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GATEAUX DERIVATIVE OF $B(H)$ NORM

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ABSTRACT. We prove that for Hilbert space operators X and Y, it follows that

$$\lim_{t\to 0^+}\frac{||X+tY||-||X||}{t}=\frac{1}{||X||}\inf_{\varepsilon>0}\sup_{\varphi\in H_\varepsilon,||\varphi||=1}\operatorname{Re}\langle Y\varphi,X\varphi\rangle,$$

where $H_\varepsilon=E_{X^*X}((||X||-\varepsilon)^2,||X||^2)$. Using the concept of φ-Gateaux derivative, we apply this result to characterize orthogonality in the sense of James in $B(H)$, and to give an easy proof of the characterization of smooth points in $B(H)$.

1. Introduction

Let H be a Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. Also, let $\mathfrak{S}_p$, $p\geq 1$, denote the Schatten ideal, i.e. $\mathfrak{S}_p=\{X\in B(H)\mid \sum s_j^p(X)<+\infty\}$, where $s_j(X)=\lambda_j(X^*X)^{1/2}$ are singular numbers. The ideal $\mathfrak{S}_p$ is a Banach space with respect to the norm $||X||_p=(\sum s_j^p(X))^{1/p}$. For more details concerning $\mathfrak{S}_p$, the reader is referred to [8], or [3], Chapter III.

It is well known that $B(H)$ and all $\mathfrak{S}_p$, except $\mathfrak{S}_2$, have a Banach space structure which does not admit an inner product generating the initial norm. Nevertheless, we can define some kind of orthogonality, called orthogonality in the sense of James [4]. This concept is closely related to the concept of smoothness of the norm and Gateaux derivative.

Definition 1.1. Let X be a Banach space, and $x,y\in X$.

a) We say that y is orthogonal to x if for all complex λ we have

$$||x+\lambda y||\geq||x||. \tag{1}$$

b) We say that x is a smooth point of the sphere $K(0,||x||)$ in X if there exists a unique functional F_x, called the support functional, such that $||F_x||=1$ and $F_x(x)=||x||$.

Remark 1.2. If X is a Hilbert space, from (1) we can easily derive $\langle x,y\rangle=0$. In general, such orthogonality is not symmetric in Banach spaces. To see this, consider vectors $(-1,0)$ and $(1,1)$ in the space $\mathbf{C}^2$ with the max-norm.

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Proposition 1.3. *a) If $x \in X$ is the smooth point of the corresponding sphere, then the limit* $\lim\limits_{\mathbf{R}\ni t\to 0}\frac{||x+ty||-||x||}{t}$ *always exists and it is equal to* $\operatorname{Re} F_x(y)$.

b) In this case y is orthogonal to x if and only if $F_x(y) = 0$.

This proposition is well known, see [1], [4], and it was used in characterizing the orthogonality in the sense of James in $\mathfrak{S}_p$ spaces.

Proposition 1.4. *Let $\mathfrak{S}_p$ be the Schatten ideal, for $1 < p < +\infty$, and $X, Y \in \mathfrak{S}_p$. The operator Y is orthogonal to X if and only if* $\operatorname{tr}(|X|^{p-1}U^*Y) = 0$, *where $X = U|X|$ is the polar decomposition of X.*

Proof. From the Clarkson-McCarthy inequalities it follows that the dual space $\mathfrak{S}_p^* \cong \mathfrak{S}_q$ is strictly convex. From this we can derive that every nonzero point in $\mathfrak{S}_p$ is a smooth point of the corresponding sphere. So we can check what is the unique support functional F_X. For details see [1], Theorem 2.3. □

However, if dual space is not strictly convex, there are many points which are not smooth. For instance, this happens in $\mathfrak{S}_1$, $\mathfrak{S}_\infty$ and $B(H)$. In [5] the concept of φ-Gateaux derivative was developed in order to substitute the usual concept of Gateaux derivative at points which are not smooth.

