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FREE PRODUCTS ARISING FROM ELEMENTS OF FINITE ORDER IN SIMPLE RINGS

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ABSTRACT. Our main result implies that, if R is a simple artinian ring which is not a matrix ring over an absolute field, then any noncentral element of R, of prime order not dividing the characteristic, is a factor in a free product with a unit which has infinite order in R. Unexpected consequences follow for division rings and group algebras.

1. The main theorem and its consequences

Let R be an associative algebra over a field k. If the group of units $\mathcal{U}(R)$ contains a free product $C_n * C_\infty$ of cyclic groups, then obviously R contains a non-central element of order n. The converse is evidently false in this degree of generality. For example, let P be a finite non-abelian p-group, and consider the group ring $R = k[P]$ over a field k of characteristic p. Then R contains non-central p-power elements, and it is well known that $\mathcal{U}(R)$ is nilpotent. Any possible converse also needs to take into account the situation encountered in the following example: Let D be a division ring the center k of which contains a primitive p-th root of unity (for some prime p). Then any element of D of multiplicative order p is central, so $D^* = D \setminus \{0\}$ cannot contain a product $C_p * C_\infty$. In particular, if a is a non-central element of D of order p^2, then D^* does not contain $C_{p^2} * C_\infty$, and the best one can hope for is a group $G = \langle a, u \rangle \subseteq D^*$, with a central subgroup Z of order p, such that $G/Z \cong C_p * C_\infty$.

For simple artinian rings R, the situation is not completely clear when the characteristic divides the order of the element a. Nevertheless, we have the following result (recall that an absolute field is a subfield of the algebraic closure of a finite field).

Theorem 1.1. *Let $R = D^{t\times t}$, where D is a division ring with center k, and assume that D is not an absolute field. Let $a \in R$ be a non-central element of finite order*

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n. Set $\langle a\rangle \cap k^* = \langle a^r\rangle$, where $r \mid n$. Then R contains infinitely many units u of infinite order such that $\langle a, u\rangle/\langle a^r\rangle \cong C_r * C_\infty$ in the following cases:

(a) *$\operatorname{char} D$ does not divide n (in particular, this is the case when either $\operatorname{char} D = 0$, or $t = 1$).*

(b) *D is commutative.*

(c) *D is locally finite-dimensional over its center.*

(We refer to the above situation by saying that a is largely free in *R, and that $\langle a, u\rangle$ is a* free product modulo the center*.)*

Plainly, the result cannot hold when D is an absolute field, because $\mathrm{GL}(n, D)$ is then locally finite. Also, any attempt to generalize Theorem 1.1 can fail in the presence of non-trivial central idempotents.

Lemma 1.2. *Let $R = \bigoplus_{i=1}^{m} R_i$ be semisimple artinian with $m \geq 2$, and assume that none of the simple rings R_i is a matrix ring over an absolute field. Then any element of finite prime-power order not dividing $\operatorname{char} R$ is largely free in R. The conclusion can fail* (a) *if the element has composite order; or* (b) *if one of the R_i is a non-commutative matrix ring over an absolute field.*

Proof. Let $a \in R$ have order $n = p^m$, where p does not divide $\operatorname{char} R_i$ for any i. If $R_i = Re_i$, where e_i is an idempotent, then there exists an i such that ae_i has order exactly n in R_i. By Theorem 1.1, there exists a unit u_i of infinite order in R_i such that $\langle ae_i, u_i\rangle$ is a free product modulo center. If $u = u_i + (1 - e_i)$, then it is trivial to verify that $\langle a, u\rangle$ is also a free product over the center.

As an example for (a), let $R = R_1 \oplus R_2$ for ease of notation, and suppose that for $i = 1, 2$, the subring R_i contains an element a_i of order n_i modulo the center of R_i, and that the least common multiple of n_1 and n_2 is $n > \max\{n_1, n_2\}$. Then $a = a_1 + a_2$ has finite order n modulo the center of R. Let u be any unit of R, and let w be the group commutator $(a^{n_1}, u^{-1}a^{n_2}u)$. Then the projection of w into each R_i is e_i, so in fact $w = 1$ in R. But $w = a^{-n_1}u^{-1}a^{-n_2}ua^{n_1}u^{-1}a^{n_2}u$, so $\langle a, u\rangle$ is *not* a free product modulo its center $\langle a^n\rangle$.

