PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 6, Pages 1837–1841
S 0002-9939(04)07765-2
Article electronically published on December 20, 2004

# A STOCHASTIC DELAY FINANCIAL MODEL

GEORGE STOICA

(Communicated by Richard C. Bradley)

Abstract. We compute the logarithmic utility of an insider when the financial market is modelled by a stochastic delay equation. Although the market does not allow free lunches and is complete, the insider can draw more from his wealth than the regular trader. We also offer an alternative to the anticipating delayed Black-Scholes formula, by proving stability of European call option prices when the delay coefficients approach the nondelayed ones.

## 1. The model

We investigate a continuous time financial market model with economic agents possessing different information levels (cf. [2]). The regular trader expects the dynamics of the stock price to be given by a Black-Scholes diffusion process. On the other hand, the insider knows that both the drift and the volatility of the stock price process are influenced by certain events that happened before the trading period started (cf. [5]). Our assumption is that the insider expects the stock price to follow a stochastic delay diffusion process; the latter class was first introduced in [4], but its use in mathematical finance is extremely new (see [1]).

Therefore, for the insider, the stock price is given by

$$\begin{cases} dY_t = b_t f(Y_{t-a})Y_t dt + \sigma_t g(Y_{t-b})Y_t dW_t, 0 \le t \le T, \\ Y_t = \phi(t), -r \le t \le 0, \end{cases} \tag{1}$$

on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions, and where $r = \max\{a, b\}$ for some $a, b > 0$.

The processes $b_t$ and $\sigma_t$ are progressively measurable, $b_t$ and $\sigma_t^2$ are a.s. integrable on $[0, T]$, $b_t \ge 0$ a.s. and $\sigma_t \ne 0$ a.s. for any $t$. The function $f : \mathbb{R} \to (0, \infty)$ is locally Lipschitz with sublinear growth, and $g : \mathbb{R} \to \mathbb{R}^*$ is continuous. $W_t, 0 \le t \le T$, is a one-dimensional standard Brownian motion adapted to $(\mathcal{F}_t)_{0\le t\le T}$, and the initial process $\phi(t)$ is continuous in $t$ a.s. and $\mathcal{F}_0$-measurable for all $t$.

**Theorem 1.1.** *Under the above hypothesis, equation* (1) *has an a.s. continuous adapted pathwise unique solution* $Y_t, 0 \le t \le T$.

Received by the editors January 8, 2004 and, in revised form, February 20, 2004.
2000 *Mathematics Subject Classification.* Primary 91B28; Secondary 91B26.
*Key words and phrases.* Stochastic delay equation, maximal logarithmic utility, insider, no-arbitrage, complete market.
The first author was supported in part by NSERC Canada Grant #249730.

*Proof.* The required solution can be found by using forward induction steps of length $\min\{a,b\}$. More precisely, if $0 \le t \le \min\{a,b\}$, then

$$Y_t = \phi(0)\mathcal{E}(M_t),$$

where $\mathcal{E}(M_t)$ is the Doléans-Dade exponential of the semimartingale

$$M_t = \int_0^t b_s f(\phi(s-a))ds + \int_0^t \sigma_s g(\phi(s-b))dW_s.$$

In general, if $k\min\{a,b\} \le t \le (k+1)\min\{a,b\}$ for some $k \ge 1$, then

$$Y_t = Y_{k\min\{a,b\}}\mathcal{E}(M_t),$$

with

$$M_t = \int_{k\min\{a,b\}}^t b_s f((Y_{s-a})ds + \int_{k\min\{a,b\}}^t \sigma_s g(Y_{s-b})dW_s$$

(in the above formula $Y_{s-a}$ and $Y_{s-b}$ are known from the previous induction steps).