Proposition 1.5. *Let X be a Banach space, $x, y \in X$, and $\varphi \in [0, 2\pi)$.*

a) The function $\alpha : \mathbf{R} \to \mathbf{R}$, $\alpha(t) = ||x + te^{i\varphi}y||$ is convex. The limit $D_{\varphi,x}(y) = \lim\limits_{t\to 0^+}\frac{||x+te^{i\varphi}y||-||x||}{t}$ *always exists. The number $D_{\varphi,x}(y)$ we shall call the φ-Gateaux derivative of the norm at the vector x, in the y and φ directions.*

b) The vector y is orthogonal to x in the sense of James if and only if the inequality $\inf_\varphi D_{\varphi,x}(y) \geq 0$ *holds.*

Proof. This proposition was proved in [5], but for the convenience of the reader we shall outline the proof.

a) We have, for $\theta \in [0, 1]$, $\alpha(\theta t + (1-\theta)s) = ||\theta(x + te^{i\varphi}y) + (1-\theta)(x + se^{i\varphi}y)|| \leq \theta\alpha(t) + (1-\theta)\alpha(s)$. $D_{\varphi,x}(y)$ is the right derivative of the function α at the origin, and it always exists since α is a convex function.

b) If y is orthogonal to x in the sense of James, then the fraction $\frac{||x+te^{i\varphi}y||-||x||}{t}$ is positive for all $t > 0$, and passing to the limit we obtain $D_{\varphi,x}(y) \geq 0$, for all φ. On the other hand, if $D_{\varphi,x}(y) \geq 0$ for all φ, then $\alpha(t) - \alpha(0) \geq (t-0)D_{\varphi,x}(y)$, by convexity of α, and therefore $||x + te^{i\varphi}y|| \geq ||x||$ for all $t > 0$, and for all $\varphi \in [0, 2\pi)$. □

The Gateaux derivative and the φ-Gateaux derivative have also been used in global minimizing problems; see for instance [6], [7], and the references therein.

The main result in this note is to determine the φ-Gateaux derivative of the $B(H)$ norm, as it is written in the abstract. As a consequence we give a characterization of orthogonality in the sense of James in the space $B(H)$.

Further, if X attains its norm on two linearly independent vectors, or if $||X||$ is a point of the continuous part of $\sigma(|X|)$, then we shall construct two different support functionals. From this we derive a characterization of the smoothness in $B(H)$.

2. Results

Before we prove the main result, we need three technical lemmas.

Lemma 2.1. *Let X, and Y be selfadjoint operators, $X \geq 0$, and let $0 < \varepsilon < ||X||$ and $0 < \delta < 1/4$ be fixed real numbers such that $||Y|| \leq \varepsilon\delta$. Also, let $H_\varepsilon = E_X(||X|| - \varepsilon, ||X||)$, where E_X is the spectral measure of the operator X, and $H_{\varepsilon,\gamma} = \{f \in H \mid f = f_1 + f_2, f_1 \in H_\varepsilon, f_2 \in H_\varepsilon^\perp, ||f_2|| \leq \gamma ||f_1||\}$. Then, we have $||X+Y|| = \sup_{||\varphi||=1, \varphi \in H_{\varepsilon,\gamma}} \langle (X+Y)\varphi, \varphi\rangle$, where $\gamma = \sqrt{2\delta/(1-4\delta)}$.*

Proof. The operator $X+Y$ is selfadjoint, and therefore we have $||X+Y|| = \sup_{||\varphi||=1} \langle (X+Y)\varphi, \varphi\rangle$. Let $\beta > 0$ be arbitrary, and let φ be a unit vector such that $\langle (X+Y)\varphi, \varphi\rangle \geq ||X+Y|| - \beta$. It is enough to prove that φ is in $H_{\varepsilon,\gamma}$, for β small enough.

Indeed, let $\varphi = \varphi_1 + \varphi_2$, $\varphi_1 \in H_\varepsilon$, $\varphi_2 \in H_\varepsilon^\perp$. Then we have $\langle (X+Y)\varphi, \varphi\rangle = \langle (X+Y)\varphi_1, \varphi_1\rangle + \langle (X+Y)\varphi_2, \varphi_2\rangle + 2\operatorname{Re}\langle (X+Y)\varphi_1, \varphi_2\rangle$. Taking into account that $\langle X\varphi_1, \varphi_2\rangle = 0$ and $||\varphi_1||^2 + ||\varphi_2||^2 = 1$, we get

$$\langle (X+Y)\varphi_1, \varphi_1\rangle \leq ||X+Y||\, ||\varphi_1||^2,$$
$$\langle (X+Y)\varphi_2, \varphi_2\rangle \leq (||X|| - \varepsilon + \varepsilon\delta)||\varphi_2||^2 = (||X|| - (1-\delta)\varepsilon)||\varphi_2||^2,$$
$$2\operatorname{Re}\langle (X+Y)\varphi_1, \varphi_2\rangle \leq 2\varepsilon\delta ||\varphi_1||\, ||\varphi_2||,$$

