THE HOMOLOGICAL DETERMINANT
OF QUANTUM GROUPS OF TYPE A

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ABSTRACT. Let \( R \) be a Hecke symmetry depending algebraically on a parameter \( q \in \mathbb{C} \). We show that the homology of the Koszul complex associated with \( R \) is one-dimensional when \( q \) is not a root of unity. A generator of this homology group then induces the homological determinant of the quantum group associated with \( R \).

INTRODUCTION

Let \( V \) be a vector space over a field \( k \) and \( GL(V) \) the general linear group. It is well known that elements of \( GL(V) \) act on the \( n \)-th homogeneous component of the exterior algebra over \( V \) by means of the determinant. More precisely, let \( x_1, x_2, \ldots, x_d \) be a basis of \( V \). Then \( \wedge_d(V) \) is one-dimensional and a non-zero vector is \( x_1 \wedge x_2 \wedge \ldots \wedge x_d \). If \( g \in GL(V) \) has the matrix \( A \) with respect to this basis, then

\[
g \cdot (x_1 \wedge x_2 \wedge \ldots \wedge x_d) = \det A \cdot x_1 \wedge x_2 \wedge \ldots \wedge x_d.
\]

Now let \( V \) be a vector superspace of dimension \( (r|s) \), \( r + s = d \). The super group \( GL(V) \) is defined as follows. Let \( x_1, x_2, \ldots, x_d \) be a homogeneous basis of \( V \), where the parity of the first \( r \) elements is even and the parity of the rest is odd. Let \( z^i_j \) be the endomorphism that maps \( x_i \) to \( x_j \) and other basis elements to zero. We consider \( z^i_j \) as a generator with parity being the sum of those of \( x_i \) and \( x_j \). The super semi-group \( \text{End}(V) \) is the spectrum of the super commutative algebra

\[
M := k\langle \{ z^i_j \}_{1 \leq i,j \leq d} \rangle / (z^i_j z^k_l = (-1)^{(i+j)(k+l)} z^k_l z^i_j)
\]

(where \( k\langle \{ z^i_j \}_{1 \leq i,j \leq d} \rangle \) denotes the free non-commutative algebra and \( i \) denotes the parity of \( x_i \)). Thus, for a super commutative algebra \( K \), an endomorphism of \( V_K := V \otimes K \) is a \( K \)-point of \( M \), i.e. an algebra homomorphism \( M \rightarrow K \).

The invertibility of a super matrix can be given in terms of the super determinant or Berezinian, which was introduced by Berezin. Let \( K \) be a super commutative algebra and \( Z \) be a \( K \)-point of \( \text{End}(V) \). The matrix \( Z = (z^i_j) \) has the following form: \( Z = (A B) \) where \( A, D \) are square matrices of dimension \( m \times m \) and \( n \times n \), respectively, whose entries’ parities are even, and \( B, D \) are matrices of types \( m \times n \)
and \( n \times m \), whose entries’ parities are odd. The super determinant of \( Z \) is defined to be

\[
\text{Ber} Z = \det T^{-1} \det (A - CD^{-1}B).
\]

It is shown that the matrix \( Z \) is invertible iff its super determinant is and that the super determinant is multiplicative. Thus, the invertible super matrix forms a group \( GL(V) \), which is an algebraic super-subgroup of \( \text{End}(V) \). It is however not clear why the definition of Ber is independent of the choice of bases (our basis is a distinguished basis).

In [17] Manin suggested the following construction to define the super determinant. Let \( V^* \) denote the vector space dual to \( V \) with the dual basis \( \xi_1, \xi_2, \ldots, \xi_n \), \( \xi^i(x_j) = \delta^i_j \). Manin introduced the following Koszul complex: its \((k, l)\)-term is given by \( K^{k,l} := \wedge^k \otimes S^l \), where \( \wedge^n \) and \( S_n \) are the \( n \)-th homogeneous components of the exterior and the symmetric tensor algebra over \( V \). The differential \( d^{k,l} : K^{k,l} \rightarrow K^{k+1,l+1} \) is given by

\[
d^{k,l}(h \otimes \phi) = \sum_i h x_i \otimes \xi^i \wedge \phi.
\]

It is easy to check that \( d_{k,l} \) is \( GL(V) \)-equivariant; hence the homology groups of this complex are representations of \( GL(V) \). On the other hand, one can show that this complex is exact everywhere except at the term \((m, n)\), where the homology group is one-dimensional; thus, it defines a one-dimensional representation of \( GL(V) \). It turns out that elements of \( GL(V) \) act on this representation by means of its super determinant; in other words, the definition of the super determinant is basis free.