For (b), suppose that some $R_i = k^{t\times t}$ is a matrix ring over an absolute field k. Let α be any non-central element of R_i, and set $a = \alpha e_i + (1 - e_i)$. Then, for any unit u of R, we have $\langle a, u\rangle e_j = \langle ue_j\rangle$ if $j \neq i$, and $\langle a, u\rangle e_i \subseteq \mathcal{U}(R_i)$, which is locally finite. Thus, $\langle a, u\rangle$ cannot be a free product modulo the center for any u. □

We derive a number of consequences of Theorem 1.1 first. Our first corollary is the following counterpart to a recent result of Passman and Gonçalves on integral units [3].

Corollary 1.3. *Let k be a non-absolute field of characteristic $p \geq 0$, and let G be a finite group. Let a be an element of prime order in G. Then a is largely free in the group ring $k[G]$ if and only if a is non-central in $G/O_p(G)$ (recall that $O_p(G)$ is the largest normal p-subgroup of the group G, and $O_0(G) = 1$).*

Proof. Let J denote the Jacobson radical of $S = k[G]$. If $x \in G$ becomes central in S/J, then the group commutator $(x, G) \subseteq (1 + J) \cap G = O_p(G)$, so x is central in $G/O_p(G)$. Therefore, if a is non-central modulo $O_p(G)$, then it remains non-central in the semisimple ring S/J, and hence in at least one of the simple components R of S/J. All such components contain the non-absolute field k in their center. By Theorem 1.1, the projection b of a in R is largely free in R, so there exists a unit $v \in R$ of infinite order such that $\langle b, v\rangle = \langle b\rangle * \langle v\rangle$ is a free product. After replacing

v by its extension to a unit of S/J, if necessary, v can be lifted to a unit u of S over the nilpotent ideal J. The elements u and a generate a free product in S, so a is largely free.

Conversely, suppose that a is central in $G/O_p(G)$. If u is any unit of S, then the commutator $(a,u)=1$ in S/J, since a is central modulo J. But the group $1+J$ is nilpotent, and so $\langle a,u\rangle$ is soluble. In other words, no free product in S can contain $\langle a\rangle$ as a factor. □

In another direction, Theorem 1.1 has the following striking corollary.

Corollary 1.4. *Let R be as in Theorem* 1.1*, and let N be a normal subgroup of $\mathcal{U}(R)$. Then N contains a non-central element of prime order $p \neq$ char R if and only if N contains a free product of a countable number of copies of C_p.*

Proof. Suppose N contains a non-central element a of finite order n coprime to $\operatorname{char} D$. Then, in the notation of Theorem 1.1, N contains the subgroup $\langle a^{u^i} : i \in \mathbb{Z}\rangle$, which, modulo its central subgroup $\langle a^r\rangle$, is a free product of countably many copies of C_r. The stated result is the case $n=r=p$. □

The previous corollary provides a much-simpler proof of a result of Gorbounov *et al.* ([4], Proposition 3.7) regarding the existence of free products in the Morava stabilizer group (a certain normal subgroup of the p-adic division algebra of index $n=p-1$ and Hasse invariant $1/n$). Starting with a non-central element α of prime order, the authors obtain a free product of a finite number of conjugates of $\langle\alpha\rangle$ by using Nagao's Theorem on the amalgamated free product structure of $\mathrm{GL}(2,K[t])$, and the uncountability of the p-adic field.

We note in passing that Corollary 1.4 is also a generalization of Herstein's well-known result that a non-central element of finite order in a division ring has an infinite number of conjugates (see, e.g., [8], Theorem 13.26).

2. Proof of the main theorem

The proof of Theorem 1.1 is based on two special cases. The first requires the following technical result.

Lemma 2.1. *Let $R=\sum_{i=0}^{m-1} y^iK$ be a cyclic algebra, where $Gal(K/E)=\langle\sigma\rangle$ has order m, $y^m=b$, and E is not a finite field. Let $a\in K$ such that $a, a^\sigma,\dots,a^{\sigma^{m-1}}$ are distinct. Then there exists a non-zero rational function $\phi_a\in E(x_0,\dots,x_{m-1})$, where $x_0,\dots,x_{m-1}$ are commuting indeterminates over the field K, with the following property: If $w=\sum y^iw_i\in E[y]$, then $w^{-1}aw$ has full support if and only if $\phi_a(w_0,\dots,w_{m-1})\neq 0$ (as in* [2]*, we say that an element $\sum_{i=0}^{m-1} y^ib_i\in R$ has full support if every $b_i\neq 0$).*

Proof. Set $S=R\otimes_E E(x_0,\dots,x_{m-1})$, and let $x=\sum_{i=0}^{m-1} y^ix_i$ be the generic element of S. Write $y^m=b\in E$. It is convenient to have $b^{m!}\neq 1$. If this is not the case, replace y by αy, where $\alpha\in E^*$ is such that $\alpha^{m!}\neq 1$. This replaces y^i by α^iy^i, and so has no effect on the size of the support of the elements of R.