Using the above explicit formulas for $Y_t$, we obtain in particular

**Corollary 1.2.** *If in addition to the hypothesis in Theorem 1 we also have $\phi(0) > 0$ a.s., then $Y_t > 0$ a.s for all $0 \le t \le T$.*

## 2. Logarithmic utility and wealth of an insider

By Theorem 1.1 it follows that the predictable process,

$$\mu_t^Y = \frac{b_t f(Y_{t-a})}{\sigma_t g(Y_{t-b})}, 0 \le t \le T,$$

is well defined by forward induction steps of length $\min\{a,b\}$ and $(\mu_t^Y)^2$ is a.s. integrable on $[0,T]$. Remark that $g \ne 0$ and, by continuity, $g$ has constant sign on its domain.

The following result is a Girsanov theorem for equation (1), and can be proved in the same manner as in [1], Theorem 2.

**Lemma 2.1.** *Under the hypothesis of Theorem 1.1, define a new probability $\mathbb{Q}^Y$ whose density with respect to $\mathbb{P}$ on $\mathcal{F}_t$ is given by*

$$\frac{d\mathbb{Q}^Y}{d\mathbb{P}} = \exp\left(\int_0^t \mu_s^Y dW_s - \frac{1}{2}\int_0^t (\mu_s^Y)^2 ds\right).$$

*Then*

$$W = \tilde{W} + \int_0^{\cdot} \mu_t^Y dt$$

*is a $\mathbb{Q}^Y$-semimartingale with a $\mathbb{Q}^Y$-Brownian motion $\tilde{W}$.*

*Remark* 2.2. As the filtrations generated by $\{Y_s, s \le t\}$ and $\{\tilde{W}_s, s \le t\}$ coincide, by the martingale representation property (see [1], Theorem 3) it follows that every integrable contingent claim is attainable in model (1); hence the probability $\mathbb{Q}^Y$ in Lemma 2.1 is actually an equivalent martingale measure. In other words, the financial market model (1) satisfies the no-arbitrage property and is complete (see [3] for definitions); hence there are no free lunches for the insider.

Among the predictable portfolio processes $\pi_t, 0 \leq t \leq T$, in the sequel we distinguish the class of *admissible* portfolios, that is, for which

$$b_t\pi_t \text{ and } (\sigma_t\pi_t)^2 \text{ are a.s. integrable on } [0,T].$$

For the insider trader, the value process $V$ of such a portfolio $\pi$ is given by

$$\begin{cases} \dfrac{dV_t}{V_t} = \pi_t \dfrac{dY_t}{Y_t}, 0 < t \leq T, \\ V_0 = \phi(0). \end{cases} \tag{2}$$

We choose the logarithmic utility function to measure the utility a trader draws from his wealth at the end of the trading interval. The expected maximal logarithmic utility of the insider is therefore given by

$$\max_{\pi} \mathbb{E}_{\mathbb{Q}^Y}(\ln V_T), \tag{3}$$

where the max is taken in the class of all admissible portfolios $\pi$ described before, during the trading interval $[0,T]$, and $\mathbb{E}_{\mathbb{Q}^Y}$ denotes expectation with respect to the probability measure $\mathbb{Q}^Y$.

The main result of this section is as follows.

**Theorem 2.3.** *If $\mu_t^Y$ is the process defined in Lemma 2.1, then the solution to the maximization problem* (3) *is given by*

$$\frac{1}{2}\mathbb{E}_{\mathbb{Q}^Y}\left[\int_0^T (\mu_t^Y)^2 dt\right].$$

*Proof.* Indeed, let $0 < t \leq T$ and denote by $k$ the integer part of $t/\min\{a,b\}$; by Theorem 1.1 and equation (2) we have

$$\begin{aligned} V_t = V_{k\min\{a,b\}} \exp\Bigg[&\int_{k\min\{a,b\}}^t \pi_s\sigma_s g(Y_{s-b})dW_s \\ &+ \int_{k\min\{a,b\}}^t \left(\pi_s b_s f(Y_{s-a}) - \frac{1}{2}\pi_s^2\sigma_s^2 g^2(Y_{s-b})\right)ds\Bigg]. \end{aligned}$$

