AN EXTREMAL FUNCTION
FOR THE CHANG-MARSHALL INEQUALITY
OVER THE BEURLING FUNCTIONS

VALENTIN V. ANDREEV

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Abstract. S.-Y. A. Chang and D. E. Marshall showed that the functional
\[ \Lambda(f) = \frac{1}{\pi} \int_0^{2\pi} \Phi(|f(e^{i\theta})|^2) d\theta \]
is bounded on the unit ball \( B \) of the space \( D \) of analytic functions in the unit disk with \( f(0) = 0 \) and Dirichlet integral not exceeding one. Andreev and Matheson conjectured that the identity function \( f(z) = z \) is a global maximum on \( B \) for the functional \( \Lambda \). We prove that \( \Lambda \) attains its maximum at \( f(z) = z \) over a subset of \( B \) determined by kernel functions, which provides a positive answer to a conjecture of Cima and Matheson.

1. Introduction

Let \( D \) be the Dirichlet space of functions \( f \) analytic on the unit disk \( D \), with \( f(0) = 0 \) and a finite Dirichlet integral
\[ \|f\|_D^2 = \frac{1}{\pi} \int_D |f'(z)|^2 dx dy. \]
It is well known that \( D \) is a Hilbert space with inner product
\[ \langle f, g \rangle_D = \frac{1}{\pi} \int_D f'(z) \overline{g'(z)} dx dy. \]
Let \( B = \{ f \in D : \|f\|_D \leq 1 \} \) be its closed unit ball.

We shall be concerned with functionals \( \Lambda_\Phi \) on \( B \) defined by
\[ \Lambda_\Phi(f) = \frac{1}{\pi} \int_0^{2\pi} \Phi(|f(e^{i\theta})|) d\theta, \]
for \( f \in B \) and \( \Phi : (-\infty, \infty) \to \mathbb{R} \) being a continuous convex nondecreasing function.

A function \( f \) is a maximum for \( \Lambda_\Phi \) if \( f \in B \) and \( \Lambda_\Phi(f) \geq \Lambda_\Phi(g) \) for all \( g \in B \).

Chang and Marshall [3] proved that if \( \Phi_{\alpha}(t) = e^{\alpha t^2} \) for \( \alpha > 0 \), then \( \Lambda_{\Phi_{\alpha}} \) is bounded on \( B \) if and only if \( \alpha \leq 1 \). In their proof they compared functions in \( B \) to the Beurling functions
\[ B_a(z) = \frac{\log \frac{1}{1-|a|^2}}{\sqrt{\log \frac{1}{1-|a|^2}}}, \]
for \( a \in B \).
for \(a \in \mathbb{D} \setminus \{0\}\), where the branch of the logarithm is chosen so that \(B_\alpha(a)\) is real. The denominator assures that \(\|B_\alpha\|_D = 1\). Up to a normalizing factor, the \(B_\alpha\) are the kernel functions for \(D\). We shall denote by \(B_0\) the set of all Beurling functions and by \(\bar{B}_0\) its closed convex hull.

A shorter proof of this fact has since been found by Marshall [9]. A significantly more general and stronger inequality has been found by Essén [7]. Andreev and Matheson [1] showed that the identity function \(f(z) = z\) is a local maximum on \(B\) and conjectured that it is also a global maximum. Cima and Matheson [4] showed that the identity function is a local maximum on the set \(\bar{B}_0\) and that \(\Lambda_\Phi\), when restricted to \(\bar{B}_0\), is not weakly continuous at 0, and thus it is an open question whether there exists a global maximum for \(\Lambda_\Phi\) on \(B\). Matheson and Pruss [10] studied the regularity of the extremal functions. We refer the reader to their paper for an excellent discussion of this and other related problems and for a list of open problems.

Our principle result is:

**Theorem 1.1.** The inequality
\[
\Lambda_{\Phi_1}(f) < \Lambda_{\Phi_1}(z)
\]
holds true for all \(f \in \bar{B}_0\).

Our result proves Conjecture 1 of Cima and Matheson in [4].

