TENSOR PRODUCTS
OF σ-WEAKLY CLOSED NEST ALGEBRA SUBMODULES

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ABSTRACT. In this paper we prove that for any unital σ-weakly closed algebra \( A \) which is σ-weakly generated by finite-rank operators in \( A \), every σ-weakly closed \( A \)-submodule has Property \( S_\sigma \). In the case of nest algebras, if \( L_1, \cdots , L_n \) are nests, we obtain the following \( n \)-fold tensor product formula:

\[
U_{\phi_1} \otimes \cdots \otimes U_{\phi_n} = U_{\phi_1} \otimes \cdots \otimes U_{\phi_n},
\]

where each \( U_{\phi_i} \) is the σ-weakly closed \( \text{Alg} L_i \)-submodule determined by an order homomorphism \( \phi_i \) from \( L_i \) into itself.

1. INTRODUCTION

One of the central results in the theory of tensor products of von Neumann algebras is Tomita’s commutation formula:

\[
M' \otimes N' = (M \otimes N)',
\]

where \( M \) and \( N \) are von Neumann algebras. It was observed in [2] that if we let \( L_1 \) and \( L_2 \) denote the projection lattices of \( M \) and \( N \) respectively, then (1) can be rewritten as

\[
\text{Alg} L_1 \overline{\otimes} \text{Alg} L_2 = \text{Alg}(L_1 \otimes L_2).
\]

This version of Tomita’s theorem makes sense for any pair of reflexive algebras \( \text{Alg} L_1 \) and \( \text{Alg} L_2 \). It remains a deep open question whether the tensor product formula (2) is valid for general reflexive algebras, or even general CSL algebras. However, (2) has been verified in a number of special cases ([2], [4], [5], [6], [7]). In particular, it is known that if \( L_1 \) is a commutative subspace lattice that is either completely distributive [8] or of finite width [4], then (2) is valid for \( L_1 \) and any subspace lattice \( L_2 \).

The main purpose of this paper is to study tensor products of σ-weakly closed submodules of some reflexive algebras (in particular, of nest algebras). Section 1 of this paper is devoted to notation and preliminaries. In Section 2, we make use of slice maps to show that if \( A \) is a σ-weakly closed algebra which is σ-weakly generated by finite-rank operators in \( A \), then every σ-weakly closed \( A \)-submodule has Property \( S_\sigma \). As a corollary, we obtain \( U_{\tau_1} \overline{\otimes} U_{\tau_2} = U_{\tau_1} \otimes U_{\tau_2} \), where each \( U_{\tau_i} \) is...
a \( \sigma \)-weakly closed \( \text{Alg} L_i \)-submodule and \( L_i \) is a nest. However, the 2-fold tensor product formula cannot be generalized to the \( n \)-fold formula by induction (see the beginning of Section 3). So in Section 3, we use another method to prove the \( n \)-fold tensor product formula \( U_{\phi_1} \otimes \cdots \otimes U_{\phi_n} = U_{\phi_1 \otimes \cdots \otimes \phi_n} \), where each \( U_{\phi_i} \) is a \( \sigma \)-weakly closed \( \text{Alg} L_i \)-submodule and \( L_i \) is a nest. The key to this proof is \cite[Theorem 2]{B} and \cite[Proposition 2.4]{A}.

In this paper, all Hilbert spaces will be separable. Let \( B(\mathcal{H}) \) be the algebra of bounded operators on \( \mathcal{H} \) and \( \mathcal{F}(\mathcal{H}) \) be the set of finite-rank operators on \( \mathcal{H} \). A sublattice \( L \) of the projection lattice of \( B(\mathcal{H}) \) is said to be a subspace lattice if it contains 0 and 1 and is strongly closed, where we identify projections with their ranges. If the elements of \( L \) pairwise commute, \( L \) is a commutative subspace lattice (CSL). A nest is a totally ordered subspace lattice. If \( L \) is a subspace lattice, \( \text{Alg} L \) denotes the set of operators in \( B(\mathcal{H}) \) that leave the elements of \( L \) invariant. Note that \( \text{Alg} L \) is a \( \sigma \)-weakly closed subalgebra of \( B(\mathcal{H}) \). If \( L \) is a CSL, \( \text{Alg} L \) is said to be a CSL algebra. If \( L \) is a nest, \( \text{Alg} L \) is said to be a nest algebra.

If \( A \) is a subset of \( B(\mathcal{H}) \), then \( \text{Lat} A \), the set of projections left invariant by each element of \( A \), is a subspace lattice. A subalgebra \( A \) of \( B(\mathcal{H}) \) is reflexive if \( A = \text{Alg} \text{Lat} A \). The reflexive algebras are precisely the algebras of the form \( \text{Alg} L \), where \( L \) is a subspace lattice. If \( L_i \subseteq B(\mathcal{H}_i) \) \((i=1, \ldots, n)\) are subspace lattices, \( L_1 \otimes \cdots \otimes L_n \) is the subspace lattice in \( B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n) \) generated by \( \{P_1 \otimes \cdots \otimes P_n : P_i \in L_i, i = 1, \ldots, n\} \). If \( S_i \subseteq B(\mathcal{H}_i) \) \((i=1, \ldots, n)\) are \( \sigma \)-weakly closed subspaces, then \( \overline{S_1 \otimes \cdots \otimes S_n} \) denotes the \( \sigma \)-weakly closed linear span of \( \{S_1 \otimes \cdots \otimes S_n : S_i \in S_i\} \) in \( B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n) \).