The quantum semigroup of type \( A \) is the “spectrum” of the bialgebra

\[
E := k\{z_{i}^{m} \}_{1 \leq i, j \leq d}/(R_{uv}^{i}_{k}z_{i}^{u}z_{i}^{v} = z_{i}^{u}z_{i}^{v}R_{uv}^{i}_{k})
\]

where \( R \) is a Hecke symmetry (see [1]). The Hecke symmetry resembles the usual flipping operator \( a \otimes b \mapsto b \otimes a \) or \( a \otimes b \mapsto (-1)^{ab} b \otimes a \) (\( a, b \) are homogeneous) in super symmetry.

In [5, 14], a Koszul complex is defined for \( R \). For that, one first has to define the quantum exterior and quantum symmetric tensors by means of certain projectors on \( V^{\otimes n} \). It is still an open question whether this complex has the homology group concentrated at a certain term and its dimension is one. Some efforts have been made. Gurevich [5] showed this for even Hecke symmetries (i.e., those that induce a finite-dimensional exterior algebra); Lyubashenko and Sudbery [14] showed this for Hecke sums of an odd and an even Hecke symmetry.

In this paper, assuming that \( R \) depends algebraically on \( q \), where \( q \) runs in \( C \), we give the affirmative answer to this question for an algebraically dense set of values of \( q \). Our tactic is first to use a new result of Deligne [1] to check the case \( q = 1 \). Then using a standard argument we show that for a dense set of values of \( q \), the homology group of \( K \) has dimension less than that of the corresponding homology groups when \( q = 1 \). In other words, for an algebraically dense subset of \( C \), the homology group has dimension at most 1. It remains to show the non-vanishing of the homology.
1. HECKE SYMMETRIES AND THE ASSOCIATED QUANTUM GROUPS

We work over an algebraically closed field $k$ of characteristic zero. Let $V$ be a vector space over $k$ of dimension $d$. Let $R : V \otimes V \rightarrow V \otimes V$ be an invertible operator. $R$ is called a Hecke symmetry if the following conditions are fulfilled:

- $R_1 R_2 R_3 = R_2 R_1 R_2$, where $R_1 := R \otimes \text{id}_V$, $R_2 := \text{id}_V \otimes R$,
- $(R + 1)(R - q) = 0$ for some $q \in k$.
- The half adjoint to $R$, $R^i : V^* \otimes V \rightarrow V \otimes V^*$, $\langle R^i(\xi \otimes v), w \rangle = \langle \xi, R(v \otimes w) \rangle$, is invertible.

Throughout this work we will assume that $q$ is not a root of unity other than the unity itself. If $q = 1$, $R$ is called vector symmetry. Vector symmetries were introduced by Lyubashenko [13] and generalized to Hecke symmetries by Gurevich [5].

Let us fix a basis $x_1, x_2, \ldots, x_d$ of $V$. Then $R$ can be given in terms of a matrix, also denoted by $R$, $R(x_i \otimes x_j) = x_k \otimes x_i R_{ij}^{kl}$, where we adopt the convention of summing over the indices that appear in both the lower and upper places. The matrix $R_{ij}^{kl}$ is given by $R_{ij}^{kl} = R_{kl}^{ji}$. Therefore, the invertibility of $R^i$ can be expressed as follows: there exists a matrix $P$ such that

$$P_{jm}^{in} R_{ml}^{nk} = \delta_i^j \delta_k^l, \quad R_{jm}^{in} P_{ml}^{nk} = \delta_i^j \delta_k^l. \quad (1)$$

Consider the following algebra:

$$E_R := k \langle \{ z_j^i \}_{1 \leq i,j \leq d} / (z_m^i z_n^j R_{mn}^{ij} = R_{pq}^{ij} z_p^q \rangle, \quad (\text{coproduct given by } \Delta(z_j^i) = z_k^i \otimes z_j^k \text{ and the counit given by } \varepsilon(z_j^i) = \delta_j^i).$$

The coassociative structure is given by $r(z_j^i, z_k^j) = R_{jk}^{i}$. The bialgebra $E$ is called the “function algebra” on the corresponding quantum endomorphism space or the matrix quantum semigroup.