and from this we obtain

$$\begin{aligned} ||X+Y|| - \beta &\leq \langle (X+Y)\varphi, \varphi\rangle \\ &\leq ||X+Y||\, ||\varphi_1||^2 + (||X|| - (1-\delta)\varepsilon)||\varphi_2||^2 + 2\varepsilon\delta||\varphi_1||\, ||\varphi_2|| \end{aligned}$$

and also

$$||X+Y||\, ||\varphi_2||^2 \leq \beta(||\varphi_1||^2 + ||\varphi_2||^2) + (||X|| - (1-\delta)\varepsilon)||\varphi_2||^2 + 2\varepsilon\delta||\varphi_1||\, ||\varphi_2||$$

or, equivalently,

$$(||X+Y|| - ||X|| + (1-\delta)\varepsilon)||\varphi_2||^2 \leq (\beta + \varepsilon\delta)(||\varphi_1||^2 + ||\varphi_2||^2).$$

However, since $||X+Y|| \geq ||X|| - \varepsilon\delta$, it follows that $((1-3\delta)\varepsilon - \beta)||\varphi_2||^2 \leq (\beta + \varepsilon\delta)||\varphi_1||^2$, and hence

$$\frac{||\varphi_2||^2}{||\varphi_1||^2} \leq \frac{\beta + \varepsilon\delta}{(1-3\delta)\varepsilon - \beta} \leq \frac{2\varepsilon\delta}{(1-4\delta)\varepsilon} = \frac{2\delta}{1-4\delta} = \gamma^2,$$

for all $\beta < \varepsilon\delta$. The proof is complete. □

Lemma 2.2. *Let A, B and C be selfadjoint operators such that A and C are positive. Also, let $H_\varepsilon = E_A(||A|| - \varepsilon, ||A||)$, where E_A is the spectral measure of the operator A. Then*

$$\limsup_{t \to 0^+} \frac{||A + tB + t^2 C|| - ||A||}{t} \leq \sup_{||\varphi||=1, \varphi \in H_\varepsilon} \langle B\varphi, \varphi\rangle.$$

Proof. By using Lemma 2.1, we have

$$\begin{aligned} ||A + tB + t^2C|| &= \sup_{||\varphi||=1, \varphi \in H_{\varepsilon,\gamma}} \left\langle (A + tB + t^2C)\varphi, \varphi\right\rangle \\ &\leq ||A|| + t \sup_{||\varphi||=1, \varphi \in H_{\varepsilon,\gamma}} \langle B\varphi, \varphi\rangle + t^2 \sup_{||\varphi||=1, \varphi \in H_{\varepsilon,\gamma}} \langle C\varphi, \varphi\rangle, \end{aligned}$$

for t small enough. From this we obtain

$$\begin{aligned}\limsup_{t\to 0^+} &\frac{||A+tB+t^2C||-||A||}{t}\\ &\le \lim_{t\to 0^+}\left(\sup_{||\varphi||=1,\varphi\in H_{\varepsilon,\gamma}}\langle B\varphi,\varphi\rangle + t\sup_{||\varphi||=1,\varphi\in H_{\varepsilon,\gamma}}\langle C\varphi,\varphi\rangle\right)\\ &= \sup_{||\varphi||=1,\varphi\in H_{\varepsilon,\gamma}}\langle B\varphi,\varphi\rangle .\end{aligned}$$

Since γ is arbitrary, we get

$$\limsup_{t\to 0^+}\frac{||A+tB+t^2C||-||A||}{t}\le \inf_{\gamma}\sup_{||\varphi||=1,\varphi\in H_{\varepsilon,\gamma}}\langle B\varphi,\varphi\rangle .$$

