (r(z) < \delta).
\end{align}

Proof. We only prove (2.1), the proof of (2.2) being similar.

Since $r(z) > \delta$, we choose $k$ such that $2^k \delta < r(z) \leq 2^{k+1}\delta$. Then $\frac{2^{k+1}\delta}{r(z)} \leq 2$ and $k \leq \log_2 \left(\frac{r(z)}{\delta}\right)$, and

\[
|f_R - f_{Q_0}| = \left| \int_R (f(\xi) - f_{Q_0}) d\sigma(\xi) \right| \leq C(r(z))^{-n} \int |f(\xi) - f_{Q_0}| d\sigma(\xi) \leq C(r(z))^{-n} \int_{Q_{k+1}} |f(\xi) - f_{Q_0}| d\sigma(\xi).
\]

As in the proof of Lemma 2.3 we have

\[
\int_{Q_{k+1}} |f(\xi) - f_{Q_0}| d\sigma(\xi) \leq C(k + 2)|Q_{k+1}|^{1 + \frac{\alpha}{n}} \|f\|_{\text{Lip}_\alpha} \leq C(k + 2)|2^{k+1}\delta|^{n+\alpha} \|f\|_{\text{Lip}_\alpha}.
\]

Thus

\[
|f_R - f_{Q_0}| \leq C(k + 2)|2^{k+1}\delta|^{n+\alpha} (r(z))^{-n} \|f\|_{\text{Lip}_\alpha} \leq C2^{n+\alpha}(2 + |\log_2 \left(\frac{r(z)}{\delta}\right)|)(r(z))^{\alpha} \|f\|_{\text{Lip}_\alpha}.
\]

Since $1 + |\ln \left(\frac{r(z)}{\delta}\right)|$ is equivalent to $2 + |\log_2 \left(\frac{r(z)}{\delta}\right)|$, (2.1) is proved, which completes the proof of the lemma. \qed

Lemma 2.5. Let $R'$ be a nonisotropic ball of radius $4\alpha r(z)$ and center at $\eta^*$, and let $f \in \text{Lip}_\alpha(S)$. Then

\[
\int_S \frac{|f(\xi) - f_{R'}|}{|1 - \langle z, \xi \rangle|^{2n+\gamma}} d\sigma(\xi) \leq C(r(z))^{-(n+\gamma-\alpha)} \|f\|_{\text{Lip}_\alpha}.
\]

Proof. Argue as in the proof of Lemma 2.3. \qed

Lemma 2.6. Let $Q_0$ be a nonisotropic ball of radius $\delta$ and center at $\eta^*$. If $\frac{\alpha}{2}r(z) \geq \frac{1}{10}, |1 - \langle \eta, \eta^* \rangle| < \frac{1}{10}, |1 - \langle z, \eta \rangle| < \frac{\alpha}{2}r(z)$, we have

\[
\int_S \frac{|f(\xi) - f_{Q_0}|}{|1 - \langle z, \xi \rangle|^{2n+\gamma}} d\sigma(\xi) \leq C(r(z))^{-(n+\gamma-\alpha)}(1 + |\ln \left(\frac{r(z)}{\delta}\right)|) \|f\|_{\text{Lip}_\alpha}.
\]

Proof.

\[
\int_S \frac{|f(\xi) - f_{Q_0}|}{|1 - \langle z, \xi \rangle|^{2n+\gamma}} d\sigma(\xi) \leq \int_S \frac{|f(\xi) - f_{R'}|}{|1 - \langle z, \xi \rangle|^{2n+\gamma}} d\sigma(\xi) + |f_{R'} - f_{Q_0}| \int_S \frac{d\sigma(\xi)}{|1 - \langle z, \xi \rangle|^{2n+\gamma}}.
\]
By Lemma 2.4 and Lemma 2.5, we have
\[
\int_S \frac{|f(\xi) - f_{Q_0}|}{|1 - \langle z, \xi \rangle|^{2n+\gamma}} d\sigma(\xi) \leq C(r(z))^{-(n+\gamma-\alpha)} \|f\|_{Lip_\alpha},
\]
\[
|f' - f_{Q_0}| \leq C(r(z))^{\alpha} (1 + \ln \left[ \frac{r(z)}{\delta} \right]) \|f\|_{Lip_\alpha}.
\]
We also have (see [1, P17])
\[
\int_S \frac{d\sigma(\xi)}{|1 - \langle z, \xi \rangle|^{2n+\gamma}} \approx r(z)^{-(n+\gamma)}.
\]
Hence
\[
\int_S \frac{|f(\xi) - f_{Q_0}|}{|1 - \langle z, \xi \rangle|^{2n+\gamma}} d\sigma(\xi)
\]
\[
\leq C(r(z))^{-(n+\gamma-\alpha)} \|f\|_{Lip_\alpha} + C(r(z))^{-(n+\gamma-\alpha)} (1 + \ln \left[ \frac{r(z)}{\delta} \right]) \|f\|_{Lip_\alpha}
\]
\[
\leq C(r(z))^{-(n+\gamma-\alpha)} (1 + \ln \left[ \frac{r(z)}{\delta} \right]) \|f\|_{Lip_\alpha}.
\]
\[
\square
\]