There is a right coaction of $E_R$ on $V^*$, given by $\delta(x_i) = x_j \otimes z_j^i$. This coaction induces actions of $E_R$ on $V^* \otimes V$ for $n \geq 1$. The braiding on $V \otimes V$ induced from the coassociative structure $r$ is precisely the operator $R$. There is a natural $\mathbb{N}$-grading on $E_R$, where the $n$-th homogeneous component consists of homogeneous polynomials of total degree $n$ and is denoted by $E_n$. Then $E_n$ is a coalgebra and coacts on $V^* \otimes V$ from the right; hence its dual $E_n^*$ acts on $V^* \otimes V$ from the left.

The Hecke algebra $H_n = H_{q,n}$ has generators $T_i, 1 \leq i \leq n - 1$, subject to the relations: $T_i T_j = T_j T_i, |i - j| \geq 2; T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, 1 \leq i \leq n - 2; T_i^2 = (q - 1)T_i + q$. There is a $k$-basis in $H_n$ indexed by permutations of $n$ elements: $T_w$, $w \in \mathfrak{S}_n$ ($\mathfrak{S}_n$ is the permutation group), in such a way that $T_{w(i,i+1)} = T_i$ and $T_u T_v = T_{uv}$ if the length of $uv$ is equal to the sum of the length of $u$ and the length of $v$. If $q$ is not a root of unity of degree greater than 1, $H_n$ is a semisimple algebra. For more details, the reader is referred to [2, 13].

The Hecke symmetry $R$ induces an action of the Hecke algebra $H_n = H_{q,n}$ on $V^* \otimes V$, $T_i \mapsto R T_i = \text{id}^{i-1} \otimes R \otimes \text{id}^{n-i-1}$ that commutes with the coaction of $E_R$. The action of $T_w$ will be denoted by $R_w$. We have the following “Double centralizer theorem” [6 Thm. 2.1].

1.1. The algebras $\rho_n(H_n)$ and $E_n^*$ are centralizers of each other in $\text{End}_k(V^* \otimes V)$.

Consequently, simple $E_n^*$-modules (and hence simple $E_n$-comodules) can be given as the image of primitive idempotents of $H_n$, and conjugate idempotents determine isomorphic (co)modules. Since conjugate classes of primitive idempotents of $H_n$ are
indexed by partitions of \( n \), simple subcomodules of \( V^\otimes n \) are indexed by a subset of partitions of \( n \). Thus \( E \) is cosemisimple, and its simple comodules are indexed by a subset of partitions.

Let \( I_\lambda \) denote the simple comodule corresponding to the partition \( \lambda \). Then \( I_\lambda \) and \( I_\mu \) can be realized as the images of two primitive idempotents \( e_\lambda \in \mathcal{H}_r \) and \( e_\mu \in \mathcal{H}_s \). Thus \( I_\lambda \otimes I_\mu \) is the image of a (not necessarily primitive) idempotent in \( \mathcal{H}_{r+s} \). This idempotent decomposes into an orthogonal sum of primitive idempotents, which yields a decomposition of \( I_\lambda \) and \( I_\mu \) into a direct sum of simple subcomodules.

Taking into account that conjugate idempotents define isomorphic comodules, we have \([7]\)

\[
I_\lambda \otimes I_\mu \cong \bigoplus_\gamma I_\gamma^{\otimes c_{\gamma\lambda\mu}}
\]

where the \( c_{\gamma\lambda\mu} \) are the Littlewood-Richardson coefficients describing the multiplicity of the Schur function \( s_\gamma \) in the product of two other Schur functions \( s_\lambda \) and \( s_\mu \) (cf. \([15]\)).