Take logarithms in the above formula and sum from $k=0$ to $n$, where $n$ is the integer part of $T/\min\{a,b\}$; by Lemma 2.1, the solution to problem (3) is deduced from the maximization problem

$$\max_{\pi} \sum_{k=0}^{n} \int_{k\min\{a,b\}}^{\min\{(k+1)\min\{a,b\};T\}} \left(\pi_s b_s f(Y_{s-a}) - \frac{1}{2}\pi_s^2\sigma_s^2 g^2(Y_{s-b})\right)ds.$$

The latter is nothing else than

$$\max_{\pi} \int_0^T \left(\pi_s b_s f(Y_{s-a}) - \frac{1}{2}\pi_s^2\sigma_s^2 g^2(Y_{s-b})\right)ds,$$

and whose solution is obtained for

$$\pi_s = \frac{b_s f(Y_{s-a})}{\sigma_s^2 g^2(Y_{s-b})};$$

the conclusion now follows.

We saw earlier that free lunches are excluded in the financial market (1) and every integrable claim has a unique price. However, Theorem 2.3 implies that the

insider can draw more from his wealth than the regular trader on a time period. Indeed, the dynamics for the regular agent are given by

$$\begin{cases} dX_t = b_t X_t dt + \sigma_t X_t dW_t, 0 < t \leq T, \\ X_0 \text{ is } \mathcal{F}_0\text{-measurable;} \end{cases}$$

in this case (see [6]), the expected maximal logarithmic utility of the regular trader equals

$$\frac{1}{2}\mathbb{E}_{\mathbb{Q}^X}\left[\int_0^T \frac{b_t^2}{\sigma_t^2} dt\right],$$

where $\mathbb{Q}^X$ is the equivalent probability measure whose density with respect to $\mathbb{P}$ on $\mathcal{F}_t$ is given by

$$\frac{d\mathbb{Q}^X}{d\mathbb{P}} = \exp\left(\int_0^t \frac{b_t}{\sigma_t} dW_s - \frac{1}{2}\int_0^t \frac{b_t^2}{\sigma_t^2} ds\right).$$

We have the following result.

**Corollary 2.4.** *If*

$$(\mu_t^Y)^2 \frac{d\mathbb{Q}^Y}{d\mathbb{P}} \geq \frac{b_t^2}{\sigma_t^2} \frac{d\mathbb{Q}^X}{d\mathbb{P}} \; dt \otimes d\mathbb{P} - a.e.,$$

*then the insider draws more from his wealth than the regular trader on* $[0, T]$.

*Proof.* Use the explicit form of the maximal logarithmic utilities of the insider and the regular agent, and replace the densities with integrals with respect to the original probability $\mathbb{P}$.

For instance, the hypothesis in Corollary 2.4 is satisfied if both $f$ and $g$ are (piecewise) constant, and $f \geq g$ a.e. on $\mathbb{R}$.

## 3. Stability of European call options

An explicit formula for the European call option prices written on the delay model (1) can be deduced as in [1], Theorem 4. However, it becomes rather tricky to apply in practice, as the corresponding random Black-Scholes partial differential equation is *anticipating* with respect to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. Actually, the fair price in a general delayed stochastic model depends not only on the stock price $Y_t$ at the present time $t$, but also on the whole "past" segment $Y_s, t - \min\{a, b\} \leq s \leq T - \min\{a, b\}$.

Instead, in this section we prove stability of European call option prices of (1) when the delay coefficients $f$ and $g$ become uniformly close to 1; more precisely, the delay prices approach the classical Black-Scholes price.

Assume that the interest rate $r$ is deterministic and denote by $k$ the strike price of the European call. The option price of the asset

$$\exp(-rT)\mathbb{E}_{\mathbb{Q}^Y}(Y_T - k)_+$$

is consistent with our results on completeness and no arbitrage (see also [6]), and the conditional expectation

$$\mathbb{E}_{\mathbb{Q}^Y}[\exp(-r(T-t))(Y_T - k)_+ | \mathcal{F}_t]$$

is the portfolio valuation process.