3. The proof of Theorem 1.1

Let \( h_Q(\xi) = (f(\xi) - f_Q) \chi_{S \setminus \overline{Q}}(\xi) \). To complete the proof of Theorem 1.1, we need the following theorem.

**Theorem 3.1.** Let \( f \in Lip_\alpha(\xi) \), let \( Q \) be a nonisotropic ball of radius \( \delta \) and center at \( \eta^* \), and let \( \frac{1}{\delta} Q = \{ \xi \in S : |1 - \langle \xi, \eta^* \rangle| < \frac{1}{\delta} \} \). Suppose that there is an \( \eta' \in \frac{1}{\delta} Q \) so that \( S_\beta(h_Q)(\eta') < \infty \). Then there exists a constant \( C \), depending only on \( n \) and \( \beta \), such that \( S_\beta(h_Q)(\eta) < \infty \) and \( |S_\beta(h_Q)(\eta) - S_\beta(h_Q)(\eta')| \leq C \delta^\alpha \|f\|_{Lip_\alpha} \) for all \( \eta \in \frac{1}{\delta} Q \).

**Proof.** Let \( D^-_\beta(\eta) = D_\beta(\eta) \cap \{ \beta \frac{r(z)}{2} \leq \frac{\delta}{16} \} \) and \( D^+_\beta(\eta) = D_\beta(\eta) \cap \{ \beta \frac{r(z)}{2} > \frac{\delta}{16} \} \).

We have
\[
S_\beta(h_Q)(\eta) = \left\{ \int_{D^-_\beta(\eta)} |\nabla| \int_S P(z, \xi) h_Q(\xi) d\sigma(\xi)|^2 r(z)^{-(n+1)} d\nu(z) \right\}^{\frac{1}{2}}
\]
\[
= \left\{ \int_{D^-_\beta(\eta)} |\nabla| \int_S P(z, \xi) h_Q(\xi) d\sigma(\xi)|^2 r(z)^{-(n+1)} d\nu(z) \right\}^{\frac{1}{2}} + \left\{ \int_{D^+_\beta(\eta)} |\nabla| \int_S P(z, \xi) h_Q(\xi) d\sigma(\xi)|^2 r(z)^{-(n+1)} d\nu(z) \right\}^{\frac{1}{2}}
\]
\[
(3.1)
\]
\[
= \left\{ \int_{D^-_\beta(\eta)} \left( \int_S |\nabla P(z, \xi)| \|h_Q(\xi)\| d\sigma(\xi) \right)^2 r(z)^{-(n+1)} d\nu(z) \right\}^{\frac{1}{2}}
\]
\[
+ \left\{ \int_{D^+_\beta(\eta)} \left( \int_S |\nabla P(z, \xi)| \|h_Q(\xi)\| d\sigma(\xi) \right)^2 r(z)^{-(n+1)} d\nu(z) \right\}^{\frac{1}{2}} := I_1 + I_2.
\]
By Lemma 2.1, we have
\[
I_1 \leq n\left\{ \int_{D_\beta'(\eta)} \int_{S \setminus Q} \frac{(1 - |z|^2)^n}{|1 - (z, \xi)|^{2n}} + \frac{(1 - |z|^2)^{n+1}}{|1 - (z, \xi)|^{2n+1}} + \frac{(1 - |z|^2)^n}{|1 - (z, \xi)|^{2n+1}} \times |f(\xi) - f_Q|d\sigma(\xi)\right)^2 r(z)^{-(n+1)}d\nu(z) \right\}^{\frac{1}{2}}.
\]