**2. Proof of Theorem 1.1**

It is natural to set \(B_0(z) = z\) (see [4]). A function \(\Phi(x)\) continuous on \(-\infty < x < \infty\) is said to be **convex** if \(\Phi((x+y)/2) \leq (\Phi(x) + \Phi(y))/2\), and **strictly convex** if strict inequality holds whenever \(x \neq y\). Theorem 1.1 is a consequence of the following result.

**Theorem 2.1.** Let \(\Phi(x)\) be a convex nondecreasing function on \(-\infty < x < \infty\). For all \(\alpha_0, \alpha \in \mathbb{D} \setminus \{0\}\) such that \(0 \leq |\alpha_0| < |\alpha| < 1\), we have
\[
\int_0^{2\pi} \Phi(\log |B_\alpha(re^{i\theta})|)d\theta \leq \int_0^{2\pi} \Phi(\log |B_{\alpha_0}(re^{i\theta})|)d\theta,
\]
where \(0 < r < 1\). If \(\Phi\) is strictly convex, then the inequality is strict for all \(r\).

**Proof.** Our proof is based on the deep results of Albert Baernstein [2, Theorem 1] on integral means of univalent functions (see also Chapter 7 of Duren’s book [5]). In particular, we need the following proposition [2, Proposition 3].

**Proposition 2.2.** For \(g, h \in L^1(-\pi, \pi)\), the following statements are equivalent.

(a) For each function \(\Phi(s)\) convex and nondecreasing on \(-\infty < s < \infty\),
\[
\int_{-\pi}^{\pi} \Phi(g(x))dx \leq \int_{-\pi}^{\pi} \Phi(h(x))dx.
\]

(b) For each \(t \in \mathbb{R}\),
\[
\int_{-\pi}^{\pi} [g(x) - t]^+ dx \leq \int_{-\pi}^{\pi} [h(x) - t]^+ dx.
\]

(c) \(g^+(\theta) \leq h^+(\theta)\), \(0 \leq \theta \leq \pi\).
Here for each \( r \in (r_1, r_2) \) and \( u(re^{i\theta}) \in L^1(0, 2\pi) \) the Baernstein star-function of \( u \) is defined as
\[
(2.2) \quad u^*(re^{i\theta}) = \sup_{|E|=2\pi} \int_E u(re^{i\theta}) \, dt,
\]
0 \leq \theta \leq \pi, where \( |E| \) denotes the Lebesgue measure of the set \( E \subset [-\pi, \pi] \).

In view of Proposition 2.2, we want first to show that
\[
(2.7) \quad u < r < 0
\]
with a continuous extension to the closed unit disk
\[
(2.4) \quad B_u \circ 0.
\]
Extend it to a continuous function in the punctured plane by setting
\[
(2.8) \quad u < r < 0
\]
at the origin.

in the open upper half-plane and continuous in the closed upper half-plane, except

\[ |a| \leq 1, \text{ for each } 0 < a < 1. \]

Notice that
\[
B_u(0) = 0 \quad \text{is subharmonic}.
\]

the Green’s function of
\[
B_u \circ 0
\]
with pole at 0.

the inequality (2.3) can be recast in the form
\[
\int_{-\pi}^{\pi} [u_a(re^{i\theta}) + \log r]^+ \, d\phi \leq \int_{-\pi}^{\pi} [u_a(re^{i\theta}) + \log r]^+ \, d\phi,
\]
0 < r < 1, 0 < \rho < \infty. By Proposition 2.2, this is implied by the inequality
\[
(2.7) \quad u^*_a(\rho e^{i\phi}) \leq u^*_a(\rho e^{i\phi}),
\]
0 < \rho < \infty, 0 \leq \phi \leq \pi.