The main technical tool in Section 2 is the use of slice maps. Slice maps were introduced by Tomiyama in \cite{C} and have been used extensively in the study of tensor products of \( C^* \)-algebras and tensor products of von Neumann algebras. We recall some definitions and results from \cite{D} and refer the reader to \cite{D} for further results and motivation. If \( M \) and \( N \) are von Neumann algebras, and \( \phi \) is in the predual \( M_\varepsilon \) of \( M \), then the right slice map \( R_\phi \) is the unique \( \sigma \)-weakly continuous linear map from \( M \overline{\otimes} N \to N \) such that

\[
\langle X, \phi \otimes \psi \rangle = \langle R_\phi(X), \psi \rangle, \quad \forall X \in M \overline{\otimes} N, \psi \in N_\varepsilon.
\]

If \( X = A \otimes B \) \((A \in M, B \in N)\), then \( R_\phi(X) = \langle A, \phi \rangle B \). The left slice map \( L_\psi : M \overline{\otimes} N \to M, \psi \in N_\varepsilon \), is similarly defined. If \( S \subseteq M \) and \( T \subseteq N \) are \( \sigma \)-weakly closed subspaces, let

\[
F(S, T) = \{X \in M \overline{\otimes} N : R_\phi(X) \in T \text{ and } L_\psi(X) \in S, \forall \phi \in M_\varepsilon, \psi \in N_\varepsilon\}.
\]

As noted in \cite{D}, we can replace \( M \) by \( B(\mathcal{H}_1) \) and \( N \) by \( B(\mathcal{H}_2) \) without affecting \( F(S, T) \). Moreover \( S \overline{\otimes} T \subseteq F(S, T) \). Tomiyama proved in \cite{F} that if \( S \) and \( T \) are von Neumann algebras, then

\[
S \overline{\otimes} T = F(S, T).
\]

His proof uses Tomita’s theorem and, in fact, Tomita’s theorem (1) is equivalent to the validity of (3) for von Neumann algebras. Hence (3) can be considered as a possible general version of Tomita’s theorem for \( \sigma \)-weakly closed subspaces.

A \( \sigma \)-weakly closed subspace \( S \subseteq B(\mathcal{H}) \) is said to have Property \( S_\circ \) if

\[
\{X \in S \overline{\otimes} N : R_\phi(X) \in T \text{ for all } \phi \in B(\mathcal{H}_1) \} = S \overline{\otimes} T
\]
for all pairs \( \{ T, N \} \), where \( T \) is a \( \sigma \)-weakly closed subspace of a von Neumann algebra \( N \). \( S \) has \( \text{Property} \ S_\sigma \) if and only if \( F(S, T) = S \overline{\otimes} T \) for all \( \sigma \)-weakly closed subspaces \( T \) of each von Neumann algebra \( N \) \[\text{[7, Remark 1.5]}\].

2. Property \( S_\sigma \)

Let \( \mathcal{A} \) be a reflexive subalgebra of \( B(\mathcal{H}) \). Suppose that \( E \rightarrow \tau(E) \) is an order homomorphism of \( \text{Lat} \mathcal{A} \) into itself (i.e., \( E \leq F \) implies \( \tau(E) \leq \tau(F) \)). Then the set \( \mathcal{U} = \{ T \in B(\mathcal{H}) : (I - \tau(E))TE = 0, \forall E \in \text{Lat} \mathcal{A} \} \) is clearly a \( \sigma \)-weakly closed \( \mathcal{A} \)-submodule of \( B(\mathcal{H}) \). We denote \( \mathcal{U} \) by \( \mathcal{U}_\tau \).

Erdos and Power in \[\text{[1]}\] proved that any \( \sigma \)-weakly closed \( \mathcal{A} \)-submodule of \( B(\mathcal{H}) \) for a nest algebra \( \mathcal{A} \) is of the above form. Here the following result is due to Han Deguang \[\text{[3]}\]:

**Theorem H.** Let \( \mathcal{A} \) be a unital \( \sigma \)-weakly closed subalgebra which is \( \sigma \)-weakly generated by rank-one operators in \( \mathcal{A} \), and let \( \mathcal{U} \) be a \( \sigma \)-weakly closed \( \mathcal{A} \)-submodule of \( B(\mathcal{H}) \). Then \( \mathcal{U} \) has the form

\[
\mathcal{U} = \{ T \in B(\mathcal{H}) : (I - \tau(E))TE = 0, \forall E \in \text{Lat} \mathcal{A} \},
\]

where \( E \rightarrow \tau(E) = [\mathcal{U}E] \) is an order homomorphism of \( \text{Lat} \mathcal{A} \) into itself.

**Theorem 2.1.** Let \( \mathcal{A} \) be a unital \( \sigma \)-weakly closed subalgebra of \( B(\mathcal{H}) \) with the property that the finite-rank operators of \( \mathcal{A} \) are \( \sigma \)-weakly dense in \( \mathcal{A} \). Then every \( \sigma \)-weakly closed \( \mathcal{A} \)-submodule has Property \( S_\sigma \).