**Example (Quantum symmetrizers).** The primitive idempotent

\[
X_n := \frac{1}{|\mathfrak{S}_n|} \sum_{w \in \mathfrak{S}_n} R_w
\]
determines a simple comodule \( S_n \) called the \( n \)-th quantum symmetric tensor power, and the primitive idempotent

\[
Y_n := \frac{1}{|\mathfrak{S}_n|/q} \sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} R_w
\]
determines a simple comodule \( \wedge_n \) called the \( n \)-th quantum anti-symmetric tensor power. Notice that \( S_n = I_{(n)} \) and \( \wedge_n = I_{(n^\uparrow)} \).

Let us briefly recall here the Littlewood-Richardson algorithm for computing the coefficients \( c_{\gamma\lambda\mu} \) \([15]\). Let \( \gamma \) and \( \lambda \) be partitions with \( \gamma_i \geq \lambda_i \) for all \( i \). We define the skew diagram \( [\gamma \setminus \lambda] := \{(i, j) : (i, j) \in [\gamma], \lambda_i < j \leq \gamma_i \} \). The \( i \)-th row of the diagram consists of nodes \( (i, j) \) with fixed \( i \), and the \( j \)-th column consists of nodes \( (i, j) \) with fixed \( j \).

Let \( \mu \) be a partition. A sequence of positive integers is said to have type \( \mu \) if each \( i \) occurs \( \mu_i \) times. Such a sequence is said to be “good” if for any term \( i > 1 \), the number of previous \( i - 1 \) in the sequence is strictly greater than the number of previous \( i \). For example, the good sequences of type \( \mu = (2, 1) \) are \( 112 \), \( 121 \).

The coefficient \( c_{\lambda\mu\gamma} \) where \( \lambda \) is a partition of \( r \), \( \mu \) is a partition of \( s \) and \( \gamma \) is a partition of \( r + s \), can be computed as follows:

(i) if \( \lambda_i > \gamma_i \) for some \( i \), then \( c_{\lambda\mu\gamma} = 0 \);

(ii) if \( \lambda_i \leq \gamma_i \) for every \( i \), then \( c_{\lambda\mu\gamma} \) is the number of ways of replacing the nodes \( (i, j) \) of \( [\gamma \setminus \lambda] \) by integers, such that

- each \( k \) occurs \( \mu_k \) times;
- the numbers are non-decreasing along rows and strictly increasing down columns;
- when reading from right to left in successive rows, we have a good sequence of type \( \mu \).
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\end{array} \). There are two good sequences of type \( \mu = (2, 1) : (112), (121) \). We have the following possibilities for \( \gamma \) for which \( c_\lambda^\gamma \mu \neq 0 \):

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algebra associated to $R$ is a factor algebra of the free non-commutative algebra over entries of $Z$ and $T$:

$$ H_R := T \{ t_{ij} \}_{1 \leq i, j \leq d} / (z_{m,n} = p_{pq}^n q_{kl}^m, t_{ij} = s_{ij}^t = s_{ij}^-). $$

$H_R$ is a Hopf algebra, the antipode is given by $S(z_j^i) = t_{ij}^*$, and the coquasitriangular structure on $E_R$ can be extended to $H_R$ thanks to the closedness of $R$: $r(z_j^i, t_{ij}^*) = P_{ji}^{kl} r(t_{ij}^*, z_k^l) = R^{-1 i j} [S, \text{Thm. 2.1.1}].$

If the Hecke symmetry $R$ is even of rank $r$, $H_R$ is cosemisimple and its simple comodules can be parameterized by sequences of $r$ non-decreasing integers $[S, \text{Thm. 3.2.1}].$ A similar statement holds for odd Hecke symmetries.

If the Hecke symmetry $R$ is neither even nor odd, the structure of $H_R$-comodules is more complicated than the structure of $E_R$-comodules. In particular, the category $H_R$-comod is not semisimple. We have, however, the following result $[S, \text{Thm. 2.3.5}].$

1.3. The natural map $E_R \longrightarrow H_R$ is injective. Consequently, every simple $E_R$-comodule is a simple $H_R$-comodule.