The function \( u(\zeta) \) is continuous in \( 0 < |\zeta| < \infty \), it is positive and harmonic in
\( D_a \), and identically zero outside \( D_a \). Thus it is subharmonic in \( 0 < |\zeta| < \infty \). Hence

Theorem A] and the definition (2.2) of the star-function, \( u^*_a \) is subharmonic in the open upper half-plane and continuous in the closed upper half-plane, except

at the origin.
Since \( B_a^{-1}(\zeta) = (1 - e^{-A\zeta})/a \), then, near the origin, \( u_a \) has the form

\[
(2.8) \quad u_a(\zeta) = -\log |\zeta| - \log \frac{A}{a} + u_{1a}(\zeta),
\]

where \( u_{1a} \) is harmonic and \( u_{1a}(0) = 0 \). Thus

\[
u^*_a(\rho e^{i\phi}) + 2\phi \log \rho \rightarrow -2\phi \log \frac{A}{a}
\]
as \( \rho \rightarrow 0 \) for \( 0 \leq \phi \leq \pi \). Similarly, near the origin, \( u_{a_0} \) has the form

\[
u_{a_0}(\zeta) = -\log |\zeta| - \log \frac{A_0}{a_0} + u_{1a_0}(\zeta),
\]

where \( u_{1a_0} \) is harmonic and \( u_{1a_0}(0) = 0 \). Thus

\[
u_{a_0}(\rho e^{i\phi}) + 2\phi \log \rho \rightarrow -2\phi \log \frac{A_0}{a_0}
\]
as \( \rho \rightarrow 0 \) for \( 0 \leq \phi \leq \pi \). It follows that

\[
[u^*_a(\rho e^{i\phi}) - u^*_{a_0}(\rho e^{i\phi})] \rightarrow -2\phi \log \frac{a_0 A}{A_0}
\]
as \( \rho \rightarrow 0 \) for \( 0 \leq \phi \leq \pi \). It is easy to see that \( a_0 A/(a A_0) > 1 \) for \( a_0 < a \) and hence that \(-2\pi \log \frac{a_0 A}{A(a_0)} \leq -2\pi \log \frac{A_0}{a_0} \leq 0 \) for \( a_0 < a \).

Hence \( (u^*_a - u^*_{a_0}) \) is subharmonic in the upper half-plane and continuous in its closure except at the origin, where it has a bounded discontinuity: for \( \phi = 0 \),

\[
\lim_{\rho \rightarrow 0} (u^*_a(\rho) - u^*_{a_0}(\rho)) = 0,
\]

and for \( \phi = \pi \),

\[
\lim_{\rho \rightarrow 0} (u^*_a(-\rho) - u^*_{a_0}(-\rho)) = -2\pi \log \frac{a_0 A}{A_0}.
\]

We want to show that \( (u^*_a - u^*_{a_0}) < 0 \) in the open upper half-plane. Since \( u^*_a - u^*_{a_0} \) is discontinuous at the origin, we cannot apply the maximum principle for subharmonic functions to \( u^*_a - u^*_{a_0} \) at this point. The proof of the inequality \( (u^*_a - u^*_{a_0}) < 0 \) for \( 3\zeta > 0 \) will be based on the following four steps (a)–(d).

(a) On the positive real axis, by definition, \( u^*_a(\zeta) = v^*(\zeta) = 0 \) for \( \zeta > 0 \).

(b) Next let \( d_a \) be the distance from 0 to the complement of \( D_a \). It is obvious that \( \Re(1 - a e^{i\theta})^{-1} > 0 \). Since the branch of the logarithm was chosen so that \( B_a(a) \) is real, then

\[
|B_a(e^{i\theta})| = \frac{1}{A} \left\{ |\log \frac{1}{1 - a e^{i\theta}}| \right\}^2 + |\arg \frac{1}{1 - a e^{i\theta}}|^2 \right]^{1/2}.
\]

Since \( \max |1 - a e^{i\theta}| = |1 - a e^{i\pi}| = 1 + a \) and \( |\arg \frac{1}{1 - a e^{i\theta}}|^2 = 0 \), it is easy to see that

\[
-\frac{1}{A} \log \frac{1}{1 + a} \leq |B_a(e^{i\theta})| \leq \frac{1}{A} \log \frac{1}{1 - a}
\]

for \( 0 < a < 1 \). Thus \( d_a = -\frac{1}{A} \log \frac{1}{1 + a} \). We want to show that \( d_a \) is a decreasing function of \( a \) for \( 0 < a < 1 \). It is clear that \( d_a \rightarrow 1 \) as \( a \rightarrow 0 \). Let

\[
f(a) = \frac{\log(1 + a)}{A}.
\]