**Theorem 3.1.** *If $X_0 = \phi(0)$ and $\lim f(x) = \lim g(x) = 1$ uniformly in $x \in \mathbb{R}$, then*

$$\lim \mathbb{E}_{\mathbb{Q}^Y}(Y_T - k)_+ = \mathbb{E}_{\mathbb{Q}^X}(X_T - k)_+$$

*and*

$$\lim \mathbb{E}_{\mathbb{Q}^Y}[(Y_T - k)_+ | \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}^X}[(X_T - k)_+ | \mathcal{F}_t]$$

*for any $0 < t \leq T$.*

*Proof.* Using the explicit formulas for $Y_t$ and dividing $[0, T]$ into forward steps of length $\min\{a, b\}$, we obtain

$$\lim \mathbb{E}_{\mathbb{P}}\left(\sup_{0 \leq t \leq T} |Y_t - X_t|\right) = 0. \tag{4}$$

Then we have

$$\mathbb{E}_{\mathbb{Q}^X}|(Y_T - k)_+ - (X_T - k)_+| \leq \mathbb{E}_{\mathbb{Q}^X}|Y_T - X_T| = \mathbb{E}_{\mathbb{P}}\left(|Y_T - X_T| \frac{d\mathbb{Q}^X}{d\mathbb{P}}\right);$$

the latter approaches 0 by (4) and the Cauchy-Schwarz inequality as $d\mathbb{Q}^X/d\mathbb{P} \in L^2$.

On the other hand, as $d\mathbb{Q}^Y/d\mathbb{P} \to d\mathbb{Q}^X/d\mathbb{P}$ uniformly, we have

$$\mathbb{E}_{\mathbb{P}}\left((Y_T - k)_+ \frac{d\mathbb{Q}^Y}{d\mathbb{P}}\right) - \mathbb{E}_{\mathbb{P}}\left((Y_T - k)_+ \frac{d\mathbb{Q}^X}{d\mathbb{P}}\right) \to 0;$$

that is,

$$\mathbb{E}_{\mathbb{Q}^Y}(Y_T - k)_+ - \mathbb{E}_{\mathbb{Q}^X}(Y_T - k)_+ \to 0.$$

The conclusion now follows by writing

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}^Y}(Y_T - k)_+ - \mathbb{E}_{\mathbb{Q}^X}(X_T - k)_+ \\ &= \mathbb{E}_{\mathbb{Q}^Y}(Y_T - k)_+ - \mathbb{E}_{\mathbb{Q}^X}(Y_T - k)_+ + \mathbb{E}_{\mathbb{Q}^X}(Y_T - k)_+ - \mathbb{E}_{\mathbb{Q}^X}(X_T - k)_+. \end{aligned}$$

Convergence of portfolio valuation processes follows using the same method as above.

## References


1. M. Arriojas, Y. Hu, S.-E. Mohammed, G. Pap, *A delayed Black and Scholes formula*, preprint 2003. Available at: http://salah.math.siu.edu/
2. K. Back, *Insider trading in continuous time*, Rev. Fin. Stud. **5** (1992), 387–409.
3. F. Delbaen, W. Schachermayer, *A general version of the fundamental theorem of asset pricing*, Math. Ann. **300** (1994), 463–520. MR1304434 (95m:90022b)
4. K. Itô, M. Nisio, *On stationary solutions of a stochastic differential equation*, J. Math. Kyoto Univ. **41** (1964), 1–75. MR0177456 (31:1719)
5. I. Karatzas, I. Pikovsky, *Anticipative portfolio optimization*, Adv. Appl. Prob. **28** (1996), 1095–1122. MR1418248 (98b:90020)
6. I. Karatzas, S. Shreve, *Brownian Motion and Stochastic Calculus*, Springer Verlag, New York-Berlin, 1987.



Department of Mathematical Sciences, University of New Brunswick, P.O. Box 5050, Saint John, New Brunswick, Canada E2L 4L5

*E-mail address*: stoica@unbsj.ca