If \( z \in D_\delta(\eta) \), by Lemma 2.3, we get
\[
\left( \int_{S \setminus Q} \frac{(1 - |z|^2)^n}{|1 - (z, \xi)|^{2n}} |f(\xi) - f_Q|d\sigma(\xi) \right)^2 \leq C(r(z))^{2n} \delta^{-2(n-\alpha)} \|f\|^2_{Lip},
\]
\[
\left( \int_{S \setminus Q} \frac{(1 - |z|^2)^{n+1}}{|1 - (z, \xi)|^{2n+1}} |f(\xi) - f_Q|d\sigma(\xi) \right)^2 \leq C(r(z))^{2n+2} \delta^{-2(n+1-\alpha)} \|f\|^2_{Lip},
\]
\[
\left( \int_{S \setminus Q} \frac{(1 - |z|^2)^{n+\frac{1}{2}}}{|1 - (z, \xi)|^{2n+1}} |f(\xi) - f_Q|d\sigma(\xi) \right)^2 \leq C(r(z))^{2n+1} \delta^{-2(n+\frac{1}{2}-\alpha)} \|f\|^2_{Lip}.
\]

It follows that
\[
I_1 \leq C\left\{ \left( (r(z))^{2n} \delta^{-2(n-\alpha)} + (r(z))^{2n+2} \delta^{-2(n+1-\alpha)} \right) \int_{D_\beta'(\eta)} (r(z))^{n-1}dv(z) + \int_{D_\beta'(\eta)} (r(z))^{n+1}dv(z) \right\}^{\frac{1}{2}}
\]
\[
= C\left( \delta^{-2(n-\alpha)} \int_{D_\beta'(\eta)} (r(z))^{n-1}dv(z) + \delta^{-2(n+1-\alpha)} \int_{D_\beta'(\eta)} (r(z))^{n+1}dv(z) \right) \|f\|_{Lip}.
\]
arguing as in [2] P116, we get
\[
\int_{D_\beta'(\eta)} dv(z) \leq C\delta^{n+1}.
\]  
Hence
(3.2) \[
I_1 \leq C\delta^n \|f\|_{Lip}.
\]  
Next, we estimate \( I_2 \):
\[
I_2 = \left\{ \int_{D_\beta'(\eta)} \int_S \nabla P(z, \xi) |h_Q(\xi)|d\sigma(\xi)\right)^2 r(z)^{-(n+1)}d\nu(z) \right\}^{\frac{1}{2}}
\]
\[
\leq \left\{ \int_{D_\beta'(\eta)^c \cup D_\beta'(\eta')} \int_S \nabla P(z, \xi) |h_Q(\xi)|d\sigma(\xi)\right)^2 r(z)^{-(n+1)}d\nu(z) \right\}^{\frac{1}{2}}
\]
\[
+ \left\{ \int_{D_\beta'(\eta) \setminus D_\beta'(\eta')} \int_S \nabla P(z, \xi) |h_Q(\xi)|d\sigma(\xi)\right)^2 r(z)^{-(n+1)}d\nu(z) \right\}^{\frac{1}{2}}.
\]
For the first term above, we have
\[
\left\{ \int_{D^+_{\beta}(n)\setminus D_{\beta}(n')} \left( \int_S |\nabla P(z, \xi)||h_Q(\xi)|d\sigma(\xi) \right)^2 r(z)^{-(n+1)} dv(z) \right\}^{\frac{1}{2}}
\]
(3.4)
\[
\leq \left\{ \int_{D_{\beta}(n')} \left( \int_S |\nabla P(z, \xi)||h_Q(\xi)|d\sigma(\xi) \right)^2 r(z)^{-(n+1)} dv(z) \right\}^{\frac{1}{2}}
\]
\[= S_{\beta}(h_Q)(n').\]