Among $H_R$-comodules that are not $E$-comodules, the super determinant plays an important role. The well-known tool for defining the quantum super determinant serves the Koszul complex (of second type) introduced by Manin $[17].$ This is a (bi-)complex, whose $(k,l)$ term is $\wedge_k \otimes S_l^*.$ The differential is induced from the dual basis map. The homology group of this complex is an $H_R$-comodule; if it is one dimensional over $k$, it defines a group-like element in $H_R$ called the homological determinant or quantum super determinant or, in some cases, the quantum Berezinian.

2. The Koszul complex

We begin with the description of the Koszul complex. We first recall the dual comodule of a tensor product of two comodules. For two (finite-dimensional) comodules $V, W$, the dual to $V \otimes W$ is isomorphic to $W^* \otimes V^*$, with the pairing given by $(\varphi \otimes \psi)(v \otimes w) := \varphi(v) \psi(w), \varphi \in W^*, \psi \in V^*, v \in V, w \in W$. The dual to longer tensor products is defined in a similar way.

Fix a basis $x_1, x_2, \ldots, x_d$ of $V$ and let $\xi^1, \xi^2, \ldots, \xi^d$ be the dual basis in $V^*$. Let $ev_V,$ be the evaluation map $ev_V(\varphi \otimes v) = \varphi(v)$ and $db_V$ be the dual basis map defined as follows: $db : k \longrightarrow V \otimes V^*, db(1) = \sum_i x_i \otimes \xi^i$. These maps clearly do not depend on the choice of basis and are maps of $H_R$-comodules. The term $K^{k,l}$ of the Koszul complex associated to $R$ is $\wedge_k \otimes S_l^*$, and the differential $d_{k,l}$ is given by:

$$ \wedge_k \otimes S_l^* \longrightarrow V \otimes X_k \otimes S_{l+1}^{*+1} \otimes V \otimes Y_k \otimes X_{l+1}^{*+1} \wedge_{k+1} \otimes S_{l+1}^*, $$

where $X_k, Y_k$ are the $q$-symmetrizer operators introduced in the previous section. Thus we have in fact a collection of diagonal subcomplexes, each of which contains the terms $K^{k,l}$ with $k - l$ equal to a fixed number. One defines another differential $d'$ as follows:

$$ \wedge_k \otimes S_l^* \longrightarrow V \otimes X_k \otimes S_{l+1}^{*+1} \otimes V \otimes Y_k \otimes X_{l+1}^{*+1} \wedge_{k+1} \otimes S_{l+1}^*, $$

where $\tau_{V,V}$ denotes the braiding on $V \otimes V^*$ induced from the coquasitriangular structure on $H_R$, its matrix is given by $P$, the inverse to the half-adjoint of $R$. 


Since all vector spaces are $H_R$-comodules and all maps are $H_R$-comodule maps, we have in fact complexes in $H_R$-comod.

The differentials $d$ and $d'$ satisfy

$$(qdd' + d'd) |_{K^{s,i}} = q^k \left( \text{rank}_q R + [l - k]_q \right) \text{id},$$

where $\text{rank}_q R := P^i_{ij}$, and $P$ is given in [1]. Hence, if $\text{rank}_q R \neq -[l - k]_q$, the cohomology group at the term $(k,l)$ vanishes.

**Theorem 1.** Let $R$ be a Hecke symmetry of birank $(r,s)$. Then

(i) $\text{rank}_q R = -[s - r]_q$;

(ii) the simple comodule $I_\lambda$ is injective and projective in the category of $H_R$-comodules if and only if $\lambda_r \geq s$;

(iii) the homology of the Koszul complex at the term $(r,s)$ is non-vanishing.