Then

\[
f'(a) = -\frac{[(1 - a) \log(1 - a) + \log(1 + a)]}{(1 - a^2) A^3}.
\]
Let
\[ f_1(a) = (1 - a) \log(1 - a) + \log(1 + a). \]
An easy computation shows that \( f_1'(a) > 0 \) for \( 0 < a < 1 \). Thus \( f_1' \) is an increasing function of \( a \), and it follows that \( f_1'(a) > 0 \) for \( 0 < a < 1 \) since \( f_1'(0) = 0 \). Therefore \( f_1' \) is an increasing function of \( a \) for \( 0 < a < 1 \) and \( f_1(a) > 0 \) since \( f_1(0) = 0 \). Finally, this implies that \( f'(a) < 0 \) for \( 0 < a < 1 \), and thus \( f \) is a decreasing function of \( a \). Therefore \( d_{a_0} > d_a \) for all \( a, a_0 < a < 1 \).

In the disk \( |\zeta| < d_a \), \( u_a(\zeta) \) has the form \( (2.8) \), where \( u_1a \) is harmonic in \( |\zeta| < d_a \) and \( u_{1a}(0) = 0 \). Thus
\[ u_a^*(\rho e^{i\pi}) = -2\pi \log \frac{1}{\rho} - 2\pi \log \frac{A}{a} \]
and, similarly,
\[ u_{a_0}^*(\rho e^{i\pi}) = -2\pi \log \frac{1}{\rho} - 2\pi \log \frac{A_0}{a_0} \]
for \( 0 < \rho < d_a \). Hence \( u_a^*(\zeta) < u_{a_0}^*(\zeta) \) for \( -d_a \leq \zeta < 0 \).

(c) Since \( u_{1a}(\zeta) \) and \( u_{1a_0}(\zeta) \) are harmonic in \( |\zeta| < d_a \) and \( u_{1a}(0) = u_{1a_0}(0) = 0 \), then for every \( \epsilon > 0 \) there is a \( \rho_0, \rho_0 = |\zeta_0| < d_a \), such that \( |u_{1a}(\zeta)| < \epsilon/2 \) and \( |u_{1a_0}(\zeta)| < \epsilon/2 \) for all \( \zeta, |\zeta| < \rho_0 \). Thus
\[
\begin{align*}
    u_a^*(\rho e^{i\phi}) &= \sup_{|E|=2\phi} \int_E u_a(\rho e^{it}) \, dt \\
    &= -2\phi \log \rho - 2\phi \log \frac{A}{a} + \sup_{|E|=2\phi} \int_E u_{1a}(\rho e^{it}) \, dt \\
    &\leq -2\phi \log \rho - 2\phi \log \frac{A}{a} + \phi \epsilon
\end{align*}
\]
and
\[
\begin{align*}
    u_{a_0}^*(\rho e^{i\phi}) &= \sup_{|E|=2\phi} \int_E u_{a_0}(\rho e^{it}) \, dt \\
    &= -2\phi \log \rho - 2\phi \log \frac{A_0}{a_0} + \sup_{|E|=2\phi} \int_E u_{1a_0}(\rho e^{it}) \, dt \\
    &\geq -2\phi \log \rho - 2\phi \log \frac{A_0}{a_0} - \phi \epsilon
\end{align*}
\]
for \( 0 < \rho \leq \rho_0 \) and \( 0 < \phi < \pi \). Now choose \( \epsilon < \log(Aa_0/aA_0) \). Then
\[
    u_a^*(\rho e^{i\phi}) - u_{a_0}^*(\rho e^{i\phi}) \leq -2\phi \log \frac{Aa_0}{aA_0} + 2\phi \epsilon < 0
\]
for all \( 0 < \rho \leq \rho_0 \) and \( 0 < \phi < \pi \). Hence \( u_a^*(\zeta) < u_{a_0}^*(\zeta) \) for \( |\zeta| \leq \rho_0 < d_a \) and \( 0 < \phi < \pi \).