For the second term above, we have
\[
\left\{ \int_{D^+_{\beta}(n)\setminus D_{\beta}(n')} \left( \int_S |\nabla P(z, \xi)||h_Q(\xi)|d\sigma(\xi) \right)^2 r(z)^{-(n+1)} dv(z) \right\}^{\frac{1}{2}}
\]
\[
\leq n \left\{ \int_{D^+_{\beta}(n)\setminus D_{\beta}(n')} \left( \int_{S'Q} \left( \frac{1-|z|^2}{1-\langle z, \xi \rangle} \right)^{n+1} \left( \frac{1-|z|^2}{1-\langle z, \xi \rangle} \right)^{n+1} + \left( \frac{1-|z|^2}{1-\langle z, \xi \rangle} \right)^{n+1} \right. \right. \]
\[
\left. \left. \times |f(\xi) - f_Q(d\sigma(\xi))|^2 r(z)^{-(n+1)} dv(z) \right) \left. \right\}^{\frac{1}{2}}.
\]

Using Lemma 2.6, it follows that
\[
\left( \int_{S'Q} \left( \frac{1-|z|^2}{1-\langle z, \xi \rangle} \right)^{n+1} |f(\xi) - f_Q(d\sigma(\xi))|^2 \right) \leq C(r(z))^{2\alpha} (1 + |\ln \frac{r(z)}{\delta}|)^2 \|f\|^2_{Lip_a},
\]
\[
\left( \int_{S'Q} \left( \frac{1-|z|^2}{1-\langle z, \xi \rangle} \right)^{n+1} |f(\xi) - f_Q(d\sigma(\xi))|^2 \right) \leq C(r(z))^{2\alpha} (1 + |\ln \frac{r(z)}{\delta}|)^2 \|f\|^2_{Lip_a},
\]
\[
\left( \int_{S'Q} \left( \frac{1-|z|^2}{1-\langle z, \xi \rangle} \right)^{n+1} |f(\xi) - f_Q(d\sigma(\xi))|^2 \right) \leq C(r(z))^{2\alpha} (1 + |\ln \frac{r(z)}{\delta}|)^2 \|f\|^2_{Lip_a}.
\]

Hence
\[
\left\{ \int_{D^+_{\beta}(n)\setminus D_{\beta}(n')} \left( \int_S |\nabla P(z, \xi)||h_Q(\xi)|d\sigma(\xi) \right)^2 r(z)^{-(n+1)} dv(z) \right\}^{\frac{1}{2}}
\]
\[
\leq C \left\{ \int_{D^+_{\beta}(n)\setminus D_{\beta}(n')} \left( 1 + |\ln \frac{r(z)}{\delta}| \right)^2 r(z)^{-(n+1-2\alpha)} \|f\|^2_{Lip_a} dv(z) \right\}^{\frac{1}{2}}.
\]

Notice that \(|D^+_{\beta}(n) \setminus D_{\beta}(n') \cap \{r(z) = c'\}| \leq C\delta^\alpha\) for all \(0 < c' < 1\) (see [28] P128) and \(\frac{\beta r(z)}{2} > \frac{1}{16}\). So we have \(|D^+_{\beta}(n) \setminus D_{\beta}(n') \cap \{r(z) = c'\}| \leq C\delta(r(z))^{-n-1}\). Thus it follows that
\[
\left\{ \int_{D^+_{\beta}(n)\setminus D_{\beta}(n')} \left( 1 + |\ln \frac{r(z)}{\delta}| \right)^2 r(z)^{-(n+1-2\alpha)} \|f\|^2_{Lip_a} dv(z) \right\}^{\frac{1}{2}}
\]
(3.5)
\[
\leq C \int_{\frac{\delta}{2}}^{1} \left( 1 + |\ln \frac{r(z)}{\delta}| \right)^2 r(z)^{2\alpha-2} dr(z) \|f\|_{Lip_a}
\]
\[
\leq C\delta^\alpha \int_{\frac{\delta}{2}}^{1} (1 + |\ln t|) t^{-2} dt \|f\|_{Lip_a}
\]
\[
\leq C\delta^\alpha \|f\|_{Lip_a}.
\]