**Proof.** Since $R$ has birank $(r,s)$, the simple $E$-comodule $I_\lambda \neq 0$ iff $\lambda_{r+1} \leq s$. Using this fact and the Littlewood-Richardson formula, we can easily show that (Hom means $\text{Hom}^{H_R}$):

Hom$(I_{((s+1)r)}', I_{(s+1)'} \otimes \Lambda_r \otimes S_{s}^*) \cong \text{Hom}(I_{((s+1)r)} \otimes S_{s}, I_{(s+1)} \otimes \Lambda_r) = k$,

Hom$(I_{((s+1)r)}', I_{(s+1)} \otimes \Lambda_{r-1} \otimes S_{s-1}^*) \cong \text{Hom}(I_{((s+1)r)} \otimes S_{s-1}, I_{(s+1)} \otimes \Lambda_{r-1}) = 0$,

Hom$(I_{((s+1)r)}', I_{(s+1)} \otimes \Lambda_{r+1} \otimes S_{s+1}^*) \cong \text{Hom}(I_{((s+1)r)} \otimes S_{s+1}, I_{(s+1)} \otimes \Lambda_{r+1}) = 0$.

As a consequence, $I_{(s+1)} \otimes \Lambda_r \otimes S_{s}^*$ contains $I_{((s+1)r)}'$ as a subcomodule while the comodules $I_{(s+1)} \otimes \Lambda_{r-1} \otimes S_{s-1}^*$, $I_{(s+1)} \otimes \Lambda_{r+1} \otimes S_{s+1}^*$ do not.

Assume that $\text{rank}_q R \neq -[s - r]_q$. Then the complex is exact at $K^{r,s}$ and $dd' + d'd = q^k \left( \text{rank}_q R + [s - r]_q \right) \text{id} \neq 0$. On the other hand, since $I_{((s+1)r)}'$ is a submodule of $I_{(s+1)} \otimes \Lambda_{r+1} \otimes S_{s+1}^*$, the restriction of $\text{id}_{I_{((s+1)r)}'} \otimes d^{r,s}$ to it should be zero. Analogously, the restriction of $\text{id}_{I_{((s+1)r)}'} \otimes d^{m,n}$ to $I_{((s+1)r)}'$ is zero. Thus, the restriction of $dd' + d'd$ on $I_{(s+1)}'$ is zero, a contradiction. Therefore, $\text{rank}_q R = -[s - r]_q$.

According to [11] Thm. 3.2 if $\text{rank}_q R = -[s - r]_q$, then $H$ possesses a non-zero integral (i.e. an $H$-comodule homomorphism $H \rightarrow k$, where $H$ coacts on itself by the coproduct and on $k$ by the unit map). Then, according to [10] Prop. 5.1 and to [11] Thm. 3.1, $I_\lambda$ is injective and projective in $H$-comod iff $\lambda_r \geq s$. Thus, $I_{((s+1)r)}'$ is projective and injective. Therefore, if $I_{((s+1)r)}'$ is a subquotient of a comodule, it is a direct summand; hence it cannot be a subquotient of $I_{(s+1)} \otimes \Lambda_{m-1} \otimes S_{n-1}^*$, and in particular, it cannot be a subcomodule of $I_{(s+1)} \otimes \text{Im}d^{r-1,s-1}$. Consequently,

$I_{(s+1)} \otimes \text{Im}d^{r-1,s-1} \neq I_{(s+1)} \otimes \text{Ker}d^{m,n}$.

Thus, the sequence

$$\cdots \rightarrow I_{(s+1)} \otimes \Lambda_{r-1} \otimes S_{s-1}^* \rightarrow I_{(s+1)} \otimes \Lambda_r \otimes S_s^* \rightarrow I_{(s+1)} \otimes \Lambda_{r+1} \otimes S_{r+1}^* \rightarrow \cdots$$

which is obtained by tensoring $K^+$ with $I_{(s+1)}$, is not exact at the term $(r,s)$, whence neither is $K^+$. \[ \square \]

3. The case $q = 1$

Assume in this section that $q = 1$; thus, $R^2$ is the identity map and $H$-comod is a tensor category (i.e., symmetric rigid monoidal). By a theorem of Deligne [1], there exists a faithful and exact tensor (i.e. symmetric monoidal) functor $\mathcal{F}$ from $H$-comod to the category of vector superspaces. Under this functor, $V$ is mapped
to a certain vector superspace $V$ and $R$ is mapped to the supersymmetry on $V \otimes V$, denoted by $T$.