(d) To establish the inequality on \( -\infty < \zeta < -d_a \), we fix \( \epsilon > 0 \) and consider the function
\[
    Q(\zeta) = u_a^*(\zeta) - u_{a_0}^*(\zeta) - \epsilon \phi,
\]
\( \zeta = \rho e^{i\phi} \), which is subharmonic in \( A = \{ \zeta : \rho_0 < |\zeta|, 0 < 3\zeta \} \) and continuous in the closure of \( A \). Let \( M \) be the maximum of \( Q(\zeta) \) in \( \overline{A} \). Then \( M \geq 0 \) and, according to the maximum principle for subharmonic functions, the maximum is attained somewhere on the boundary of \( A \). Suppose \( M > 0 \). Since \( u_a^*(\zeta) \leq u_{a_0}^*(\zeta) \) on the set \( \{ \zeta : -d_a \leq \zeta \leq \rho_0 \} \cup \{ \zeta : |\zeta| = \rho_0, 3\zeta > 0 \} \cup \{ \zeta : \rho_0 \leq \zeta < \infty \}, \) there
is some point $-\zeta_1 = -\rho_1$ for which $-\infty < \zeta_1 < -d_a$ and $Q(\zeta_1) = M$. Let $G_a(\phi)$ denote the symmetric decreasing rearrangement of $u_a(\rho_1 e^{i\phi})$. Then
\[ \frac{\partial u_{a}^*(\rho_1 e^{i\phi})}{\partial \phi} = 2G_a(\phi) \]
for $0 < \phi \leq \pi$ by [2, Proposition 2]. But because $\rho_1 > d_a$, there is some point on the circle $|\zeta| = \rho_1$ that lies outside $D_a$, so
\[ G_a(\pi) = \inf_{0 \leq \phi \leq \pi} u_a(\rho_1 e^{i\phi}) = 0. \]
Applying the same argument to $u_{a_0}$ we obtain
\[ \frac{\partial u_{a_0}^*(\rho_1 e^{i\phi})}{\partial \phi} = 2G_{a_0}(\phi) \]
for $0 \leq \phi \leq \pi$. If $d_a < \rho_1 \leq d_{a_0}$, then
\[ G_{a_0}(\phi) = \inf_{0 \leq \phi \leq \pi} \{ t : \lambda(t) \leq 2\phi \}, \]
where $\lambda$ is the distribution function of $u_{a_0}$, $\lambda(t) = |\{ \phi : u_{a_0}(\rho_0 e^{i\phi}) > t \}|$, and
\[ G_{a_0}(\pi) = \lim_{\phi \to -\pi} -G_{a_0}(\phi). \]
Hence $G_{a_0}(\pi) \geq 0$ if $d_a < \rho_1 \leq d_{a_0}$. If $d_{a_0} < \rho_1$, there is some point on the circle $|\zeta| = \rho_1$ that lies outside $D_{a_0}$, so
\[ G_{a_0}(\pi) = \inf_{0 \leq \phi \leq \pi} u_{a_0}(\rho_1 e^{i\phi}) = 0. \]
Therefore
\[ \frac{\partial Q}{\partial \phi}(\zeta_1) \leq -\epsilon < 0, \]
which contradicts the assumption that $Q(\zeta)$ has a relative maximum at $\zeta_1$. Hence $M = 0$ and
\[ u_{a}^*(\zeta) \leq u_{a_0}^*(\zeta) + \epsilon \phi \leq u_{a_0}^*(\zeta) + \epsilon \pi \]
for $\zeta \in A$. Letting $\epsilon \to 0$ we obtain that
\[ u_{a}^*(\rho e^{i\phi}) \leq u_{a_0}^*(\rho e^{i\phi}) \]
for $\zeta \in A$.

We are in a position now to prove that $u_{a}^*(\zeta) < u_{a_0}^*(\zeta)$ in the open upper half-plane. Combining (a)–(d) we obtain (2.7). Furthermore, $u_{a}^*(\zeta) < u_{a_0}^*(\zeta)$ on the set $\{ \zeta : -d_a \leq \zeta \leq \rho_0 \} \cup \{ \zeta : |\zeta| = \rho_0, \Im \zeta > 0 \}$ by (b) and (c). Hence $u_{a}^* - u_{a_0}^*$ is a subharmonic function on $A$ that is not identically equal to zero there and, by the maximum principle, this implies that $u_{a}^*(\zeta) < u_{a_0}^*(\zeta)$ everywhere in $A$. Also, $u_{a}^*(\zeta) < u_{a_0}^*(\zeta)$ for $\{ \zeta : 0 < |\zeta| \leq \rho_0 \leq d_a, 0 < \Im \zeta \}$ by (c). Therefore,
\[ u_{a}^*(\zeta) < u_{a_0}^*(\zeta) \]
in the open upper half-plane.