Using (3.3) - (3.5), we have
\[
(3.6) \quad I_2 \leq S_{\beta}(h_Q)(n') + C\delta^\alpha \|f\|_{Lip_a}.
\]
Combining (3.1) with (3.2), (3.6), it follows that
\[ S_\beta(h_Q)(\eta) \leq S_\beta(h_Q)(\eta') + C\delta^\alpha \|f\|_{Lip_\alpha}. \]
Thus \( S_\beta(h_Q)(\eta) \) is finite. Reversing the roles of \( \eta \) and \( \eta' \), we obtain
\[ |S_\beta(h_Q)(\eta) - S_\beta(h_Q)(\eta')| \leq C\delta^\alpha \|f\|_{Lip_\alpha}. \]
This completes the proof of Theorem 3.1. \( \square \)

**The proof of Theorem 1.1.** Write \( f \) as
\[ f(\xi) = f_Q + (f(\xi) - f_Q)\chi_{Q}(\xi) + (f(\xi) - f_Q)\chi_{S\setminus Q}(\xi) \]
\[ = f_Q(\xi) + g_Q(\xi) + h_Q(\xi). \]
Since \( f_Q \) is a constant, \( S_\beta(f_Q) = 0 \). Thus \( S_\beta(f_Q) \) is in \( Lip_\alpha \) with the \( Lip_\alpha \) norm equal to 0. Therefore
\[ S_\beta(f) \leq S_\beta(g_Q) + S_\beta(h_Q), \]
\[ S_\beta(h_Q) \leq S_\beta(g_Q) + S_\beta(f). \]

By the boundedness of \( S_\beta(f) \) on \( L^p \) \((1 < p < \infty)\) and the Cauchy-Schwarz inequality, we have
\[ \int_Q |S_\beta(g_Q)|d\sigma(\xi) \leq |Q|^{\frac{1}{2}}\left( \int_Q |S_\beta(g_Q)|^2d\sigma(\xi) \right)^{\frac{1}{2}} \]
\[ \leq C|Q|^{\frac{1}{2+\delta}} \|f\|_{Lip_\alpha}. \]
It also follows from (3.9) that \( S_\beta(g_Q) \) is finite almost everywhere. Therefore \( S_\beta(h_Q) < +\infty \) at almost every point such that \( S_\beta(f) < +\infty \).

Now, we prove \( ||S_\beta(f)||_{Lip_\alpha} \leq C||f||_{Lip_\alpha} \).

To show the boundedness of \( S_\beta \) on \( Lip_\alpha \), by Lemma 2.2 and the triangle inequality, it suffices to show that for every \( f \in Lip_\alpha(S) \), there is a constant \( \lambda = \lambda(Q,f) \) such that
\[ \frac{1}{|Q|^{\frac{1}{2+\delta}}} \int_Q |S_\beta(f(\xi) - \lambda)|d\sigma(\xi) \leq C||f||_{Lip_\alpha}. \]
Let \( Q' \subset S \) be any nonisotrophic ball, and \( Q = 16Q' \) (that is \( Q' = \frac{1}{16}Q \)). Since \( S_\beta(h_Q)(\eta) < +\infty \) at almost every point, we choose a point \( \eta' \in \frac{1}{16}Q \) so that \( S_\beta(h_Q)(\eta') \) is finite. Then, by (3.9) and Theorem 3.1,
\[ \frac{1}{|Q'|^{\frac{1}{2+\delta}}} \int_{Q'} |S_\beta(f(\xi) - S_\beta(h_Q)(\eta'))|d\sigma(\xi) \]
\[ = \frac{1}{|Q'|^{\frac{1}{2+\delta}}} \int_{Q'} |S_\beta(g_Q + h_Q)(\xi) - S_\beta(h_Q)(\xi) + S_\beta(h_Q)(\xi) - S_\beta(h_Q)(\eta')|d\sigma(\xi) \]
\[ = \frac{1}{|Q'|^{\frac{1}{2+\delta}}} \int_{Q'} |S_\beta(g_Q)(\xi)|d\sigma(\xi) + \frac{1}{|Q'|^{\frac{1}{2+\delta}}} \int_{Q'} |S_\beta(h_Q)(\xi) - S_\beta(h_Q)(\eta')|d\sigma(\xi) \]
\[ \leq C||f||_{Lip_\alpha}. \]
Since \( Q' \) is arbitrary, the proof is completed. \( \square \)

**Remark 3.2.** The referee asked whether the following proposition is true or not:
\[ S_\alpha f \in Lip_\alpha(S) \Rightarrow f \in Lip_\alpha(S) \]
The authors cannot give a positive proof right now.
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