We can therefore reconstruct a super bialgebra $E$ and a Hopf super algebra $H$ from $V$ and $T$. We will show that this Hopf superalgebra is isomorphic to the function algebra over the general linear supergroups $GL(r|s)$, where $(r, s)$ is the birank of $R$, or, in other words, the super dimension of $V$ is $(r|s)$. Indeed, $E$ is the function algebra on $\text{End}(V)$ and the images of $I_\lambda$ under the embedding $F$ are simple $E$-comodules. Since $F$ is faithful and exact and since $I_\lambda \neq 0 \Leftrightarrow \lambda_{r+1} < s$, we conclude that $E$ is isomorphic to the function algebra on $M(r|s)$. Hence $H$ is isomorphic to the function algebra on $GL(r|s)$, by virtue of [13].

Let $K^-$ denote the image of the complex $K^-$. Then the homology of $K^-$ is concentrated at the term $(r, s)$, and is one-dimensional; it defines the super determinant. As a consequence, the homology of $K^-$ is also concentrated at the term $(r, s)$, for $F$ is faithful and exact. Let $D$ denote the homology of $K^-$. Then $D$, the image of $D$ under $F$, is one-dimensional and hence invertible; consequently,

$$\mathcal{F}(D^\ast \otimes D) \cong \mathcal{F}(D^\ast) \otimes \mathcal{F}(D) \cong D^- \otimes D \cong k,$$

where the last isomorphism is given by the evaluation morphism, that is, the image of $ev_D : D^\ast \otimes D \rightarrow k$ under $\mathcal{F}$. Since $\mathcal{F}$ is faithful and exact, we conclude that $D^\ast \otimes D \cong k$, that is $D$ is invertible, hence one-dimensional. Thus, we have proved:

**Theorem 2.** Let $R$ be a vector symmetry of birank $(r, s)$. Then the associated Koszul complex is exact everywhere except at the term $(r, s)$ where it has a one-dimensional homology group, which determines a group-like element called the homological determinant.

4. **The case $q$ generic**

Using the result of the previous section we show in this section that given a Hecke symmetry of birank $(r, s)$ that depends algebraically on $q$, then, for a dense set of values $q$, the associated Koszul complex is exact everywhere except at the term $(r, s)$, where it has a one-dimensional homology group and thus determines a group-like element in $H_{R_q}$, called the homological determinant. In this section $k$ will be assumed to be the field $\mathbb{C}$ of complex numbers.

Thus let $R = R_q$ be a Hecke symmetry depending on a parameter $q \in \mathbb{C}$. We first observe that the dimension of $\Lambda_{q,k}$ does not depend on $q$, so far as $q$ is not a root of unity. Indeed, $\Lambda_{q,k}$ is the image of a projection, and its dimension can be given as the trace of a matrix that depends algebraically on $q$. Since $\mathbb{C}$ without the set of roots of unity is still connected, we conclude that this trace, being always integral, must be a constant. The same happens with $S_{q,l}$. Thus, the terms of $K^-$ have dimension not depending on $q$.

On the other hand, observe that the rank of the operator $d_{q,i}^k$, for almost any $q$ (that is, except for a finite number of values of $q$) is larger than the rank of $d_{q,i}^k$ and for the kernel of $d_{q,i}^k$, we have the reversed inequality. Consequently, the dimension over $k$ of the homology group $H(K_{q,i}^k)$ for almost any $q$ is less than or equal to the dimension of $H(K_{q,i}^k)$. According to Theorems 1 and 2, we conclude that for an algebraically dense set of values of $q$, $H(K_{q,i}^k) = 0$, for all $(k, l) \neq (r, s)$ and $H(K_{q,i}^r)$ is one-dimensional.
Theorem 3. Let $R = R_q$ be a Hecke symmetry over $\mathbb{C}$, depending algebraically on $q$. Then there is an algebraically dense set of values of $q$ for which the homology of the Koszul complex is one-dimensional and concentrated at the term $(r, s)$, where $(r, s)$ is the birank of $R$.

References


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