**Proof.** Suppose that \( \mathcal{U} \) is a \( \sigma \)-weakly closed \( \mathcal{A} \)-submodule. Let \( T \) be a \( \sigma \)-weakly closed subspace of a von Neumann algebra \( N \), and suppose that \( X \in \mathcal{U} \overline{\otimes} N \) and \( R_\phi(X) \in \mathcal{T} \) for all \( \phi \in B(\mathcal{H})_\phi \). It suffices to show that \( X \in \mathcal{U} \overline{\otimes} T \). Let \( \pi \) be the normal \( * \)-isomorphism of \( B(\mathcal{H}) \) into \( B(\mathcal{H}) \overline{\otimes} N \) defined by \( \pi(A) = A \otimes I \) for \( A \in B(\mathcal{H}) \). If \( F_1, F_2 \in \mathcal{A} \cap \mathcal{F}(\mathcal{H}) \) and \( \phi \in B(\mathcal{H})_\phi \), a routine calculation shows that \( R_\phi(\pi(F_1)X\pi(F_2)) = R_{F_2\phi F_1}(X) \), where \( F_2\phi F_1 \in B(\mathcal{H})_\phi \) is defined by \( (A, F_2\phi F_1) = (\langle F_1AF_2, \phi \rangle, A) \in B(\mathcal{H})_\phi \). Hence \( R_\phi(\pi(F_1)X\pi(F_2)) \) is in \( T \) for all \( \phi \in B(\mathcal{H})_\phi \). Since \( \pi(F_0)(\mathcal{U} \overline{\otimes} N)\pi(F_2) = F_1\mathcal{U}F_2 \overline{\otimes} N \) and \( F_1\mathcal{U}F_2 \) has Property \( S_\sigma \) by \[\text{[7, Proposition 1.7]}\], \( \pi(F_0)X\pi(F_2) \) is in \( F_1\mathcal{U}F_2 \overline{\otimes} T \). But \( F_1\mathcal{U}F_2 \subseteq \mathcal{U} \); thus \( \pi(F_0)X\pi(F_2) \in \mathcal{U} \overline{\otimes} T \). Let \( \{ F_0 \} \) be a net in \( \mathcal{A} \cap \mathcal{F}(\mathcal{H}) \) converging \( \sigma \)-weakly to the identity map \( I \). Then \( \pi(F_0)X\pi(F) \) converges \( \sigma \)-weakly to \( X\pi(F) \) for all \( F \in \mathcal{A} \cap \mathcal{F}(\mathcal{H}) \), and so \( X\pi(F) \in \mathcal{U} \overline{\otimes} T \) for all \( F \in \mathcal{A} \cap \mathcal{F}(\mathcal{H}) \). Finally, \( X\pi(F_0) \) converges \( \sigma \)-weakly to \( X \), and so \( X \in \mathcal{U} \overline{\otimes} T \). Hence \( \mathcal{U} \) has Property \( S_\sigma \). \( \square \)

It is known from \[\text{[11]}\] that a commutative subspace lattice \( \mathcal{L} \) is completely distributive if and only if the rank-one subalgebra of \( \text{Alg} \mathcal{L} \) is \( \sigma \)-weakly dense in \( \text{Alg} \mathcal{L} \). Thus we have the following result:

**Corollary 2.2.** If \( \mathcal{L} \) is a completely distributive CSL, then every \( \sigma \)-weakly closed \( \text{Alg} \mathcal{L} \)-submodule has Property \( S_\sigma \).

If \( \mathcal{L} \) is a completely distributive CSL, it follows from Theorem H that every \( \sigma \)-weakly closed \( \text{Alg} \mathcal{L} \)-submodule is of the form \( \mathcal{U}_\tau \), where \( E \rightarrow \tau(E) \) is an order homomorphism of \( \mathcal{L} \) into itself.

**Corollary 2.3.** Suppose that \( \mathcal{L}_i \ (i = 1, 2) \) are completely distributive CSLs, and that \( \mathcal{U}_{\tau_i} \ (i = 1, 2) \) are \( \sigma \)-weakly closed \( \text{Alg} \mathcal{L}_i \)-submodules respectively. Then \( \mathcal{U}_{\tau_1} \overline{\otimes} \mathcal{U}_{\tau_2} = F(\mathcal{U}_{\tau_1}, \mathcal{U}_{\tau_2}) \).
Proof. A $\sigma$-weakly closed subspace $S$ has Property $S_\sigma$ if and only if $S \subseteq T = F(S, T)$ for all $\sigma$-weakly closed subspaces $T$ ([2] Remark 1.5). Thus the corollary follows from Corollary 2.2.

In the case of nest algebras, we can say more about tensor products of $\sigma$-weakly closed nest algebra submodules. In the rest of this paper, we suppose that $L_i$ ($i = 1, 2, \cdots, n$) are nests on separable complex Hilbert spaces $H_i$ and $\tau_i$ are order homomorphisms of $L_i$ into $L_i$.