Write $x^{-1}=f^{-1}\sum_{i=0}^{m-1} y^if_i$, where $f,f_i\in E[x_0,\dots,x_{m-1}]^*$. The equation $1=x^{-1}x$ immediately implies that

$$x_0f_0+b\sum_{i=1}^{m-1} x_if_{m-i}=f, \tag{2.1}$$

and for $k = 1, \ldots, m-1$,

$$\sum_{i=0}^{k} x_i f_{k-i} + b \sum_{i=k+1}^{m-1} x_i f_{m+k-i} = 0. \tag{2.2}$$

If we write $x^{-1}ax = f^{-1}\sum_{t=0}^{m-1} y^t g_t$, then it is trivial to verify that

$$g_t = \sum_{i=0}^{t} a^{\sigma^i} x_i f_{t-i} + b \sum_{i=t+1}^{m-1} a^{\sigma^i} x_i f_{m+t-i} \tag{2.3}$$

for $t = 0, \ldots, m-1$. Use (2.1) or (2.2) as appropriate to eliminate the term involving x_0 from (2.3). This yields

$$g_0 = af + b \sum_{i=1}^{m-1} (a^{\sigma^i} - a) x_i f_{m-i}, \tag{2.4}$$

and for $t \geq 1$,

$$g_t = \sum_{i=1}^{t} (a^{\sigma^i} - a) x_i f_{t-i} + b \sum_{i=t+1}^{m-1} (a^{\sigma^i} - a) x_i f_{m+t-i}. \tag{2.5}$$

We claim that $\phi_a = f g_0 \ldots g_{m-1} \neq 0$. The easiest way to see this is to find suitable specializations of the values x_i for which the various $g_t \neq 0$. For example, consider the values $x_0 = 1$, $x_i = 0$ for $i \geq 1$. This specializes $x, x^{-1} \mapsto 1$, and by (2.4), specializes g_0 to $af(1, 0, \ldots, 0) = a \neq 0$. In particular, g_0 is not identically zero. Similarly, consider some $t \in [1, m-1]$. Let the least common multiple of m and t be n, so $n = tu = mv$. Then $1 - b^v = 1 - y^n = 1 - (y^t)^u$. Therefore, $(1 - b^v)(1 - y^t)^{-1} = (1 - (y^t)^u)(1 - y^t)^{-1} = \sum_{j=0}^{u-1} y^{jt}$. Consider the specialization $x_0 = 1, x_t = -1$, and all other $x_i = 0$. This yields $x \mapsto 1 - y^t$, and from (2.5) we obtain $g_t \mapsto (a^{\sigma^t} - a) x_t f_0 \mapsto -(a^{\sigma^t} - a)(1 - b^v)^{-1} \neq 0$. The result follows. □

The following is the first special case of Theorem 1.1.

Corollary 2.2. *Let $R = (K/E, \sigma, b)$, be a cyclic algebra, where $Gal(K/E) = \langle \sigma \rangle$ has order m, and the field $K = E[a]$ is generated by a primitive n-th root of unity a over the center E. Assume that E is the quotient field of a Dedekind domain distinct from E. Then a is largely free in R.*

Proof. Write $R = \bigoplus_{i=0}^{m-1} y^i K$ and $b = y^m \in E$. As in [2], for a fixed non-trivial discrete valuation ν of K, write $V_\nu \subset K$ for the set of those elements $x \in K$ such that the set of values $\{\nu(x^{\sigma^i}) : i = 0, \ldots, m-1\}$ has a unique maximum and a unique minimum, and let $T_\nu \subset R$ be the set of those non-zero elements $w = \sum_{i=0}^{m-1} y^i w_i$ of R such that the value $\nu(w_i^{\sigma^j})$ is independent of i and j.

The elements $a, a^2, \ldots, a^{r-1}$ are all distinct (recall that r is the least integer such that a^r is central). Let ϕ be the product of the finitely many rational functions $\phi_a, \phi_{a^2}, \ldots, \phi_{a^{r-1}}$ provided by Lemma 2.1. Since the field E is infinite, there exist m-tuples $(e_0, \ldots, e_{m-1}) \in E^{(m)}$ for which $\phi(e_0, \ldots, e_{m-1})$ is defined and non-zero. Setting $z = \sum y^i e_i$, we find that all the conjugates $c^i = z^{-1} a^i z$, $i = 1, \ldots r-1$, have full support.