We have to eliminate the number γ. For a unit vector $\varphi\in H_{\varepsilon,\gamma}$ there exists a unit vector $\psi\in H_\varepsilon$ such that $||\varphi-\psi||^2\le 2(1-1/\sqrt{1+\gamma^2})$. (It is enough to put $\psi=\varphi_1/||\varphi_1||$, where $\varphi=\varphi_1+\varphi_2$, $\varphi_1\in H_\varepsilon$, $\varphi_2\in H_\varepsilon^\perp$.) Then

$$\begin{aligned}\langle B\varphi,\varphi\rangle &= \langle B(\varphi-\psi),\varphi\rangle + \langle B\psi,\varphi-\psi\rangle + \langle B\psi,\psi\rangle\\ &\le 2||B||\,||\varphi-\psi|| + \sup_{||\psi||=1,\psi\in H_\varepsilon}\langle B\psi,\psi\rangle\\ &\le \sup_{||\psi||=1,\psi\in H_\varepsilon}\langle B\psi,\psi\rangle + 2||B||\sqrt{2(1-1/\sqrt{1+\gamma^2})}.\end{aligned}$$

Thus,

$$\sup_{||\varphi||=1,\varphi\in H_{\varepsilon,\gamma}}\langle B\varphi,\varphi\rangle\le \sup_{||\psi||=1,\psi\in H_\varepsilon}\langle B\psi,\psi\rangle + 2||B||\sqrt{2(1-1/\sqrt{1+\gamma^2})},$$

and we obtain the result by taking the infimum over all positive γ. □

Lemma 2.3. *Let X and Y be bounded operators. Then it follows that*

$$\lim_{t\to 0^+}\frac{||X+tY||-||X||}{t}\le\frac{1}{||X||}\sup_{||\varphi||=1,\varphi\in H_\varepsilon}\operatorname{Re}\langle Y\varphi,X\varphi\rangle ,$$

where $H_\varepsilon = E_{X^*X}((||X||-\varepsilon)^2,||X||^2)$.

Proof. Indeed, by using Lemma 2.2, we have

$$\begin{aligned}\lim_{t\to 0^+}\frac{||X+tY||-||X||}{t} &= \lim_{t\to 0^+}\frac{||X+tY||^2-||X||^2}{t(||X+tY||+||X||)}\\ &= \frac{1}{2||X||}\lim_{t\to 0^+}\frac{||X^*X+t(Y^*X+X^*Y)+t^2Y^*Y||-||X^*X||}{t}\\ &\le \frac{1}{2||X||}\sup_{\varphi\in H_\varepsilon,||\varphi||=1}\langle (Y^*X+X^*Y)\varphi,\varphi\rangle\\ &= \frac{1}{||X||}\sup_{\varphi\in H_\varepsilon,||\varphi||=1}\operatorname{Re}\langle Y\varphi,X\varphi\rangle .\end{aligned}\tag{2}$$

□

Theorem 2.4. *Let X and Y be in $B(H)$. We have*

$$\lim_{t\to 0^+}\frac{||X+tY||-||X||}{t}=\frac{1}{||X||}\inf_{\varepsilon>0}\sup_{\varphi\in H_\varepsilon,||\varphi||=1}\operatorname{Re}\langle Y\varphi,X\varphi\rangle ,$$

where $H_\varepsilon = E_{X^*X}((||X|| - \varepsilon)^2, ||X||^2)$, and E_{X^*X} stands for the spectral measure of the operator X^*X.

Proof. First of all, note that $\inf_{\varepsilon>0}$ can be replaced by $\lim_{\varepsilon\to 0^+}$, since the mapping $\varepsilon \mapsto \sup_{\varphi\in H_\varepsilon, ||\varphi||=1} \operatorname{Re}\langle Y\varphi, X\varphi\rangle$ is decreasing.

Next, given $\varepsilon > 0$, choose $\varphi_\varepsilon \in H_\varepsilon$, $||\varphi_\varepsilon|| = 1$, such that $\operatorname{Re}\langle Y\varphi_\varepsilon, X\varphi_\varepsilon\rangle \geq \sup_{\varphi\in H_\varepsilon, ||\varphi||=1} \operatorname{Re}\langle Y\varphi, X\varphi\rangle - \varepsilon$. It is obvious that $\langle X\varphi_\varepsilon, X\varphi_\varepsilon\rangle \mapsto ||X||^2$, as $\varepsilon \to 0^+$, and hence