If $L \in L_1 \otimes \cdots \otimes L_n$, it follows from [2] Proposition 2.4 that

$$L = \vee \{ E_1 \otimes \cdots \otimes E_n : E_1 \otimes \cdots \otimes E_n \leq L \}.$$ 

Thus we can define

$$\tau(L) = \vee \{ \tau_1(E_1) \otimes \cdots \otimes \tau_n(E_n) : E_1 \otimes \cdots \otimes E_n \leq L \}.$$ 

Obviously, $(\tau_1 \otimes \cdots \otimes \tau_n)(E_1 \otimes \cdots \otimes E_n) = \tau_1(E_1) \otimes \cdots \otimes \tau_n(E_n)$. Thus $\tau_1 \otimes \cdots \otimes \tau_n$ is a well-defined order homomorphism of $L_1 \otimes \cdots \otimes L_n$ into itself and $U_{\tau_1 \otimes \cdots \otimes \tau_n}$ is a $\sigma$-weakly closed $\Alg(L_1 \otimes \cdots \otimes L_n)$-submodule. Hence the equality $\Alg(L_1 \otimes \cdots \otimes \Alg(L_n) = \Alg(L_1 \otimes \cdots \otimes L_n)$ of [2] Theorem 2.6] can be rewritten as

$$U_{\tau_1} \otimes \cdots \otimes U_{\tau_n} = U_{\tau_1 \otimes \cdots \otimes \tau_n},$$

where $I_i$ is the identity map of $L_i$ into $L_i$.

**Lemma 2.4.** Let $L_i$ ($i = 1, 2$) be nests on separable Hilbert spaces $H_i$ and $\tau_i$ ($i = 1, 2$) be order homomorphisms of $L_i$ into $L_i$. Then $U_{\tau_1 \otimes \tau_2} = F(U_{\tau_1}, U_{\tau_2})$.

**Proof.** Suppose that $X \in U_{\tau_1 \otimes \tau_2} \subseteq B(H_1 \otimes H_2)$. If $E_2 \in L_2$ and $\phi \in B(H_2)_*$, it follows from [7] (1.3) that

$$\tau_2(E_2)R_{\phi}(X)E_2 = R_{\phi}((I_1 \otimes \tau_2(E_2))X(I_1 \otimes E_2)) = R_{\phi}((I_1 \otimes \tau_2(E_2))(\tau_1(I_1) \otimes \tau_2(E_2))X(I_1 \otimes E_2)) = R_{\phi}(E_2X(I_1 \otimes E_2)) = R_{\phi}(X_1 \otimes E_2).$$

So $R_{\phi}(X) \in U_{\tau_1 \otimes \tau_2}$. Similarly, $L_{\phi}(X) \in U_{\tau_1}$ for all $\psi \in B(H_2)_*$. Hence by the definition of $F(U_{\tau_1}, U_{\tau_2})$, we have $U_{\tau_1 \otimes \tau_2} \subseteq F(U_{\tau_1}, U_{\tau_2})$.

Conversely, suppose that $X \in F(U_{\tau_1}, U_{\tau_2})$. If $E_2 \in L_2$ and $\phi \in B(H_2)_*$, then $\tau_2(E_2)R_{\phi}(X)E_2 = R_{\phi}(X_1 \otimes E_2)$. Thus $R_{\phi}((I_1 \otimes \tau_2(E_2))(\tau_1(I_1) \otimes \tau_2(E_2))X(I_1 \otimes E_2)) = R_{\phi}(X_1 \otimes E_2)$ for all $\phi \in B(H_2)_*$. It follows from [7] (1.5) that

$$X(E_1 \otimes E_2) = (\tau_1 \otimes \tau_2(E_2))X(I_1 \otimes E_2).$$

Similarly, if $E_1 \in L_1$, we have that $X(E_1 \otimes I_2) = (\tau_1 \otimes \tau_2(I_2))X(I_1 \otimes E_2)$. Therefore,

$$X(E_1 \otimes E_2) = X(E_1 \otimes I_2)(I_1 \otimes E_2),$$

and

$$X(E_1 \otimes E_2) = \tau_1(E_1) \otimes \tau_2(E_2)X(E_1 \otimes E_2).$$

Thus, by virtue of [2] Proposition 2.4, it is easy to show that $X \in U_{\tau_1 \otimes \tau_2}$ for each $L \in L_1 \otimes L_2$. Hence $X \in U_{\tau_1 \otimes \tau_2}$ and $U_{\tau_1 \otimes \tau_2} = F(U_{\tau_1}, U_{\tau_2})$.

**Theorem 2.5.** Let $L_i$ and $\tau_i$ be as in the preceding lemma. Then $U_{\tau_1 \otimes \tau_2} = U_{\tau_1 \otimes \tau_2}$.