It follows from Proposition 2.2 that
\[ (2.9) \quad \int_0^{2\pi} \Phi(\log |B_a(r e^{i\theta})|) d\theta \leq \int_0^{2\pi} \Phi(\log |B_{a_0}(r e^{i\theta})|) d\theta \]
for all $0 < a_0 < a < 0$ and $0 < r < 1$. The proof of strict inequality in (2.9) is identical to the proof of strict inequality in Theorem 1 in [2, pp. 157-158] and will be omitted. This completes the proof of Theorem 2.1. □
Proof of Theorem 1.1. The choice $\Phi(x) = e^{2x}$ in (2.1) allows us to conclude that
$$\Lambda_{\Phi_1}(B_a(re^{i\theta})) < \Lambda_{\Phi_1}(B_{a_0}(re^{i\theta}))$$
for all $0 \leq a_0 < a < 0$ and $0 < r < 1$. Let
$$\|B_a(re^{i\theta})\|_p^p = \frac{1}{2\pi} \int_0^{2\pi} |B_a(re^{i\theta})|^p d\theta.$$ 
Since
$$\Lambda_{\Phi_1}(B_a(re^{i\theta})) = 1 + \frac{\infty}{n=1} \frac{\|B_a(re^{i\theta})\|_{2n}^{2n}}{n!},$$
and, by Lemma 1 of [1], $B_a \in H^p$ for $0 < p < \infty$, we can choose a sequence $r_n \to 1$ as $n \to \infty$ for which the inequalities $\Lambda_{\Phi_1}(B_a(r_n e^{i\theta})) < \Lambda_{\Phi_1}(B_{a_0}(r_n e^{i\theta}))$ hold. Hence
$$\Lambda_{\Phi_1}(B_a(re^{i\theta})) \leq \Lambda_{\Phi_1}(B_{a_0}(re^{i\theta}))$$
for all $0 < r \leq 1$ by Hardy’s convexity theorem for integral means (see, e.g., [6, Theorem 1.5]).

It now remains to demonstrate that strict inequality holds true in Theorem 1.1. According to Theorem 2 of [4], $B_0$ is a local maximum on the set of Beurling functions. Thus there is an $a_0$, $0 < a_0$, such that
$$\Lambda_{\Phi_1}(B_a(e^{i\theta})) < \Lambda_{\Phi_1}(B_0(e^{i\theta}))$$
for $0 < a \leq a_0$. (James and Matheson [8] have informed the author that, using a numerical method, they have proved the last inequality for $0 < a < 1/2$.)

Finally, combine the last inequality with the fact that $\Lambda_{\Phi_1}$ is log-convex [4, p. 387] to complete the proof of Theorem 1.1. □

It was pointed out in [1] that $B_0$ does not maximize the integral means over $\mathcal{B}$. If we choose $\Phi(x) = e^{\alpha x}$, $0 < p < \infty$, in Theorem 2.1, we obtain that $B_0$ maximizes the integral means over $\mathcal{B}_0$.

Corollary 2.3. The inequality
$$\frac{1}{2\pi} \int_0^{2\pi} |B_a(re^{i\theta})|^p d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |B_{a_0}(re^{i\theta})|^p d\theta$$
holds true for all $0 \leq |a_0| < |a| < 0$, $0 < r \leq 1$, and all $0 < p < \infty$.

It will be interesting to see if the approach in Theorem 2.1 can be extended to the univalent functions in $\mathcal{D}$. The result of this paper provides further evidence in favor of a conjecture made in [1]:

Conjecture 1. $\Lambda_{\Phi_1}$ attains its maximum on $\mathcal{B}$ at $B_0$.

REFERENCES


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Department of Mathematics, Lamar University, P. O. Box 10047, Beaumont, Texas 77710

E-mail address: andreev@math.lamar.edu