$$\begin{aligned}
\frac{||X+tY|| - ||X||}{t} &= \frac{||X^*X + t(Y^*X + X^*Y) + t^2Y^*Y|| - ||X||^2}{t(||X+tY|| + ||X||)} \\
\geq \frac{1}{||X+tY|| + ||X||} &\left[\frac{\langle X^*X\varphi_\varepsilon, \varphi_\varepsilon\rangle - ||X||^2}{t} + 2\operatorname{Re}\langle Y\varphi_\varepsilon, X\varphi_\varepsilon\rangle + t\langle Y\varphi_\varepsilon, Y\varphi_\varepsilon\rangle\right] \\
\geq \frac{1}{||X+tY|| + ||X||} &\left[\frac{\langle X^*X\varphi_\varepsilon, \varphi_\varepsilon\rangle - ||X||^2}{t} + 2\left(\sup_{\varphi\in H_\varepsilon, ||\varphi||=1} \operatorname{Re}\langle Y\varphi, X\varphi\rangle - \varepsilon\right)\right. \\
&\qquad \left. + t\langle Y\varphi_\varepsilon, Y\varphi_\varepsilon\rangle\right]
\end{aligned}$$

Taking $\liminf_{\varepsilon\to 0^+}$ we get

$$\frac{||X+tY|| - ||X||}{t} \geq \frac{2\inf_{\varepsilon>0} \sup_{\varphi\in H_\varepsilon, ||\varphi||=1} \operatorname{Re}\langle Y\varphi, X\varphi\rangle + t\liminf_{\varepsilon\to 0^+} \langle Y\varphi_\varepsilon, Y\varphi_\varepsilon\rangle}{||X+tY|| + ||X||},$$

and finally, passing to the limit as $t \to 0^+$ we obtain

$$\lim_{t\to 0^+} \frac{||X+tY|| - ||X||}{t} \geq \frac{1}{||X||} \inf_{\varepsilon>0} \sup_{\varphi\in H_\varepsilon, ||\varphi||=1} \operatorname{Re}\langle Y\varphi, X\varphi\rangle.$$

The other "$\leq$" inequality follows immediately from Lemma 2.3. □

3. Applications

Corollary 3.1. *Let X, Y be Hilbert space operators. The following three conditions are mutually equivalent.*

(i) The operator $Y \in B(H)$ is orthogonal to the operator X, in the sense of James, i.e. for all $\lambda \in \mathbf{C}$ it follows that $||X + \lambda Y|| \geq ||X||$;

(ii) For all θ, and for all $\varepsilon > 0$, it follows that

$$\sup_{\varphi\in H_\varepsilon, ||\varphi||=1} \operatorname{Re} e^{i\theta}\langle Y\varphi, X\varphi\rangle \geq 0; \tag{3}$$

(iii) There exists a sequence of unit vectors φ_n such that $||X\varphi_n|| \to ||X||$ and $\langle Y\varphi_n, X\varphi_n\rangle \to 0$ as $n \to \infty$.

Proof. $(i) \Leftrightarrow (ii)$. From Theorem 2.4, we have that

$$D_{\theta,X}(Y) = \frac{1}{||X||} \inf_{\varepsilon>0} \sup_{\varphi\in H_\varepsilon, ||\varphi||=1} \operatorname{Re}(e^{i\theta}\langle Y\varphi, X\varphi\rangle),$$

and taking into account Proposition 1.5, we are done.

$(iii) \Rightarrow (i)$ Let there exist a sequence of unit vectors φ_n such that $||X\varphi_n|| \to ||X||$ and $\langle X\varphi_n, Y\varphi_n\rangle \to 0$, and let $\lambda \in \mathbf{C}$ be an arbitrary complex number. Then

$||X+\lambda Y||^2 \geq ||(X+\lambda Y)\varphi_n||^2 = ||X\varphi_n||^2 + 2\operatorname{Re}\lambda\langle Y\varphi_n, X\varphi_n\rangle + |\lambda|^2||Y\varphi_n||^2 \geq ||X\varphi_n||^2 + 2\operatorname{Re}\lambda\langle Y\varphi_n, X\varphi_n\rangle \to ||X||^2$ as $n \to +\infty$.