**Proof.** Since every nest is a completely distributive CSL, the theorem follows from Corollary 2.3 and Lemma 2.4, obviously.
3. The n-fold tensor product formula

Since \( \mathcal{L}_1 \otimes \mathcal{L}_2 \) is not totally ordered in general, we cannot deduce the tensor product formula \( \mathcal{U}_{\tau_1} \otimes_{\tau_2} \otimes_{\tau_3} = \mathcal{U}_{\tau_1} \otimes \mathcal{U}_{\tau_2} \otimes \mathcal{U}_{\tau_3} \) by

\[
\mathcal{U}_{\tau_1} \otimes_{\tau_2} \otimes_{\tau_3} = U_{(\tau_1 \otimes \tau_2) \otimes \tau_3} = U_{\tau_1 \otimes \tau_2} \otimes U_{\tau_3} = \mathcal{U}_{\tau_1} \otimes \mathcal{U}_{\tau_2} \otimes \mathcal{U}_{\tau_3}.
\]

(In order to use Theorem 2.5, the second equality needs the totally ordered property of \( \mathcal{L}_1 \otimes \mathcal{L}_2 \).) So we cannot generalize Theorem 2.5 to \( n \)-fold tensor products for \( n > 2 \) by induction. In this section, instead of the slice maps, we shall use Theorem H to prove the \( n \)-fold tensor product formula. Let \( \mathcal{L}_i \) (\( i = 1, \ldots, n \)) be nests and let \( \mathcal{U}_i \) be \( \sigma \)-weakly closed \( \mathcal{L}_i \)-submodules. From Theorem H, it follows from \( \mathcal{U}_i = \mathcal{U}_{\tau_i} \), where \( \tau_i(E) = [\mathcal{U}_i E] \) for any \( E \in \mathcal{L}_i \). In the rest of this section, we always use \( \tau_i \) to denote these special order homomorphisms.

**Lemma 3.1.** For each \( i = 1, \ldots, n \), let \( E_i \in \mathcal{L}_i \) and \( f_i \in E_i \) such that \([Alg \mathcal{L}_i] f_i = E_i\). Then \([Alg \mathcal{L}_1] \otimes \ldots \otimes [Alg \mathcal{L}_n] (f_1 \otimes \ldots \otimes f_n) = \mathcal{U}_{\tau_1} \otimes \ldots \otimes \mathcal{U}_{\tau_n} E_n\).

**Proof.** Since \( \mathcal{U}_i \cdot Alg \mathcal{L}_i = \mathcal{U}_{\tau_i} \), \( [\mathcal{U}_i E_i] = [\mathcal{U}_{\tau_i} f_i] \). By virtue of [2] Lemma 2.2,

\[
E_1 \otimes \ldots \otimes E_n = [(Alg \mathcal{L}_1) \otimes \ldots \otimes (Alg \mathcal{L}_n)](f_1 \otimes \ldots \otimes f_n).
\]

Thus, since \((\mathcal{U}_{\tau_1} \otimes \ldots \otimes \mathcal{U}_{\tau_n})(Alg \mathcal{L}_1) \otimes \ldots \otimes (Alg \mathcal{L}_n) = \mathcal{U}_{\tau_1} \otimes \ldots \otimes \mathcal{U}_{\tau_n}\),

\[
[\mathcal{U}_1 \otimes \ldots \otimes \mathcal{U}_n] (f_1 \otimes \ldots \otimes f_n) = [\mathcal{U}_{\tau_1} \otimes \ldots \otimes \mathcal{U}_{\tau_n}(f_1 \otimes \ldots \otimes f_n)]
\]

Hence it suffices to prove that

\[
[\mathcal{U}_1 \otimes \ldots \otimes \mathcal{U}_n](f_1 \otimes \ldots \otimes f_n) = [\mathcal{U}_{\tau_1} f_1] \otimes \ldots \otimes [\mathcal{U}_{\tau_n} f_n].
\]

If \( g_i \) is any vector in \([\mathcal{U}_i f_i]\), then \( g_i \) can be norm approximated by vectors of the form \( T_i f_i \), where \( T_i \in \mathcal{U}_i \). Hence \( g_1 \otimes \ldots \otimes g_n \) can be approximated by vectors of the form \( T_1 f_1 \otimes \ldots \otimes T_n f_n = (T_1 \otimes \ldots \otimes T_n)(f_1 \otimes \ldots \otimes f_n) \). Thus any vector of the form \( g_1 \otimes \ldots \otimes g_n \) with \( g_i \in [\mathcal{U}_i f_i] \) lies in \([\mathcal{U}_1 \otimes \ldots \otimes \mathcal{U}_n](f_1 \otimes \ldots \otimes f_n]\).

Since such vectors generate \([\mathcal{U}_1 f_1] \otimes \ldots \otimes [\mathcal{U}_n f_n]\), we have \([\mathcal{U}_1 f_1] \otimes \ldots \otimes [\mathcal{U}_n f_n] \subseteq [\mathcal{U}_1 \otimes \ldots \otimes \mathcal{U}_n](f_1 \otimes \ldots \otimes f_n)\).