By Lemma 7 of [2], there exists a valuation ν of K such that $V = V_\nu$ is infinite and all the c^i belong to $T = T_\nu$. If $v \in V$, then any word of the form $c^{r_1} v^{s_1} \ldots c^{r_d} v^{s_d}$, with $1 \leq r_1, \ldots, r_d \leq r-1$, belongs to $TVTV \ldots TV$. By Lemma 6 of [2], the

latter product is contained in TV. Since TV consists of elements of full support in R, it has empty intersection with K, so the product $TVTV\dots TV$ cannot be trivial. In other words, $\langle c, v\rangle$ is a free product over its center, and hence so is $\langle c, v\rangle^{z^{-1}} = \langle a, v^{z^{-1}}\rangle$. □

The other special case of Theorem 1.1 is, in some sense, at the other extreme, and deals with the case where the underlying division ring is commutative. We need a number of preliminary results.

Lemma 2.3. *Let K be a field with a non-trivial discrete valuation ν, and let $t \geq 2$. Fix a set $\{e_{ij}\}$ of t^2 matrix units in $R = K^{t\times t}$. Consider the subsets*

$$V = V_\nu = \Big\{ \textit{diag}(r_i) \in GL(t, K) \,:\, \textit{the set } \{\nu(r_i)\} \textit{ has a unique minimum value} \Big\},$$

$$T = T_\nu = \Big\{ \sum_{i,j} t_{ij}e_{ij} \,:\, \textit{the value } \nu(r_{ij}) \textit{ is finite and independent of } i \textit{ and } j \Big\}$$

of R. Then

(i) *$TVT \subseteq T$.*

(ii) *TV and VT are sub-semigroups of R, neither of which contains a scalar matrix.*

Proof. For $t = \sum_{i,j} t_{ij}e_{ij} \in T$, write $\nu(t) = \nu(t_{ij})$. Similarly, for $w = \mathrm{diag}(w_i) \in V$, write $\nu(w) = \min\{\nu(w_i)\}$. For any i, j, we have $(wt)_{i,j} = \sum_m w_{im}t_{mj} = w_i t_{ij} \neq 0$, so the final assertion of (ii) follows. If also $t' \in T$, then for any i, j we have $(twt')_{i,j} = \sum_{m,n} t_{im}w_{mn}t'_{nj} = \sum_m t_{im}w_m t'_{mj}$. For a fixed m, we have $\nu(t_{im}w_m t'_{mj}) = \nu(t) + \nu(t') + \nu(w_m)$. Since every m occurs in the sum for $(twt')_{i,j}$, it is evident that $\nu((twt')_{i,j}) = \nu(t) + \nu(t') + \nu(w)$. By definition, this means that $twt' \in T$, so (i) is established. The rest of (ii) now follows because $(TV)(TV) = (TVT)V \subseteq TV$, and similarly $VTVT \subseteq VT$. □

Corollary 2.4. *Let K be the quotient field of a Dedekind domain not equal to K, and let $t \geq 2$. If S is any finite set of non-scalar matrices in $GL(t, K)$, then there exists a matrix $x \in GL(t, K)$, and a discrete valuation ν of K, such that S^x is contained in the set T_ν defined in Lemma 2.3. If K is a finite Galois extension of a subfield Φ, then we may choose $x \in GL(t, \Phi)$.*

Proof. Let $g \in \mathrm{GL}(t, K)$ be non-scalar, and let $x = (x_{ij})$ be the generic $t \times t$ matrix in t^2 commuting indeterminates x_{ij} over K. It is well known that $x^{-1}gx$ has all entries non-zero (see, e.g., [10], page 36, Point 1). Let $\phi_g \neq 0$ be the product of the entries of $x^{-1}gx$, so $\phi_g \in K(x_{ij} \,:\, 1 \leq i, j \leq t)$. If $Q(g)$ is the Zariski open subset of $K^{t\times t}$ where ϕ_g is defined and non-zero, then, for every $w \in Q(g)$, the conjugate $g^w = \sum_{i,j} t_{ij}e_{ij}$ has every $t_{ij} \neq 0$. In particular, any $w \in \bigcap_{g\in S} Q(g)$ has the property that all the g^w have non-zero entries. The existence of ν now follows from Lemma 7 of [2].

To obtain the result over Φ, simply replace ϕ_g by $\psi_g = \prod_{\gamma\in\Gamma}(\phi_g)^\gamma$, where $\Gamma = \mathrm{Gal}(K/\Phi)$. Then $\psi_g \in \Phi(x_{ij} \,:\, 1 \leq i, j \leq t)^*$, and the result follows as before. □

This allows us to prove another special case of Theorem 1.1.