$(ii) \Rightarrow (iii)$ The set $\{\langle Y\varphi, X\varphi\rangle \mid ||\varphi|| = 1, \varphi \in H_\varepsilon\}$ is, in fact, the numerical range of the operator X^*Y on the subspace H_ε. This set is convex by the Toeplitz-Hausdorff theorem. Its closure is, therefore, a closed convex set, and by the condition (3) it has such a position in the complex plane that it must contain at least one value with positive real part, under all rotations around the origin. So, it must contain zero, and we get a vector $\varphi \in H_\varepsilon$ such that $|\langle Y\varphi, X\varphi\rangle| < \varepsilon$. If ε runs through the set $1/\mathbf{N}$ we obtain the required sequence. □

Remark 3.2. The equivalence between (i) and (iii) was proved using quite different methods in [2].

Clearly, the above theorem is a generalization of Corollary 2, from [5]. The following statement was initially proved in [1], Theorem 3.1 (in truth for real spaces). We present a simple proof of this result.

Corollary 3.3. *The operator X is a smooth point of the sphere $K(0, ||X||)$ in the space $B(H)$ if and only if for some $\delta > 0$ the space $H_\delta = E_{|X|}(||X|| - \delta, ||X||)$ is of dimension one, where $E_{|X|}$ denotes the spectral measure of the operator $|X| = \sqrt{X^*X}$, i.e. $s_1(X) > s_2(X)$.*

Proof. If for all $\delta > 0$, the space H_δ is of dimension greater than one, then either X attains its norm on at least two linearly independent vectors, or for all $\delta > 0$, the space H_δ is infinite dimensional.

Let us, first, assume that X attains its norm on two linearly independent vectors, say f and g. Suppose that f and g are unit vectors. Then we have two functionals $F_X(Y) = \langle Xf, Yf\rangle/||X||$ and $G_X(Y) = \langle Xg, Yg\rangle/||X||$. It is easy to see that $F_X(X) = G_X(X) = ||X||$, and also $||F_X||, ||G_X|| \leq 1$. So we have two different support functionals, and X cannot be a smooth point.

Let us, now, assume that for all $\delta > 0$, the space H_δ is infinite dimensional. Consider the spaces $K_n = E_{|X|}(||X|| - \varepsilon_n, ||X|| - \varepsilon_{n+1})H$, where $E_{|X|}$ is the spectral measure of the operator $|X| = \sqrt{X^*X}$, and ε_n is a decreasing sequence tending to zero. There is no loss of generality if we assume that every space K_n is at least two dimensional. (We can do this by choosing a subsequence.) Now, we choose mutually orthogonal unit vectors $f_n, g_n \in K_n$, and we get two sequences of mutually orthogonal vectors such that $||Xf_n||, ||Xg_n|| \to ||X||$. Let glim denote the Banach generalized limit on the space c of all convergent complex sequences. Consider the following functionals: $F_X(Y) = \operatorname{glim}_{n\to+\infty}\langle Xf_n, Yf_n\rangle/||X||$, and $G_X(Y) = \operatorname{glim}_{n\to+\infty}\langle Xg_n, Yg_n\rangle/||X||$. It is easy to see that both of them are support functionals. Further, for the operator $Y = \sum_n \langle\ \ , f_n\rangle Xf_n$, we have $F_X(Y) = ||X||$ and $G_X(Y) = 0$. Thus, X cannot be a smooth point.

Finally, let us assume that X attains its norm on the unique vector f, and that the norm of the restriction $X|_L$ is equal to $||X|| - \delta$, where $L = \{f\}^\perp$. We shall prove that Y orthogonal to X implies $\langle Xf, Yf\rangle = 0$. Indeed, by Proposition 1.5, and by (2), we get

$$0 \leq D_{\varphi,X}(Y) \leq \frac{1}{||X||}\operatorname{Re}\left\langle Xf, e^{i\varphi}Yf\right\rangle,$$

since for $\varepsilon < \delta$, the space H_ε is equal to $Lin\{f\}$. Thus, for all φ we have $0 \leq \operatorname{Re} e^{-i\varphi}\langle Xf, Yf\rangle$ implying $\langle Xf, Yf\rangle = 0$.

We have a support functional $F_X(Y) = \langle Xf, Yf\rangle / ||X||$. Suppose that there is another support functional G_X. Then there exists an operator Y such that $G_X(Y) = 0$ and $F_X(Y) \neq 0$. For all complex numbers λ we get $||X|| = G_X(X) = G_X(X + \lambda Y) \leq ||X + \lambda Y||$, which means that Y is orthogonal to X, implying $F_X(Y) = 0$. This is the contradiction.

Hence, in this case, there is a unique support functional and X is a smooth point of the corresponding sphere. □

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