To prove the reverse inequality, for any \( T_i \in \mathcal{U}_i \), we have that

\[
([\mathcal{U}_1 f_1] \otimes \ldots \otimes [\mathcal{U}_n f_n])(T_1 \otimes \ldots \otimes T_n)(E_1 \otimes \ldots \otimes E_n)
\]

\[
= ([\mathcal{U}_1 E_1] \otimes \ldots \otimes [\mathcal{U}_n E_n])(T_1 \otimes \ldots \otimes T_n)(E_1 \otimes \ldots \otimes E_n)
\]

\[
= (\tau_1(E_1) \otimes \ldots \otimes \tau_n(E_n))(T_1 \otimes \ldots \otimes T_n)(E_1 \otimes \ldots \otimes E_n)
\]

\[
= \tau_1(E_1)T_1E_1 \otimes \ldots \otimes \tau_n(E_n)T_nE_n
\]

\[
= T_1E_1 \otimes \ldots \otimes T_nE_n
\]

\[
= (T_1 \otimes \ldots \otimes T_n)(E_1 \otimes \ldots \otimes E_n).
\]

This shows that

\[
([\mathcal{U}_1 f_1] \otimes \ldots \otimes [\mathcal{U}_n f_n])[\mathcal{U}_1 \otimes \ldots \otimes \mathcal{U}_n](E_1 \otimes \ldots \otimes E_n)
\]

\[
= [\mathcal{U}_1 \otimes \ldots \otimes \mathcal{U}_n](E_1 \otimes \ldots \otimes E_n)].
\]

Thus

\[
[\mathcal{U}_1 \otimes \ldots \otimes \mathcal{U}_n](f_1 \otimes \ldots \otimes f_n) \leq [\mathcal{U}_1 f_1] \otimes \ldots \otimes [\mathcal{U}_n f_n].
\]

Therefore

\[
[\mathcal{U}_1 \otimes \ldots \otimes \mathcal{U}_n](f_1 \otimes \ldots \otimes f_n) = [\mathcal{U}_1 f_1] \otimes \ldots \otimes [\mathcal{U}_n f_n].
\]

This completes the proof. \( \square \)
Theorem 3.2. Let $\mathcal{U}_i$ ($i = 1, \cdots, n$) be $\sigma$-weakly closed $\text{Alg} \mathcal{L}_i$-submodules and $\tau_i(E) = [\mathcal{U}_i E]$ for any $E \in \mathcal{L}_i$. Then $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n = \mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n$.

Proof. It is obvious that $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n$ is a $\sigma$-weakly closed $\text{Alg} \mathcal{L}_1 \otimes \cdots \otimes \text{Alg} \mathcal{L}_n$-submodule. By virtue of [2, Theorem 2.6], $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n$ is a $\sigma$-weakly closed $\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$-submodule. It follows from [2, Proposition 2.7] that $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ is a completely distributive CSL. Thus, Theorem H shows that $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n$ is determined by the order homomorphism $L \rightarrow [[\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n] L]$ of $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ into itself.

Now suppose that $E_i \in \mathcal{L}_i$. For each $i$, choose a vector $v_i \in E_i$ such that $[(\text{Alg} \mathcal{L}_i) v_i] = E_i$ (the proof of the existence of such $v_i$ is routine). It follows from Lemma 3.1 that

$$[(\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n)(E_1 \otimes \cdots \otimes E_n)] = [\mathcal{U}_1 E_1] \otimes \cdots \otimes [\mathcal{U}_n E_n] = \tau_1(E_1) \otimes \cdots \otimes \tau_n(E_n) = (\tau_1 \otimes \cdots \otimes \tau_n)(E_1 \otimes \cdots \otimes E_n).$$

If $L \in \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$, [2, Proposition 2.4] shows that

$$L = \vee\{E_1 \otimes \cdots \otimes E_n : E_1 \otimes \cdots \otimes E_n \leq L\}.$$ 

Thus,

$$[(\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n)L] = \vee\{(\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n)(E_1 \otimes \cdots \otimes E_n) : E_1 \otimes \cdots \otimes E_n \leq L\} = \vee\{(\tau_1 \otimes \cdots \otimes \tau_n)(E_1 \otimes \cdots \otimes E_n) : E_1 \otimes \cdots \otimes E_n \leq L\} = (\tau_1 \otimes \cdots \otimes \tau_n)(L).$$

Hence $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n$ and $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n$ are $\sigma$-weakly closed $\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$-submodules determined by the same order homomorphism. This shows that

$$\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n = \mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_n.$$  

\[\square\]

Given general order homomorphisms $\phi_i$ from $\mathcal{L}_i$ into $\mathcal{L}_i$, we will consider the relation between $\mathcal{U}_i \otimes \cdots \otimes \mathcal{U}_\phi_i$ and $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_\phi_i$. We need some lemmas at first.

For non-zero vectors $x, y \in \mathcal{H}$, the rank-one operator $xy^*$ is defined by the equation

$$(xy^*)(z) = \langle z, y \rangle x, \quad \forall z \in \mathcal{H}.$$  

Lemma 3.3. Suppose that $\mathcal{L}$ is a subspace lattice, and that $\mathcal{U}_\phi$ is the $\sigma$-weakly closed $\text{Alg} \mathcal{L}$-submodule determined by an order homomorphism $\phi$ from $\mathcal{L}$ into itself. Then a rank-one operator $xy^* \in \mathcal{U}_\phi$ if and only if there exists an element $N \in \mathcal{L}$ such that $x \in N$ and $y \in \phi_\rightarrow(N)^+ \backslash \phi(N)$, where $\phi_\rightarrow(N) = \vee\{G \in \mathcal{L} : \phi(G) \nsubseteq N\}$.