Corollary 2.5. *Let K be a non-absolute field. If a is any non-scalar matrix of finite order in $GL(t, K)$, then there exist infinitely many elements $u \in GL(t, K)$, of*

infinite order, such that $\langle a, u\rangle$ is a free product over the center. If K/k is a finite Galois extension, then the elements u may be chosen to belong to $GL(t, k)$.

Proof. Let Λ be the subring of K generated by the finitely many entries of a (and, if necessary, an element of K which is not a root of unity). Then Λ is a Dedekind domain which is not a field, and we may replace K by the quotient field of Λ in what follows.

Let S consist of the finitely many non-scalar powers of a. Choose a valuation ν as in Corollary 2.4, and then choose any $w \in V_\nu \cap V_\nu^{-1}$, where V_ν is defined in Lemma 2.3. By part (ii) of Lemma 2.3, there exists a conjugate b of a such that the group $\langle b, w\rangle$ is a free product modulo its center. Conjugating back, we obtain the result for a.

Finally, if k is given, then the sets T_ν and V_ν may be chosen to lie in $k^{t\times t}$, and the conjugating element may also be chosen in $\mathrm{GL}(t, k)$, by Corollary 2.4. □

Proof of Theorem 1.1. Part (b) is Corollary 2.5. For part (c), we may obviously assume that D is finite-dimensional over k. Let K/k be a Galois splitting field for R, so $R_K = R \otimes_k K \cong K^{t\times t}$ for some t. Consider the natural action of $G = \mathrm{Gal}(K/k)$ on R_K. By the last statement of Corollary 2.5, the unit u may be chosen to belong to the fixed ring of R_K under G. This fixed ring is obviously R, as required. We now turn to part (a), and we distinguish several cases.

Case 1. *$F = k[a]$ is a subfield of R*: The extension $F = k(a)$ of k is non-trivial and cyclotomic, so we can write $\mathrm{Gal}(F/k) = \langle\sigma\rangle$, where σ has finite order $m \geq 2$. By the (generalized) Skolem-Noether Theorem (e.g., [5], Theorem 4.3.1), σ is induced by conjugation by an element $y \in D$. In particular, $y^m = b \in C_R(a)$.

Let K be the quotient field of $k_0[a, b]$, where k_0 is the prime field. Then x acts on K as the automorphism σ, and $K[x] \equiv (K/E, \sigma, b)$ is a classical cyclic algebra, where E is the fixed field of K. The conclusion in this case follows from Corollary 2.2.

In the remaining cases, $F = k[a]$ is not a field. It is, however, semisimple, because the order of a is coprime to $\mathrm{char}\, k$. Let $1 = e_1 + \ldots + e_d$ be the decomposition of 1 into primitive idempotents in F.

Case 2. *k contains a primitive n-th root of unity ζ*: In this case, every $Fe_i = ke_i$. Let $V \cong D^{(t)}$ be a simple R-module. Then $V = \bigoplus_{j=1}^{d} Ve_j$. The decomposition of Ve_j into one-dimensional D-subspaces is equivalent to the decomposition of each e_j into a sum of orthogonal idempotents f_{ji}. These can be chosen to satisfy $f_{ji}e_j = e_jf_{ji} = f_{ji}$ for all i, and then the f_{ji} also commute with ae_j, since by assumption $ae_j \in k_j$. This means that the matrix of a determined by the idempotents f_{ji} is block diagonal $diag(A_1, \ldots, A_d)$, where each $A_j = \zeta^{r_j} I$ is a scalar matrix, and the least common multiple of the exponents r_j is n (what makes a non-central is, of course, the fact that not all the r_j are the same).

We also observe that, if k is an absolute field, then D contains an element b transcendental over k (since algebraic division algebras over absolute fields are commutative). Therefore, D always contains a non-absolute subfield K, and so R contains the subring $R_1 = K^{t\times t}$. Note that $a \in R_1$ by construction. The existence of a free product now follows from Corollary 2.5 and completes the proof of Theorem 1.1.

Case 3. *$F = k[a]$ is not a field, and k does not contain a primitive n-th root of unity*: Let $\Phi = k(\zeta)$, where ζ is a primitive n-th root of unity, and set $S = R \otimes_k \Phi$.

Then $S = \Delta^{s\times s}$ is simple artinian (see, e.g., [1], p. 364). By Case 2, there exists a subfield $K \supseteq \Phi$ of S which is not absolute, and such that $a \in K^{s\times s}$. Replacing K by the intersection of its finitely many conjugates under $\mathrm{Gal}(\Phi/k)$, we may assume that K is stable under $\mathrm{Gal}(\Phi/k)$. The conclusion follows from Corollary 2.5, by Galois descent. □

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