Proof. The proof is routine. We leave the details to the interested readers.  

\[\square\]

Lemma 3.4. Let $\mathcal{L}_i$ be a nest and $\phi_i$ be an order homomorphism from $\mathcal{L}_i$ into itself. Define $\psi_i : I_1 \otimes \cdots \otimes I_i \otimes \cdots \otimes I_n \rightarrow I_1 \otimes \cdots \otimes I_i \otimes \cdots \otimes I_n$ by

$$\psi_i(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n) = I_1 \otimes \cdots \otimes \phi_i(N_i) \otimes \cdots \otimes I_n, \quad \forall N_i \in \mathcal{L}_i.$$ 

Then the rank-one operator $xy^* \in \mathcal{U}_\phi$ if and only if there exists an element $N_i \in \mathcal{L}_i$ such that $x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n$ and $y \in I_1 \otimes \cdots \otimes \phi_i(N_i)^+ \otimes \cdots \otimes I_n$. 

\[\square\]
Proof. Certainly $\psi_i$ is an order homomorphism from $I_1 \otimes \cdots \otimes I_n$ into itself, and $U_{\psi_i}$ is the $\sigma$-weakly closed $\text{Alg}(I_1 \otimes \cdots \otimes I_n)$-submodule determined by $\psi_i$. By virtue of Lemma 3.3, a rank-one operator $xy^* \in U_{\psi_i}$ if and only if there exists an element $N_i \in L_i$ such that $x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n$ and $y \in \psi_i^{-1}(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)^{\perp}$. In the following, we compute $\psi_i^{-1}(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)^{\perp}$.

By the definition of $\psi_i^{-1}$, we have

$$\psi_i^{-1}(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n) = \{ I_1 \otimes \cdots \otimes G_i \otimes \cdots \otimes I_n : \psi_i(I_1 \otimes \cdots \otimes G_i \otimes \cdots \otimes I_n) \not\supseteq I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n \}$$

$$= \{ I_1 \otimes \cdots \otimes G_i \otimes \cdots \otimes I_n : \psi_i(G_i) \not\supseteq I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n \}$$

$$= \{ I_1 \otimes \cdots \otimes (\bigvee G_i \subseteq \phi_i(G_i) \not\supseteq N_i) \}$$

$$= \{ I_1 \otimes \cdots \otimes (I_1 \otimes \cdots \otimes I_n \otimes \cdots \otimes I_n) \}$$

and so $\phi_i^{-1}(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n)^{\perp} = I_1 \otimes \cdots \otimes \phi_i^{-1}(N_i)^{\perp} \otimes \cdots \otimes I_n$. □

Proposition 3.5. Let $L_i (i = 1, \ldots, n)$ be nests and $\phi_i$ be order homomorphisms from $L_i$ into itself. Then a rank-one operator $xy^* \in \text{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$ if and only if there exist $N_i \in L_i$ such that $x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n$ and $y \in \phi_i^{-1}(N_i)^{\perp} \otimes \cdots \otimes \phi_n^{-1}(N_n)^{\perp}$.

Proof. Set $F_i = I_1 \otimes \cdots \otimes L_i \otimes \cdots \otimes I_n$, and define $\psi_i : F_i \to F_i$ by

$$\psi_i(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n) = I_1 \otimes \cdots \otimes \phi_i(N_i) \otimes \cdots \otimes I_n, \quad \forall N_i \in L_i.$$

Each $\psi_i$ is an order homomorphism from $F_i$ into itself and $U_{\psi_i}$ is the $\sigma$-weakly closed $\text{Alg}(F_i)$-submodules determined by $\psi_i$. Thus we have the equation $U_{\phi_1 \otimes \cdots \otimes \phi_n} = U_{\psi_1} \cap \cdots \cap U_{\psi_n}$. In fact, by virtue of [2, Proposition 2.4],

$$L = \{ N_1 \otimes \cdots \otimes N_n : N_1 \otimes \cdots \otimes N_n \leq L \}$$

for any $L \in L_1 \otimes \cdots \otimes L_n$.

Thus it is easy to show that

$$U_{\phi_1 \otimes \cdots \otimes \phi_n} = \{ T \in B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n) : T(N_1 \otimes \cdots \otimes N_n) \subseteq \phi_1(N_1) \otimes \cdots \otimes \phi_n(N_n), \forall N_i \in L_i \},$$

and so $U_{\phi_1 \otimes \cdots \otimes \phi_n} \subseteq U_{\psi_1} \cap \cdots \cap U_{\psi_n}$. For any $T \in U_{\psi_1} \cap \cdots \cap U_{\psi_n}$, we have that for any $N_i \in L_i$,

$$T(N_1 \otimes \cdots \otimes N_n) \subseteq T(I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n) \subseteq I_1 \otimes \cdots \otimes \phi_i(N_i) \otimes \cdots \otimes I_n, \quad \forall 1 \leq i \leq n.$$

Thus $T(N_1 \otimes \cdots \otimes N_n) \subseteq \phi_1(N_1) \otimes \cdots \otimes \phi_n(N_n)$ and $T \in U_{\phi_1 \otimes \cdots \otimes \phi_n}$. Hence $U_{\phi_1 \otimes \cdots \otimes \phi_n} = U_{\psi_1} \cap \cdots \cap U_{\psi_n}$. From Lemma 3.4 it follows that for any $1 \leq i \leq n$, a rank-one operator $xy^* \in U_{\psi_i}$ if and only if there exists $N_i \in L_i$ such that $x \in I_1 \otimes \cdots \otimes N_i \otimes \cdots \otimes I_n$ and $y \in \psi_i^{-1}(N_i)^{\perp} \otimes \cdots \otimes I_n$. Therefore a rank-one operator $xy^* \in U_{\psi_1} \cap \cdots \cap U_{\psi_n} = U_{\phi_1 \otimes \cdots \otimes \phi_n}$ if and only if there exists $N_i \in L_i (1 \leq i \leq n)$ such that $x \in N_1 \otimes \cdots \otimes N_n$ and $y \in \phi_i^{-1}(N_i)^{\perp} \otimes \cdots \otimes \phi_n^{-1}(N_n)^{\perp}$. □

Lemma 3.6. Suppose that $L$ is a subspace lattice and that $U_{\phi}$ is the $\sigma$-weakly closed $\text{Alg} L$-submodule determined by an order homomorphism $\phi$ from $L$ into itself. Then $\tau \leq \phi$ and $\tau_* = \phi_*$, where $\tau(E) = [U_{\phi} E]$ for any $E \in L$. 

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Proof. It follows from the definition of $\mathcal{U}_\phi$ that
\[ \tau(E) = [\mathcal{U}_\phi E] \leq \phi(E) \quad \text{for any} \quad E \in \mathcal{L}. \]
So $\tau \leq \phi$.

Since $\tau \leq \phi$, we have $\tau_\omega \geq \phi_\omega$. So it suffices to show that $\tau_\omega \leq \phi_\omega$. If not, there exists $E \in \mathcal{L}$ such that $\tau_\omega(E) \not\leq \phi_\omega(E)$. It follows from the definition of $\tau_\omega$ that there exists $F \in \mathcal{L}$ such that $\tau(F) \not\geq E$ and $F \not\leq \phi_\omega(E)$. Thus we can choose non-zero vectors $x, y$ such that $x \in E$ and $x \not\in \tau(F)$, $y \in \phi_\omega(E)\dagger$ and $y \not\in F\dagger$. From Lemma 3.3, it follows that $x \otimes y \in \mathcal{U}_\phi$. Since $(I - \tau(F))(x \otimes y)F \neq 0$, $x \otimes y \not\in \mathcal{U}_\tau$. However it follows from the proof of Theorem H that $\mathcal{U}_\tau = \mathcal{U}_\phi$. This is a contradiction. Accordingly, $\tau_\omega \leq \phi_\omega$ and $\tau_\omega = \phi_\omega$. \hfill\Box

Now we are in the position to show the general tensor product formula of $\sigma$-weakly closed $\mathcal{L}_\tau$-submodules.

Theorem 3.7. Let $\mathcal{L}_i$ $(i = 1, \cdots, n)$ be nests and $\phi_i$ be order homomorphisms from $\mathcal{L}_i$ into itself. Then $\mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n} = \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$.

Proof. It follows from Theorem H that $\mathcal{U}_{\phi_i} = \mathcal{U}_{\tau_i}$, where $\tau_i(E) = [\mathcal{U}_{\phi_i} E]$ for any $E \in \mathcal{L}_i$. Thus by virtue of Theorem 3.2, we have that
\[ \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n} = \mathcal{U}_{\tau_1 \otimes \cdots \otimes \tau_n}. \]
So it suffices to show $\mathcal{U}_{\tau_1 \otimes \cdots \otimes \tau_n} = \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$. Since $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ is a completely distributive CSL ([2], Proposition 2.7]), it follows from [10] Theorem 3 that the rank-one operators of $\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$ are $\sigma$-weakly dense in $\text{Alg}(\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$. So it is routine to show that the linear spans of rank-one operators in $\mathcal{U}_{\tau_1 \otimes \cdots \otimes \tau_n}$ and $\mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$ are $\sigma$-weakly dense in $\mathcal{U}_{\tau_1 \otimes \cdots \otimes \tau_n}$ and $\mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$, respectively. From Proposition 3.5 and Lemma 3.6, it follows that $\mathcal{U}_{\tau_1 \otimes \cdots \otimes \tau_n}$ and $\mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$ have the same rank-one operators. Therefore $\mathcal{U}_{\tau_1 \otimes \cdots \otimes \tau_n} = \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$ and $\mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n} = \mathcal{U}_{\phi_1 \otimes \cdots \otimes \phi_n}$. \hfill\Box

Remark 3.8. Theorem 2.5 is a particular case of Theorem 3.2. In [3], Theorem 2.2 shows that $\mathcal{U}_{\tau_i}$ $(i = 1, \cdots, n)$ are reflexive subspaces. Combining the above result, we know that the tensor product of $\mathcal{U}_{\tau_i}$ is also reflexive. It is natural to ask whether the tensor product of reflexive subspaces is also reflexive. This seems a challenging problem.

References


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